Disjoint packings on $6k+5$ points

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Abstract. We prove that if $v = 4 + 7^i t$, where $i \geq 0$ and $t \in \{1, 7, 13, 19, 25, 31, 43, 67\}$, then there exist $v - 4$ disjoint packings on v points of pairs into triples, all of which have the same leave, which is a (fixed) 4-cycle.

1. Introduction

Suppose X is a set of v elements (called *points*) and $\mathcal{A}$ is a collection of w-subsets (called *blocks*) of X. Then we call $(X, \mathcal{A})$ a *w-uniform hypergraph*. A $(2,3)$-*packing* on X is a 3-uniform hypergraph $(X, \mathcal{A})$ such that every pair of points appears in at most one block. The *leave* of a $(2,3)$-packing $\mathcal{A}$ is the graph (X, E), where E consists of all the pairs which do not appear in any block of $\mathcal{A}$. A $(2,3)$-packing $(X, \mathcal{A})$ is said to be maximum if there does not exist any $(2,3)$-packing $(X, \mathcal{B})$ with $|\mathcal{A}| \leq |\mathcal{B}|$. Two $(2,3)$-packings $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are *disjoint* if $\mathcal{A} \cap \mathcal{B} = \emptyset$. Two $(2,3)$-packings $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are *compatible* if they have the same leave. A set of more than two $(2,3)$-packings is called disjoint (compatible, resp.) if each pair is disjoint (compatible, resp.). Throughout this paper, we restrict our attention to $(2,3)$-packings on v points in which every point occurs in at least one block (since, if there are points that occur in no blocks, then we have a $(2,3)$-packing on v' points for some $v' < v$).

We denote the maximum number of disjoint compatible packings on v points by $M(v)$. Determination of the numbers $M(v)$ is related to the construction of perfect threshold schemes (see, for example, [2], [10]). The following upper bounds are proved in [10].

Theorem 1.1. [10]. *$M(v) \leq v - 2$ for $v \equiv 1, 3 \pmod 6$; $M(v) \leq v - 4$ for $v \equiv 0, 2, 5 \pmod 6$; and $M(v) \leq v - 6$ for $v \equiv 4 \pmod 6$. Further, except when $v \equiv 4 \pmod 6$, the upper bound is attained only if the packings are maximum.*

Values of v for which $M(v)$ meets the upper bound are summarized in the following.

Theorem 1.2.

(1) *For* $v \equiv 1,3 \pmod 6$ *and* $v > 7$, $M(v) = v - 2$. *Also,* $M(7) = 3$. ([7],[8],[12])
(2) *For* $v \equiv 0,2 \pmod 6$ *except possibly for* $v \equiv 260, 516 \pmod{768}$, $M(v) = v - 4$. ([3],[4])
(3) *For* $v = 4 + 2 \cdot 3^k$, $k \geq 1$, $M(v) = v - 6$. ([4])
(4) *For* $v = 11, 17, 23$, $M(v) = v - 4$. ([2],[10])

In this paper we will prove the following result which provides the first infinite classes meeting the upper bound when $v \equiv 5 \pmod 6$.

Theorem 1.3. *If* $v = v + 7^i t$ *where* $i \geq 0$ *and* $t \in \{1, 7, 13, 19, 25, 31, 43, 67\}$, *then* $M(v) = v - 4$.

We should also mention that there has been some recent interest in constructing sets of disjoint packings which are not required to be compatible. Such structures have applications to the construction of constant-weight codes [1],[13]. In [5],[6], Etzion proves that there exist $6k + 3$ disjoint maximum packings on n points if $n = 6k + 4$ or $6k + 5$.

2. Some Lemmas

In this section we collect some results which will be used in the next section.

Lemma 2.1. [9],[11]. *A maximum* $(2,3)$*-packing on* $v \equiv 5 \pmod 6$ *points has* $(v(v-1) - 8/6$ *blocks and the leave is a* 4*-cycle. Such a* $(2,3)$*-packings exists for all* $v \equiv 5 \pmod 6$.

Suppose $v \equiv 5 \pmod 6$ and we have a set of disjoint compatible (maximum) $(2,3)$-packings on point set $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup Z_{v-4}$ having leave $\{\{\infty_1, \infty_2\}, \{\infty_2, \infty_3\}, \{\infty_3, \infty_4\}, \{\infty_4, \infty_1\}\}$. Then, every $(2,3)$-packings has a block $\{\infty_1, \infty_3, i\}$ for some $i \in Z_{v-4}$. Since $|\{\{\infty_1, \infty_3, i\} : i \in Z_{v-4}\}| = v - 4$, there are at most $v - 4$ disjoint compatible $(2,3)$-packings on $v \equiv 5 \pmod 6$ points, as states in Theorem 1.1.

A group-divisible design is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

(i) X is a finite set of *points*
(ii) $\mathcal{G}$ is a partition of X into subsets called *groups*
(iii) $\mathcal{B}$ is a set of subsets of X (called *blocks*), such that a group and a block contain at most one common point, and every pair of points from distinct groups occurs in exactly one block.

We abbreviate the term group-divisible design to GDD. The *type* of a GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. The type is denoted by $1^{u_1} 2^{u_2} \ldots$, where there are precisely u_i occurences of i, $i \geq 1$. In this paper, we only use GDDs with block size 3. Such a GDD is called a 3-GDD.

Two 3-GDDs with the *same* group set are said to be *disjoint* if their block sets are disjoint. A set of more than two 3-GDDs is called disjoint if each pair is disjoint. It is easy to see that the maximum number of disjoint 3-GDDs of type t^u is at most $t(u-2)$. If $s \geq t$, then the maximum number of disjoint 3-GDDs of type $t^u s^1$ is at most $t(u-1)$. We will call such a collection of disjoint 3-GDDs a *large set*, and denote it by $\mathrm{LS}(t^u)$ and $\mathrm{LS}(t^u s^1)$ respectively.

Lemma 2.2. *If $u+1$ is even, then there exists an* $\mathrm{LS}(1^{u+1}u^1)$.

Proof: A 3-GDD of type $1^{u+1}u^1$ exists if $u+1$ is even. Let $(X, \mathcal{G}, \mathcal{B})$ be such a 3-GDD and G_0 be the (unique) group of size u. It is easy to see that every block intersects G_0. Let π be a permutation acting on the points of G_0 that consists of a single cycle of length u. Applying π^i $(0 \leq i < u)$ on $\mathcal{B}$, we will obtain u disjoint block sets. Hence there exists an $\mathrm{LS}(1^{u+1}u^1)$ if $u+1$ is even.

A *parallel class* in a GDD is a set of blocks which forms a partition of X.

Lemma 2.3. *Suppse $u \equiv 1 \pmod 6$. There exists an $\mathrm{LS}(u^3)$ in which each 3-GDD has a parallel class and such that these u parallel classes form the block set of 3-GDD of type u^3.*

Proof: Let $S = \{a, b, c\}$. Let $X = S \times Z_u$ and $\mathcal{G} = \{\{s\} \times Z_u : s \in S\}$. For every $i \in Z_u$, let $\mathcal{B}_i = \{\{(a,x), (b,y), (c,z)\} : x + y + z \equiv 3_i \pmod u\}$. Then $(X, \mathcal{G}, \mathcal{B}_i), i \in Z_u$ form u disjoint 3-GDDs. Therefore they form an $\mathrm{LS}(u^3)$. Let $\mathcal{P}_i = \{\{a, x+i), (b, 2x+i), c, -3x+i)\} : x \in Z_u\}$. Then each $\mathcal{P}_i$ is parallel class in $\mathcal{B}_i$, and $(X, \mathcal{G}, \bigcup_{1 \leq i \leq u} \mathcal{P}_i)$ is a 3-GDD of type u^3.

3. The main construction

In [10], a set of 7 disjoint compatible packings on 11 points was presented. It turns out to have some special properties which are very useful. Take a point set $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup Z_7$. We list the packings in Table 1 and classify the blocks of each $\mathcal{A}_i$ into 4 types.

For $1 \leq k \leq 4$, define $\mathcal{A}_i(k)$ be the set of blocks of type k in $\mathcal{A}_i$.

The following properties will be important in our construction.

(1) The blocks in $\mathcal{A}_i(1)$ are $\{\infty_1, \infty_3, i-1\}$ and $\{\infty_2, \infty_4, i-1\}$.

(2) For $x \in \{\infty_1, \infty_2, \infty_3, \infty_4\}$, define $E_i(2, x) = \{\{a, b\} : \{x, a, b\} \in \mathcal{A}_i(2)\}$. Then $E_i(2) = \bigcup_{x \in \{\infty_1, \infty_2, \infty_3, \infty_4\}} E_i(2, x)$ forms a 6-cycle on $Z_7 \backslash \{i-1\}$. Direct edges in $E_i(2)$ so that every point has indegree one and outdegree one, and call the resulting collection of ordered pairs $D_i(2)$. Then the ordered pairs $D_i(2)$, $1 \leq i \leq 7$, can be arranged so that they are mutually disjoint. This is shown in Table 1.

(3) For $1 \leq i \leq 7$, let $\mathcal{R}_i = \{\{i, i+1, i+3\}, \{i+2, i+4, i+5\}\} \pmod 7$. Then $(\bigcup_{i \leq i \leq 7} \mathcal{R}_i) \cap (\bigcup_{1 \leq i \leq 7} \mathcal{A}_i) = \emptyset$. Further, for each i, $(1 \leq i \leq 7)$, the pairs of points of Z_7 contained in the blocks of $\mathcal{A}_i(3)$ are precisely those contained in the triples of $\mathcal{R}_i$.

Table 1

	$\mathcal{A}_1$:	$\mathcal{A}_2$:	$\mathcal{A}_3$:	$\mathcal{A}_4$:	$\mathcal{A}_5$:	$\mathcal{A}_6$:	$\mathcal{A}_7$:
type 1:	$\{\infty_1,\infty_3,0\}$	$\{\infty_1,\infty_3,1\}$	$\{\infty_1,\infty_3,2\}$	$\{\infty_1,\infty_3,3\}$	$\{\infty_1,\infty_3,4\}$	$\{\infty_1,\infty_3,5\}$	$\{\infty_1,\infty_3,6\}$
	$\{\infty_2,\infty_4,0\}$	$\{\infty_2,\infty_4,1\}$	$\{\infty_2,\infty_4,2\}$	$\{\infty_2,\infty_4,3\}$	$\{\infty_2,\infty_4,4\}$	$\{\infty_2,\infty_4,5\}$	$\{\infty_2,\infty_4,6\}$
type 2:	$\{\infty_1,4,6\}$	$\{\infty_2,5,0\}$	$\{\infty_4,6,1\}$	$\{\infty_3,0,2\}$	$\{\infty_4,1,3\}$	$\{\infty_2,2,4\}$	$\{\infty_3,3,5\}$
	$\{\infty_2,6,2\}$	$\{\infty_1,0,3\}$	$\{\infty_2,1,4\}$	$\{\infty_4,2,5\}$	$\{\infty_3,3,6\}$	$\{\infty_3,4,0\}$	$\{\infty_2,5,1\}$
	$\{\infty_3,2,3\}$	$\{\infty_3,3,4\}$	$\{\infty_1,4,5\}$	$\{\infty_2,5,6\}$	$\{\infty_1,6,0\}$	$\{\infty_2,0,1\}$	$\{\infty_4,1,2\}$
	$\{\infty_2,3,1\}$	$\{\infty_1,4,2\}$	$\{\infty_2,5,3\}$	$\{\infty_4,6,4\}$	$\{\infty_3,0,5\}$	$\{\infty_1,1,6\}$	$\{\infty_2,2,0\}$
	$\{\infty_4,1,5\}$	$\{\infty_4,2,6\}$	$\{\infty_3,3,0\}$	$\{\infty_1,4,1\}$	$\{\infty_2,5,2\}$	$\{\infty_2,6,3\}$	$\{\infty_1,0,4\}$
	$\{\infty_2,5,4\}$	$\{\infty_1,6,5\}$	$\{\infty_2,0,6\}$	$\{\infty_4,1,0\}$	$\{\infty_3,2,1\}$	$\{\infty_4,3,2\}$	$\{\infty_2,4,3\}$
type 3:	$\{\infty_1,1,2\}$	$\{\infty_2,2,3\}$	$\{\infty_4,3,4\}$	$\{\infty_3,4,5\}$	$\{\infty_4,5,6\}$	$\{\infty_4,6,0\}$	$\{\infty_3,0,1\}$
	$\{\infty_4,2,4\}$	$\{\infty_4,3,5\}$	$\{\infty_3,4,6\}$	$\{\infty_1,5,0\}$	$\{\infty_2,6,1\}$	$\{\infty_1,0,2\}$	$\{\infty_1,1,3\}$
	$\{\infty_3,4,1\}$	$\{\infty_3,5,2\}$	$\{\infty_1,6,3\}$	$\{\infty_2,0,4\}$	$\{\infty_1,1,5\}$	$\{\infty_3,2,6\}$	$\{\infty_4,3,0\}$
	$\{\infty_1,3,5\}$	$\{\infty_2,4,6\}$	$\{\infty_4,5,0\}$	$\{\infty_3,6,1\}$	$\{\infty_4,0,2\}$	$\{\infty_3,1,3\}$	$\{\infty_3,2,4\}$
	$\{\infty_3,5,6\}$	$\{\infty_3,6,0\}$	$\{\infty_1,0,1\}$	$\{\infty_2,1,2\}$	$\{\infty_1,2,3\}$	$\{\infty_1,3,4\}$	$\{\infty_4,4,5\}$
	$\{\infty_4,6,3\}$	$\{\infty_4,0,4\}$	$\{\infty_3,1,5\}$	$\{\infty_1,2,6\}$	$\{\infty_2,3,0\}$	$\{\infty_4,4,1\}$	$\{\infty_1,5,2\}$
type 4:	{0,1,6}	{1,2,0}	{2,3,1}	{3,4,2}	{4,5,3}	{5,6,4}	{6,0,5}
	{0,2,5}	{1,3,6}	{2,4,0}	{3,5,1}	{4,6,2}	{5,0,3}	{6,1,4}
	{0,3,4}	{1,4,5}	{2,5,6}	{3,6,0}	{4,0,1}	{5,1,2}	{6,2,3}

Table 2

$D_1(2)$:	$D_2(2)$:	$D_3(2)$:	$D_4(2)$:	$D_5(2)$:	$D_6(2)$:	$D_7(2)$:
(4,6)	(5,0)	(6,1)	(0,2)	(1,3)	(2,4)	(3,5)
(6,2)	(0,3)	(1,4)	(2,5)	(3,6)	(4,0)	(5,1)
(2,3)	(3,4)	(4,5)	(5,6)	(6,0)	(0,1)	(1,2)
(3,1)	(4,2)	(5,3)	(6,4)	(0,5)	(1,6)	(2,0)
(1,5)	(2,6)	(3,0)	(4,1)	(5,2)	(6,3)	(0,4)
(5,4)	(6,5)	(0,6)	(1,0)	(2,1)	(3,2)	(4,3)

This is shown in Table 2.

Now, suppose $u \equiv 1 \pmod 6$. What we will do is to prove that $M(7u + 4) = 7u$ if $M(u + 4) = u$. We construct the packings on point set $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup Z_7 \times Z_u$. Each packing will have leave $L = \{\{\infty_1, \infty_2\}, \{\infty_2, \infty_3\}, \{\infty_3, \infty_4\}, \{\infty_4, \infty_1\}\}$. The construction proceeds in several steps, each of which is related to one of the block types of i.

Step 1: For $1 \le i \le 7$ do the following. Take u disjoint compatible packings on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \{i-1\} \times Z_u$. Each packing has the leave L. Name the

u packings $\mathcal{B}(1,i,j)$ $1 \le j \le u$. (There are $(u^2+7u+4)/6$ blocks in each $(1,i,j)$.)

Step 2: For $1 \le i \le 7$ do the following. If $(a,b) \in D_i(2)$ and $\{x,a,b\} \in \mathcal{A}_i(2)$, take an LS$(1^{u+1}u^1)$ on $\{x\} \cup (\{a,b\} \times Z_u)$, with groups $\{x\}$, $\{(a,k)\}$ $(k \in Z_u)$ $\{(b,k): k \in Z_u\}$. (This large set exists by Lemma 2.2.) Name the u disjoint block sets $\mathcal{B}((a,b),j)$, $1 \le j \le u$. Let $\mathcal{B}(2,i,j) = \bigcup_{(a,b \in D_i(2)} \mathcal{B}((a,b),j)$, $1 \le j \le u$. (There are $3u(u+1)$ blocks in each $\mathcal{B}(2,i,j)$, since a 3-GDD of type $1^{u+1}u^1$ has $u(u+1)/2$ blocks.)

Step 3: Do the following for $1 \le i \le 7$. If $R = \{a,b,c\} \in \mathcal{R}_i$, then take an LS$(u^3)$ on point set $R \times Z_u$, with groups $\{a\} \times Z_u$, $\{b\} \times Z_u$, $\{c\} \times Z_u$. We use Lemma 2.3. Let $\mathcal{B}(R,j)$ $(1 \le j \le u)$ be the u disjoint block sets. Let $\mathcal{P}(R,j) \subseteq \mathcal{B}(R,j)$ be a parallel class such that $\bigcup_{1 \le j \le u} \mathcal{P}(R,j)$ forms a block set of a 3-GDD of type u^3. If $\{x,a,b\}$, $\{y,b,c\}$, $\{z,c,a\}$ are in $\mathcal{A}_i(3)$, then let

$$\mathcal{C}(R,j) = \bigcup \{\{x,(a,r),(b,s)\},\{y,(b,s),(c,t)\},\{z,(c,t),(a,r)\}\}.$$

where the union is over the set $\{(a,r),(b,s),(c,t)\} \in \mathcal{P}(R,j)$.

Since $\bigcup_{1 \le j \le u} \mathcal{P}(R,j)$ forms the block set of a 3-GDD of type u^3, it follows that $\mathcal{C}(R,j)$ $(1 \le j \le u)$ are disjoint. Let $\mathcal{B}(3,i,j) = \bigcup_{R \in \mathcal{R}_i} (\mathcal{B}(R,j) \setminus \mathcal{P}(R,j)) \cup \mathcal{C}(R,j))$, $1 \le j \le u$. (There are $2u^2+4u$ blocks in each $\mathcal{B}(3,i,j)$.)

Step 4: For $1 \le i \le 7$ do the following. If $A = \{a,b,c\} \in \mathcal{A}_i(4)$, then take an LS$(u^3)$ on point set $A \times Z_u$, with groups $\{a\} \times Z_u$, $\{b\} \times Z_u$, $\{c\} \times Z_u$. Name the u disjoint block sets $\mathcal{B}(A,j)$, $1 \le j \le u$. Let $\mathcal{B}(4,i,j) = \bigcup_{A \in \mathcal{A}_i(4)} \mathcal{B}(A,j)$. (There are $3u^2$ blocks in each $\mathcal{B}(4,i,j)$.)

Now we can present the construction. For $1 \le i \le 7$ and $1 \le j \le u$, define

$$\mathcal{B}(i,j) = \mathcal{B}(1,i,j) \cup \mathcal{B}(2,i,j) \cup \mathcal{B}(3,i,j) \cup \mathcal{B}(4,i,j).$$

We claim that $\mathcal{B}(i,j)$ $(1 \le i \le 7, 1 \le j \le u)$ are disjoint $(2,3)$-packings on X with leave L. The verification is straightforward and we leave it to the reader.

Therefore, we get the following.

Theorem 3.1. *Suppose* $u \equiv 1 \pmod 6$. *If* $M(u+4) = u$, *then* $M(7u+4) = 7u$.

It would be nice if we had more sets of $v-4$ disjoint compatible packings ($v \equiv 5 \pmod 6$) with similar properties as the set of packings on 11 points, but we have not found any more examples yet.

4. Some small examples

First we prove that $M(v) = v-4$ for $v = 29, 35, 47, 71$.

Suppose $u \equiv 1 \pmod 6$ is a prime. We take point set $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup Z_u$ where Z_u is the finite field of order u. The idea is to construct the first packing

$\mathcal{A}_1$ (a $(2,3)$-packing on X), from which the others can be developed modulo u. We will use the multiplicative structure of the field. Suppose ξ is a primitive element in Z_u. Let π be the permutation of X which multiplies each element of Z_u by ξ, but fixes $\infty_1, \infty_2, \infty_3$ and ∞_4.

In order to narrow down our search, we assume the following properties hold:

(1) $\{\infty_1, \infty_3, 0\} \in \mathcal{A}_1$ and $\{\infty_2 \infty_4, 0\} \in \mathcal{A}_1$

(2) If $\{x, a, b\} \in \mathcal{A}_1$, then $\pi^2\{x, a, b\} \in \mathcal{A}_1$, where $x = \{\infty_1, \infty_2, \infty_3, \infty_4\}$, and $a, b \in Z_u$.

(3) If $\{a, b, c\} \in \mathcal{A}_1$, then $\pi\{a, b, c\} \in \mathcal{A}_1$, where $a, b, c \in Z_u$.

For a, b, c in Z_u, there are six i's such that $\pi^i\{a, b, c\}$ has the form $\{w, w+1, w+u\}$. We denote the set of six u's thus obtained by $U(\{a, b, c\})$. Two block sets developed from $\{a, b, c\}$ and $\{a', b', c'\}$ respectively (using π and developed modulo u), where $\{a, b, c, a', b', c'\} \subseteq Z_u$, are disjoint if and only if $U(\{a, b, c\}) \cap U(\{a', b', c'\}) = \emptyset$. To check that $\mathcal{A}_1$ is a $(2,3)$-packing, we simply check the exponential differences.

We succeed with this approach for $u = 43$ and $u = 67$.

43 disjoint compatible packings on 47 points

$\mathcal{A}_1$: (ξ=3)		
blocks	developing by	U
$\{\infty_1,\infty_3,0\}$, $\{\infty_2,\infty_4,0\}$		
$\{\infty_1,1,\xi^5\}$, $\{\infty_2,\xi,\xi^6\}$ $\{\infty_3,1,\xi^7\}$, $\{\infty_4,\xi,\xi^8\}$	π^2	
$\{0,1,-1\}$	π (but orbit length 21)	2, 22, 42
$\{1,\xi^4,\xi^{12}\}$	π	3, 15, 21, 23, 29, 41
$\{1,\xi^2,\xi^{24}\}$	π	5, 12, 18, 26, 32, 39
$\{1,\xi^3,\xi^{19}\}$	π	4, 11, 14, 30, 33, 40
$\{1,\xi^6,\xi^{17}\}$	π	6, 8, 17, 27, 36, 38
$\{1,\xi^{13},\xi^{28}\}$	π	9, 16, 20, 24, 28, 35
$\{1,\xi^1,\xi^{33}\}$	π	10, 13, 19, 25, 31, 34

For $u = 25$, we use the finite field GF(25) of order 25, generated from the irreducible polynomial $x^2 + x + 2$ ($\xi = x$). Here we assume that:

(1) $\{\infty_1, \infty_3, 0\} \in \mathcal{A}_1$ and $\{\infty_2, \infty_4, 0\} \in \mathcal{A}_1$

(2) If $\{x, a, b\} \in \mathcal{A}_1$, then $\pi^8\{x, a, b\} \in \mathcal{A}_1$, where $x \in \{\infty_1, \infty_2, \infty_3, \infty_4\}$, and $a, b \in$ GF(25)

(3) If $\{a, b, c\} \in \mathcal{A}_1$, then $\pi\{a, b, c\} \in \mathcal{A}_1$, where $a, b, c \in$ GF(25).

67 disjoint compatible packings on 71 points

$\mathcal{A}_1$: (ξ=2)		
blocks	developing by	U
$\{\infty_1,\infty_3,0\}$, $\{\infty_2,\infty_4,0\}$		
$\{\infty_1,1,\xi^{17}\}$, $\{\infty_2,\xi,\xi^{18}\}$ $\{\infty_3,1,\xi^{19}\}$, $\{\infty_4,\xi,\xi^{20}\}$	π^2	
$\{0,1,-1\}$	π (but orbit length 33)	2, 34, 66
$\{1,\xi^{14},\xi^{38}\}$	π	3, 23, 33, 35, 45, 65
$\{1,\xi^{10},\xi^{40}\}$	π	4, 17, 22, 46, 51, 64
$\{1,\xi^{4},\xi^{22}\}$	π	6, 12, 28, 40, 56, 62
$\{1,\xi^{6},\xi^{35}\}$	π	5, 18, 27, 41, 50, 63
$\{1,\xi^{3},\xi^{23}\}$	π	7, 11, 20, 48, 57, 61
$\{1,\xi^{5},\xi^{21}\}$	π	8, 19, 26, 42, 49, 60
$\{1,\xi^{9},\xi^{34}\}$	π	9, 15, 25, 43, 53, 59
$\{1,\xi,\xi^{59}\}$	π	10, 16, 21, 47, 52, 58
$\{1,\xi^{12},\xi^{27}\}$	π	13, 29, 31, 37, 39, 55
$\{1,\xi^{2},\xi^{55}\}$	π	14, 24, 32, 36, 44, 54

For $u = 31$, we need some small adjustments. Here we assume

(1) $\{\infty_1, \infty_3, 0\}\mathcal{A}_1, \{\infty_2, \infty_4, 0\} \in \mathcal{A}_1$

(2) If $A \in \mathcal{A}_1$, then $\pi^2 A \in \mathcal{A}_1$.

For a, b, c in Z_u, there are three i's such that $\pi^{2i}\{a, b, c\}$ has the form $\{w, w + 1, w + u\}$. The set of three u's thus obtained is denoted by $U(\{a, b, c\})$.

Applying Theorem 3.1 and using the above resluts we have our main theorem.

Theorem 4.1. *If* $v = 4 + 7^i t$ *where* $i \geq 0$ $t \in \{1, 7, 13, 19, 25, 31, 43, 67\}$, *then* $M(v) = v - 4$.

25 disjoint compatible packings on 29 points

$\mathcal{A}_1$: ($x^2+x+2=0$)		
blocks	developing by	U
$\{\infty_1,\infty_3,0\}$, $\{\infty_2,\infty_4,0\}$		
$\{\infty_1,x^8,x^{19}\}$, $\{\infty_1,x^4,x^{23}\}$ $\{\infty_1,x^{10},x^{21}\}$, $\{\infty_1,x,x^6\}$	π^8	
$\{\infty_2,x^3,x^{22}\}$, $\{\infty_2,x^8,x^{21}\}$ $\{\infty_2,x^{17},x^{12}\}$, $\{\infty_2,x^{10},x^{23}\}$	π^8	
$\{\infty_3,x^7,x^{20}\}$, $\{\infty_3,x^{11},x^{16}\}$ $\{\infty_3,x^9,x^{22}\}$, $\{\infty_3,x^{13},x^{18}\}$	π^8	
$\{\infty_4,x^{10},x^{15}\}$, $\{\infty_4,x^9,x^{20}\}$ $\{\infty_4,1,x^5\}$, $\{\infty_4,x^{11},x^{22}\}$	π^8	
$\{0,1,x^{12}\}$	π(but orbit length 12)	x^6,x^{12},x^{18}
$\{1,x^2,x^8\}$	π	$x^7,x^8,x^{11},x^{13},x^{16},x^{17}$
$\{1,x,x^{10}\}$	π	$x^3,x^5,x^{10},x^{14},x^{19},x^{21}$
$\{1,x^{17},x^{20}\}$	π	$x,x^2,x^9,x^{15},x^{22},x^{23}$

31 disjoint compatible packings on 35 points

$\mathcal{A}_1$: ($\xi=3$)		
blocks	developing by	U
$\{\infty_1,\infty_3,0\}$, $\{\infty_2,\infty_4,0\}$		
$\{\infty_1,\xi,\xi^{18}\}$	π^2	
$\{\infty_2,1,\xi^5\}$	π^2	
$\{\infty_3,1,\xi^{15}\}$	π^2	
$\{\infty_4,\xi,\xi^{20}\}$	π^2	
$\{0,1,\xi\}$	π^2	3,11,15
$\{\xi,\xi^3,\xi^9\}$	π^2	12,18,19
$\{1,\xi^2,\xi^6\}$	π^2	17,21,29
$\{1,\xi^8,\xi^{18}\}$	π^2	5,23,25
$\{1,\xi^{14},\xi^3\}$	π^2	7,9,27
$\{\xi,\xi^5,\xi^6\}$	π^2	4,8,22
$\{\xi,\xi^{11},\xi^{14}\}$	π^2	10,24,28
$\{\xi,\xi^{13},\xi^{22}\}$	π^2	2,16,30
$\{\xi,\xi^8,\xi^{15}\}$	π^2	13,14,20

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