Orthogonal Steiner triple systems of order $6m + 3$

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Abstract. In this paper, we prove that for any $n > 27363$, $n \equiv 3$ modulo 6, there exist a pair of orthogonal Steiner triple systems of order n. Further, a pair of orthogonal Steiner triple systems of order n exist for all $n \equiv 3$ modulo 6, $3 < n \leq 27363$, with at most 918 possible exceptions. The proof of this result depends mainly on the construction of pairwise balanced designs having block sizes that are prime powers congruent to 1 modulo 6, or 15 or 27. Some new examples are also constructed recursively by using conjugate orthogonal quasigroups.

1. Introduction.

A *pairwise balanced design* (or PBD) is a pair $(X, \mathcal{A})$, where $\mathcal{A}$ is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every unordered pair of *points* (i.e. elements of X) is contained in a unique block in $\mathcal{A}$. If v is a positive integer and K is a set of positive integers, each of which is greater than or equal to 2, then we say that $(X, \mathcal{A})$ is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathcal{A}$. We will define $B(K) = \{v$: there exists a (v, K)-PBD$\}$. A set K is said to be PBD-*closed* if $B(K) = K$.

A *Steiner triple system* of order n, (or STS(n)), can be defined to be an $(n, 3)$-PBD. The necessary and sufficient condition for the existence of an STS(n) is that $n \equiv 1$ or 3 (modulo 6). Two STS(n) on the same point set, say $(X, \mathcal{A})$ and $(X, \mathcal{B})$, are said to be *orthogonal* provided the following properties are satisfied:

1) $\mathcal{A} \cap \mathcal{B} = \emptyset$
2) if $\{u, v, w\}$ and $\{x, y, w\} \in \mathcal{A}$, and $\{u, v, s\}$ and $\{x, y, t\} \in \mathcal{B}$, then $s \neq t$.

Orthogonal STS(n) will be denoted by OSTS(n). OSTS(n) can be used to construct a Room square of order n (or, equivalently, a pair of orthogonal one-factorizations of order $n+1$, or a pair of perpendicular Steiner quasigroups of order n). Indeed, OSTS(n) were originally introduced in 1968 by O'Shaughnessy [14] as a method of constructing Room squares. Although the spectrum of Room squares was determined in 1975 by Mullin and Wallis [13], the spectrum of orthogonal Steiner triple systems remains open.

OSTS(n) are known to exist if $n \equiv 1$ mod 6 is a prime power (see [8]). Also, the set $\mathcal{OSTS} = \{n$: there exists OSTS(n)$\}$ is PBD-closed (see [5]). If we define $P_{1,6}$ to be the set of prime powers congruent to 1 modulo 6, then $B(P_{1,6}) \subseteq \mathcal{OSTS}$. In [11] and [22], it was proved that $n \in B(P_{1,6})$ (and hence $n \in \mathcal{OSTS}$) if $n \equiv 1$ mod 6 and $n \geq 1927$. There remained 31 values of $n \equiv 1$ mod 6 less than 1927 for which an $(n, P_{1,6})$-PBD was not constructed, as given in the following theorem.

Theorem 1.1 [11], [22]. *If* $n \equiv 1$ *modulo* $6, n \geq 1$, *and* $n \notin \{55, 115, 145, 205, 235, 253, 265, 295, 319, 355, 391, 415, 445, 451, 493, 649, 655, 667, 685, 697, 745, 781, 799, 805, 1243, 1255, 1315, 1585, 1795, 1819, 1921\}$, *then there is an* $(n, P_{1,6})$*-PBD.*

In Section 6, we construct OSTS(253) and OSTS(685), thus reducing the number of exceptions for $n \equiv 1$ modulo 6 to 29.

Much less is known regarding OSTS(n) for $n \equiv 3$ mod 6. The only small examples of OSTS(n) (i.e. $n < 100$) known to exist are $n = 15$ ([4]) and $n = 27$ ([15]). Also, there do *not* exist OSTS(9) ([9]).

Of course, Wilson's theory of PBD-closure ([17], [18], and [21]), ensures that there exists a constant N such that, for all $n > N, n \in \mathcal{OSTS}$ if and only if $n \equiv 1$ or 3 modulo 6. However, this theory does not yield any reasonable upper bounds on N. The main result of this paper is the determination of an upper bound on the constant N; namely, that $N \leq 27363$.

Some new examples of OSTS are also obtained recursively using conjugate orthogonal quasigroups, which are discussed in Sections 5 and 6. These quasigroups seem to be of interest in their own right, and the determination of the spectrum remains an open problem.

2. Recursive constructions for PBDs.

In this section, we describe several recursive constructions for PBDs. First, we need to define some terminology.

A *group-divisible design* (or GDD), is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the following properties:

1) $\mathcal{G}$ is a partition of X into subsets called *groups*
2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point
3) every pair of points from distinct groups occurs in a unique block.

The *group-type* (or *type*) of a GDD$(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G|: G \in \mathcal{G}\}$. We usually use an "exponential" notation to describe group-types: a group-type $1^i 2^j 3^k \ldots$ denotes i occurences of 1, j occurences of 2, etc. As with PBDs, we will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathcal{A}$.

We often construct PBDs from GDDs by filling in the groups as follows.

Filling in Groups Suppose $(X, \mathcal{G}, \mathcal{A})$ is a K-GDD, where K is a PBD-closed set. If $|G| \in K$ for all $G \in \mathcal{G}$, then $|X| \in K$. If $|G| + 1 \in K$ for all $G \in \mathcal{G}$, then $|X| + 1 \in K$.

A *transversal design* TD(k, n) is a k-GDD of type n^k, i.e. a GDD with knpoints, k groups of size n, and n^2 blocks of size k. Note that every group and every block of a transversal design intersect in a point. It is well-known that a TD(k, n) is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order n. For a list of lower bounds on the number of MOLS of all orders up to 10000, we refer the reader to Brouwer [3].

We now briefly describe Wilson's Fundamental Construction for GDDs ([19]).

Fundamental Construction (FC) Suppose $(X, \mathcal{G}, \mathcal{A})$ is a GDD, and let $w: X \rightarrow Z^{+} \cup \{0\}$ be any function (we refer to w as a *weighting*). For every $x \in X$, let $s(x)$ be $w(x)$ "copies" of x. For every $A \in \mathcal{A}$, suppose that $(\cup_{x \in A} s(x), \{s(x): x \in A\}, \mathcal{B}(A))$ is a GDD. Then $(\cup_{x \in X} s(x), \{\cup_{x \in G} s(x): G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \mathcal{B}(A))$ is a GDD.

The remaining constructions are "product" type constructions. We will describe a very general type of product construction, but first we need to define the idea of incomplete designs. Informally, a TD$(k, n)-$ TD(k, m) (an *incomplete* transversal design) is a transversal design from which a sub-transversal design is missing. Formally, a TD$(k, n)-$ TD(k, m) is a quadruple $(X, Y, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

1) X is a set of cardinality kn
2) $\mathcal{G} = \{G_i: 1 \leq i \leq n\}$ is a partition of X into k *groups* of size n
3) $Y \subset X$, $|Y| = km$, and $|Y \cap G_i| = m$, for $1 \leq i \leq n$
4) $\mathcal{A}$ is a set of $n^2 - m^2$ *blocks* of size k, each of which intersects each group in a point
5) every pair of points x, y from distinct groups, such that at least one of x, y is in $X \backslash Y$, occurs in a unique block of $\mathcal{A}$.

Note that these definitions imply that no block contains two points from Y. Hence, existence of a TD$(k, n)-$ TD(k, m) and a TD(k, m) implies the existence of a TD(k, n).

We also need PBDs containing subdesigns, or flats. Let $(X, \mathcal{A})$ be a PBD. If a set of points $Y \subseteq X$ has the property that, for any $A \in \mathcal{A}$, either $|Y \cap A| \leq 1$ or $A \subseteq Y$, then we say that Y is a *subdesign* or *flat* of the PBD. The *order* of the flat is $|Y|$. If Y is a flat, then we can delete all blocks $A \subseteq Y$, replacing them by a single block, Y, and the result is a PBD. Also, any block or point of a PBD is itself a flat.

However, for the construction we are about to describe, we do not require that the flat be present: i.e. it can be "missing". Hence, we define incomplete PBDs, as follows. An *incomplete* PBD (or IPBD) is a triple $(X, Y, \mathcal{A})$, where X is a set of points, $Y \subseteq X$, and $\mathcal{A}$ is a set of blocks which satisfies the properties:

1) for any $A \in \mathcal{A}$, $|A \cap Y| \leq 1$
2) any two points x, z, not both in Y, occur in a unique block.

Equivalently, we require that $(X, \mathcal{A} \cup \{Y\})$ be a PBD. We say that $(X, Y, \mathcal{A})$ is a (v, w, K)-IPBD if $|X| = v$, $|Y| = w$, and $|A| \in K$ for every $A \in \mathcal{A}$. This is where it is important that we make the distinction between PBDs containing flats, and incomplete PBDs. It is possible that there can exist a (v, w, K)-IPBD, but that there does not exist any (v, K)-PBD containing a flat of order w. For example, it is easy to construct an $(11, 5, 3)$-IPBD, but there is no $(11, 3)$-PBD.

The following construction is referred to as the *singular indirect product* (see, for example, [7] and [10]).

Singular Indirect Product (SIP) Suppose K is a set of positive integers and $u \in K$; suppose v, w, and a are integers such that $0 \leq a \leq w \leq v$; and suppose the following designs exist:

1) a $\mathrm{TD}(u, v-a) - \mathrm{TD}(u, w-a)$,
2) a (v, w, K)-IPBD, and
3) a $(u(w-a)+a, K)$-PBD.

Then there is a $(u(v-a)+a, K)$-PBD that contains flats of order u and $u(w-a)+a$. Hence, in particular, $u(v-a)+a \in B(K)$.

If we let $w = a$ in the singular indirect product, we obtain the *singular direct product*.

Singular Direct Product (SDP) Suppose K is a set of positive integers and $u \in K$. Suppose v and w are non-negative integers such that $w \leq v$, there exists a $\mathrm{TD}(u, v)$, there is a (v, w, K)-IPBD, and there is a (w, K)-PBD. Then there is a $(u(v-w)+w, K)$-PBD that contains flats of order u, v, and w. Hence, in particular, $u(v-w)+w \in B(K)$.

If we further specialize this construction by letting $w = 0$, we obtain the *direct product*.

Direct Product (DP) Suppose K is a set of positive integers and $u, v \in K$. If there exists a $\mathrm{TD}(u, v)$, then there is a (uv, K)-PBD that contains flats of order u and v. Hence, in particular, $uv \in B(K)$.

In order to apply the Singular Indirect Product, we need incomplete transversal designs. We use constructions given in [20] to produce these.

Lemma 2.9. *Suppose the following TDs exist: a $TD(k, m)$, a $TD(k, m+1)$, and a $TD(k+1, t)$. Suppose that $0 \leq u \leq t$. Then there exists a $TD(k, mt+u) - TD(k, u)$.*

Lemma 2.10. *Suppose the following TDs exist: a $TD(k, m)$, a $TD(k, m+1)$, a $TD(k, m+2)$, a $TD(k+2, t)$, and a $TD(k, u)$. Suppose that $0 \leq v \leq t$. Then there exists a $TD(k, mt+u+v) - TD(k, v)$.*

3. A bound.

In this section, we shall prove that if $n \equiv 3$ modulo 6 and $n > 27363$, then there is an $(n, P_{1,6} \cup \{15, 27\})$-PBD. In order to apply the singular indirect product construction, we need some results on incomplete transversal designs.

Lemma 3.1. *If $t \geq 77$ and $0 \leq u \leq t$, then there exists a $TD(7, 7t + u) - TD(7, u)$.*

Proof: Since $t \geq 77$, a TD$(8, t)$ exists by [3]. There also exist TD$(7, 7)$ and TD$(7, 8)$. Apply Lemma 2.9 with $m = k = 7$ to obtain the desired incomplete TD. ∎

Lemma 3.2. *Suppose $w \equiv 1$ modulo 6, $w \geq 1927$, and a $TD(15, w)$ exists. Then there is an $(n, P_{1,6} \cup \{15\})$-PBD for all $n \equiv 3$ modulo 6, $99w \leq n \leq 105w$.*

Proof: Let $a = \frac{105w - n}{6}$; then $0 \leq a \leq w$. We apply SIP with $v = 15w$ and $u = 7$. First, there is a TD$(7, 15w - a)-$ TD$(7, w - a)$ by Lemma 3.1, since $2w \geq 77$. Next, there is a $(15w, w, P_{1,6} \cup \{15\})$-IPBD, since a TD$(15, w)$ exists. Finally, there is a $(7(w - a) + a, P_{1,6})$-PBD by Theorem 1.1, since $7(w - a) + a \geq w > 1921$ and $7(w - a) + a \equiv 1$ modulo 6. The desired PBD is obtained. ∎

We can now prove a preliminary bound.

Lemma 3.3. *If $n \equiv 3$ modulo 6 and $n \geq 357093$, then there is an $(n, P_{1,6} \cup \{15\})$-PBD.*

Proof: If $n \equiv 3$ modulo 6 and $n \geq 357093 = 99 \cdot 3607$, then there exists w such that $w \equiv 1$ modulo 6, $w \geq 3607$, and $99w \leq n \leq 105w$. For such w, a TD$(15, w)$ exists by [3]. Apply Lemma 3.2. ∎

Next, we shall lower the bound of Lemma 3.3 using the following variation of Lemma 3.2.

Lemma 3.4. *Suppose $w \equiv 1$ modulo 6 and a $TD(15, w)$ exists. Then there is an $(n, P_{1,6} \cup \{15\})$-PBD for all $n \equiv 3$ modulo 6, $98w + 1927 \leq n \leq 105w$.*

Proof: If $w \leq 271$, then $98w + 1927 > 105w$, so there is nothing to prove. Hence, assume $w \geq 277$. As in Lemma 3.2, let $a = \frac{105w - n}{6}$, and then apply SIP with $v = 15w$ and $u = 7$. A TD$(7, 15w - a)-$ TD$(7, w - a)$ exists by Lemma 3.1, since $2w \geq 77$. A $(15w, w, P_{1,6} \cup \{15\})$-IPBD is constructed from a TD$(15, w)$. Finally, there is a $(7(w - a) + a, P_{1,6})$-PBD by Theorem 1.1, since $7(w - a) + a = n - 98w \geq 1927$ and $7(w - a) + a \equiv 1$ modulo 6. ∎

Lemma 3.5. *If $n \equiv 3$ modulo 6 and $57885 \leq n \leq 357315$, then there is an $(n, P_{1,6} \cup \{15\})$-PBD.*

Proof: We apply Lemma 3.4 with w = 571, 589, 601, 619, 643, 661, 679, 703, 727, 757, 787, 811, 847, 883, 925, 967, 1015, 1063, 1117, 1177, 1237, 1303, 1375, 1453, 1537, 1627,1723, 1825, 1927, 2035, 2155, 2281, 2413, 2557, 2707, 2869, 3031, 3211, and 3403. For each w in the above list, a TD$(15, w)$ exists by [3]. Apply Lemma 3.4. It is easy to see that the resulting intervals leave no integers in the given range uncovered. For, it suffices to verify the inequality

$98w + 1927 \leq 105w' + 6$ for each $w \neq 571$ in the above list, where w' denotes the integer in the list preceding w. Hence, we cover all $n \equiv 3$ modulo 6, where $98 \cdot 571 + 1927 = 57885 \leq n \leq 105 \cdot 3403 = 357315$. ∎

Lemma 3.6. *If* $n \equiv 3$ *modulo* 6 *and* $27369 \leq n \leq 57879$, *then there is an* $(n, P_{1,6} \cup \{15, 27\})$*-PBD.*

Proof: These values of n are all obtained from SIP, writing $n = 7(v - a) + a$. Given a particular $(v, w, P_{1,6} \cup \{15, 27\})$-IPBD, we can often obtain an interval of values n by using different values for a. These intervals are listed in Table 3.1. For each interval, the following information is presented: values v and w for which a suitable IPBD exists, the resulting interval of values $n = 7(v - a) + a$ which are obtained from SIP, and the corresponding interval of values $7(w - a) + a$. The following information must be verified.

1) We need TD$(7, v - a)-$ TD$(7, w - a)$. In most cases, $(v - a) - (w - a) = (v - w) = 7t$ for some $t \geq 77$. In these cases, it is easy to check that $w - a \leq w \leq t$, whence Lemma 3.1 can be applied. The remaining incomplete TDs are obtained by Lemma 2.10. We determine an equation $v = 7t + u + w$, where $0 \leq w \leq t$, $0 \leq u \leq t$, and such that a TD$(9, t)$ and a TD$(7, u)$ both exist. Such equations are listed in Table 3.2. The existence of a TD$(9, t)$ and a TD$(7, u)$ can be checked in [3]. Then, a TD$(7, v)-$ TD$(7, w)$ exists by Lemma 2.10. Moreover, any TD$(7, v - a)-$ TD$(7, w - a)$ exists if $0 \leq a \leq w$, by using the equation $v - a = 7t + u + (w - a)$.

2) We need a $(v, w, P_{1,6} \cup \{15, 27\})$-IPBD. The examples we use come from the product constructions. In Table 3.3, we give applications of SDP and SIP. The remaining examples are all applications of DP, where $v = 15w$ or $v = 27w$, and the required TD exists by [3].

3) We need a $(7(w-a)+a, P_{1,6})$-PBD. In most cases, $7(w-a)+a \equiv 1$ modulo 6, and the required PBDs exist by Theorem 1.1. There are four other values that are required: 2517, 4281, 4293, and 4425. These are obtained from SIP in Table 3.4.

This completes the proof. ∎

So, we have proved the following theorem.

Theorem 3.7. *If* $n \equiv 3$ *modulo* 6 *and* $n \geq 27369$, *then there is an* $(n, P_{1,6} \cup \{15, 27\})$*-PBD.*

Proof: Combine the results in Lemmata 3.3, 3.5 and 3.6. ∎

Table 3.1

v	w	7(v-a)+a		7(w-a)+a		v	w	7(v-a)+a		7(w-a)+a	
8271	529	56121	57879	1927	3685	4887	181	33213	33231	271	289
8025	535	54357	56115	1927	3685	4965	331	33189	33207	751	769
7845	523	53181	54351	1927	3097	4887	181	33183	33183	241	241
7611	457	52005	53175	1927	3097	4965	331	33141	33177	703	739
7485	499	50829	51999	1927	3097	4887	181	33123	33135	181	193
7305	487	49653	50823	1927	3097	4965	331	33111	33117	673	679
7125	475	48477	49647	1927	3097	4851	735	33105	33105	4293	4293
6945	463	47301	48471	1927	3097	4965	331	33099	33099	661	661
6855	457	46713	47295	1927	2509	4851	735	33087	33093	4275	4281
6945	463	46695	46707	1321	1333	4965	331	32937	33081	499	643
6855	457	46611	46689	1825	1903	4707	157	32931	32931	1081	1081
6945	463	46185	46605	811	1231	4965	331	32895	32925	457	487
6855	457	46107	46179	1321	1393	4701	151	32883	32889	1033	1089
6945	463	46077	46101	703	727	4965	331	32859	32877	421	439
6585	439	44949	46071	1927	3049	4695	313	32601	32853	1927	2179
6495	433	44361	44943	1927	2509	4659	7	32595	32595	31	31
6339	361	43773	44355	1927	2509	4695	313	32499	32589	1825	1915
6315	421	43185	43767	1927	2509	4647	97	32493	32493	643	643
6183	229	42999	43179	1321	1501	4695	313	32475	32487	1801	1813
6207	229	42657	42993	811	1147	4641	91	32469	32469	619	619
6135	409	42009	42651	1927	2569	4695	313	32265	32463	1591	1789
6045	403	41421	42003	1927	2509	4611	61	32259	32259	409	409
5955	397	40833	41415	1927	2509	4695	313	31995	32253	1321	1579
6045	403	40815	40827	1321	1333	4605	307	31911	31989	1825	1903
5955	397	40731	40809	1825	1903	4695	313	31485	31905	811	1231
6045	403	40305	40725	811	1231	4605	307	31407	31479	1321	1393
5775	385	39657	40299	1927	2569	4563	169	31257	31401	499	643
5685	379	39069	39651	1927	2509	4605	307	30897	31251	811	1165
5595	373	38481	39063	1927	2509	4413	199	30819	30891	1321	1393
5505	367	37893	38475	1927	2509	4605	307	30789	30813	703	727
5415	361	37305	37887	1927	2509	4401	187	30759	30783	1261	1285
5373	199	37029	37299	811	1081	4395	7	30753	30753	37	37
5505	367	36777	37023	811	1057	4395	181	30747	30747	1249	1249
5415	361	36699	36771	1321	1393	4395	7	30741	30741	25	25
5505	367	36669	36693	703	727	4395	181	30309	30735	811	1237
5415	361	36639	36663	1261	1285	4335	289	30249	30303	1927	1981
5235	349	36129	36633	1927	2431	4323	7	30243	30243	31	31
5211	193	35937	36123	811	997	4335	289	30147	30237	1825	1915
5133	331	35541	35931	1927	2317	4377	163	29997	30141	499	643
5211	193	35523	35535	397	409	4335	289	29913	29991	1591	1669
5079	277	35439	35517	1825	1903	4377	163	29895	29907	397	409
5235	349	35013	35433	811	1231	4335	289	29643	29889	1321	1567
5001	199	34935	35007	1321	1393	4239	157	29385	29637	811	1063
5055	337	34851	34929	1825	1903	4335	289	29133	29379	811	1057
4989	187	34425	34845	811	1231	4245	283	29055	29127	1321	1393
5055	337	34347	34419	1321	1393	4155	277	28971	29049	1825	1903
4965	331	34263	34341	1825	1903	4245	283	28545	28965	811	1231
5055	337	33837	34257	811	1231	4077	151	28293	28539	811	1057
4887	181	33753	33831	811	889	4155	277	27957	28287	811	1141
4965	331	33699	33747	1261	1309	4065	271	27879	27951	1321	1393
4887	181	33693	33693	751	751	4155	277	27849	27873	703	727
4965	331	33687	33687	1249	1249	4065	271	27819	27843	1261	1285
4887	181	33645	33681	703	739	4077	151	27807	27813	325	331
4965	331	33249	33639	811	1201	4257	645	27801	27801	2517	2517
4887	181	33243	33243	301	301	4065	271	27369	27795	811	1237
4851	735	33237	33237	4425	4425						

Table 3.2
Construction of incomplete transversal designs

v	w	$v-w=7t+u$	v	w	$v-w=7t+u$
4077	151	7.559 + 13	4659	7	7.663 + 11
4239	157	7.583 + 1	4887	181	7.671 + 9
4323	7	7.615 + 11	5211	193	7.713 + 27
4395	7	7.625 + 13	5373	199	7.739 + 1
4563	169	7.625 + 19	6183	229	7.849 + 11

Table 3.3
Applications of the singular indirect product

$v = 7(v' - a') + a'$	w'	PBD with flat	ITD	$7(w' - a') + a'$	w
4257 = 7(645 − 43) + 43	43	645 = 15.43	602 sub 0	43	645
4323 = 7(645 − 32) + 32	43	645 = 15.43	613 = 7.86 + 11	109	7
4377 = 7(645 − 23) + 23	43	645 = 15.43	622 = 7.86 + 20	163	163
4395 = 7(645 − 20) + 20	43	645 = 15.43	625 = 7.86 + 23	181	181, 7
4401 = 7(645 − 19) + 19	43	645 = 15.43	626 = 7.86 + 24	187	187
4413 = 7(645 − 17) + 17	43	645 = 15.43	628 = 7.86 + 26	199	199
4611 = 7(675 − 19) + 19	25	675 = 25.27	656 = 7.89 + 27 + 6	61	61
4641 = 7(675 − 14) + 14	25	675 = 25.27	661 = 7.89 + 27 + 11	91	91
4647 = 7(675 − 13) + 13	25	675 = 25.27	662 = 7.89 + 27 + 12	97	97
4659 = 7(675 − 11) + 11	25	675 = 25.27	664 = 7.89 + 27 + 14	109	7
4701 = 7(675 − 4) + 4	25	675 = 25.27	671 = 7.89 + 27 + 21	151	151
4707 = 7(675 − 3) + 3	25	675 = 25.27	672 = 7.89 + 27 + 22	157	157
4851 = 7(735 − 49) + 49	49	735 = 15.49	686 sub 0	49	735
4989 = 7(735 − 26) + 26	49	735 = 15.49	709 = 7.98 + 23	187	187
5001 = 7(735 − 24) + 24	49	735 = 15.49	711 = 7.98 + 25	199	199
5079 = 7(735 − 11) + 11	49	735 = 15.49	724 = 7.98 + 38	277	277
5133 = 7(735 − 2) + 2	49	735 = 15.49	733 = 7.98 + 47	331	331
6207 = 7(915 − 33) + 33	61	915 = 15.61	882 = 7.122 + 28	229	229
6339 = 7(915 − 11) + 11	61	915 = 15.61	904 = 7.122 + 50	361	361
7611 = 7(1095 − 9) + 9	73	1095 = 15.73	1086 = 7.146 + 64	457	457
8271 = 7(1185 − 4) + 4	79	1185 = 15.79	1181 = 7.158 + 75	529	529

Table 3.4
Applications of the singular indirect product

$v = 7(v' - a') + a'$	w'	PBD with flat	ITD	$7(w' - a') + a'$
2517 = 7(375 − 18) + 18	25	375 = 15.25	357 = 7.49 + 7 + 7	67
4281 = 7(645 − 39) + 39	43	645 = 15.43	606 = 7.86 + 4	67
4293 = 7(645 − 37) + 37	43	645 = 15.43	608 = 7.86 + 6	79
4425 = 7(645 − 15) + 15	43	645 = 15.43	630 = 7.86 + 28	211

4. Values below 27369.

It remains to consider the existence of $(n, P_{1,6} \cup \{15, 27\})$-PBDs for $n \equiv 3$ modulo 6, $n \leq 27363$. These values are handled as follows. In Appendix 1, we present a table of intervals, which cover all but 1370 orders by applying the singular indirect product as in Section 3. Of these 1370 orders, we construct PBDs for 84 of them in this section. In Appendix 2, further applications of SIP are presented, using equations of the form $n = 7(w - a) + a$. There remain1039 orders for which PBDs are not constructed, which are given in Appendix 3.

In order to save space, we do not include all the details regarding these constructions, but we do indicate the values of v, w, and a used. The reader should have no difficulty in finding constructions for the necessary incomplete TDs, using the same methods as in Section 3. Any PBDs containing flats which are used recursively will have previously been constructed.

Lemma 4.1. *There is an* $(n, P_{1,6} \cup \{15, 27\})$*-PBD if* $n \in \{189, 285, 351, 375, 405, 465, 513, 555, 645, 675, 729, 735, 837, 915, 999, 1005, 1095, 1161, 1185, 1455, 1545, 1635, 1647, 1809, 1815, 2085, 2133, 2265, 2355, 2715, 2781, 2835, 2895, 2943, 2985, 3267, 3345, 3429, 3435, 3615, 4065, 4077, 4155, 4239, 4245, 4563, 4605, 4695, 5211, 5235, 5373, 5415, 5505, 5595, 5955, 6021, 6183, 6495, 6507, 6585, 7125, 7215, 7749, 8565, 8655, 8835, 9285, 9375, 9423, 9465, 10545, 11535\}$.

Proof: These are all applications of the direct product. Each value of n can be written as $n = 7v$, $n = 15v$ or $n = 27v$ for suitable v. ∎

Lemma 4.2. *There is an* $(n, P_{1,6} \cup \{15, 27\})$*-PBD if* $n = 2835, 5265, 5625$ *or* 10935.

Proof: A $\{7\}$-GDD of type 3^{15} was constructed in [1]. If there is a TD$(7, m)$, then we can give every point weight m and apply the fundamental construction, obtaining a $\{7\}$-GDD of type $(3m)^{15}$. If, further, there is a $(3m, P_{1,6} \cup \{15, 27\})$-PBD, then there is a $(45m, P_{1,6} \cup \{15, 27\})$-PBD. Taking $m = 63$, 117, 125, and 243, we construct the stated PBDs by this method. ∎

Lemma 4.3. *There is an* $(n, P_{1,6} \cup \{15, 27\})$*-PBD if* $n = 1107, 1197, 2367, 5877, 7047, 12897, 14067$ *or* 14319.

Proof: These are all applications of SIP, as given below. ∎

$1107 = 15(85 - 12) + 12,$	$w = 13$	$7047 = 13(555 - 14) + 14,$	$w = 15$
$1197 = 15(91 - 12) + 12,$	$w = 13$	$12897 = 13(1005 - 14) + 14,$	$w = 15$
$2367 = 15(169 - 12) + 12,$	$w = 13$	$14067 = 13(1095 - 14) + 14,$	$w = 15$
$5877 = 13(465 - 14) + 14,$	$w = 15$	$14319 = 13(1107 - 6) + 6,$	$w = 7$

Lemma 4.4. *There is a* $(5865, \{7, 15, 27\})$*-PBD.*

Proof: Start with a $\{7\}$-GDD of type 3^{15} [1], give every point weight 130 and apply the fundamental construction, obtaining a $\{7\}$-GDD of type $(390)^{15}$. Now, a TD(15, 27) gives rise to a $(405, 15, \{15, 27\})$-PBD. Hence, we fill in the groups of the $\{7\}$-GDD of type $(390)^{15}$, adjoining 15 new points, to produce a $(5865, \{7, 15, 27\})$-PBD. ∎

There remain 1039 orders $n \equiv 3$ modulo 6, $9 \leq n \leq 27363$, for which we have not constructed an $(n, P_{1,6} \cup \{15, 27\})$-PBD. These orders are listed in Appendix 3. There are 121 underlined values in Appendix 3; for these n, we shall construct OSTS(n) in Section 6.

5. Conjugate orthogonal quasigroups.

The connection between conjugate orthogonal quasigroups and orthogonal Steiner triple systems has been discussed by Lindner and Mendelsohn in [6]. They also give some constructions for conjugate orthogonal quasigroups. In this section, we review the results of [6] and give some improvements.

A quasigroup of order v is a pair $(Q, \otimes)$, where Q is a set of cardinality v, and $\otimes: Q \times Q \to Q$ is a binary operation such that $q \otimes r = q \otimes s$ if and only if $r = s$, and $r \otimes q = s \otimes q$ if and only if $r = s$. (The operation table of a quasigroup is a *Latin square*, and conversely, any Latin square gives rise to a quasigroup for which it is the operation table.) The quasigroup $(Q, \otimes)$ is said to be *idempotent* if $q \otimes q = q$ for all $q \in Q$. Two quasigroups of order v, $(Q, \otimes)$ and $(Q, \oplus)$, are said to be *orthogonal* if, for every ordered pair $(s, t) \in S \times S$, there is a unique ordered pair (q, r) such that $q \otimes r = s$ and $q \oplus r = t$.

Let $(Q, \otimes)$ be any quasigroup. We define on the set Q six binary operations $\otimes_{(1,2,3)}, \otimes_{(1,3,2)}, \otimes_{(2,1,3)}, \otimes_{(2,3,1)}, \otimes_{(3,1,2)}$, and $\otimes_{(3,2,1)}$, as follows: $q \otimes r = s$ if and only if

$$q \otimes_{(1,2,3)} r = s, \qquad q \otimes_{(1,3,2)} s = r, \qquad r \otimes_{(2,1,3)} q = s,$$
$$r \otimes_{(2,3,1)} s = q, \qquad s \otimes_{(3,1,2)} q = r, \qquad s \otimes_{(3,2,1)} r = q.$$

These six binary operations all define quasigroups (not necessarily distinct), called the *conjugates* of $(Q, \otimes)$. The set of conjugates of $(Q, \otimes)$ is denoted $\langle (Q, \otimes) \rangle$. Two quasigroups, $(Q, \otimes)$ and $(Q, \oplus)$, are defined to be *conjugate orthogonal* quasigroups if any quasigroup in $\langle (Q, \otimes) \rangle$ is orthogonal to any quasigroup in $\langle (Q, \oplus) \rangle$. Conjugate orthogonal quasigroups of order v are denoted COQ(v). Define $\mathcal{COQ} = \{v$: there exist COQ(v)$\}$.

Now, it is easy to see that any quasigroup in $\langle (Q, \otimes) \rangle$ is idempotent if $(Q, \otimes)$ is idempotent. Hence, we also denote two idempotent conjugate orthogonal quasigroups of order v by ICOQ(v) and define $\mathcal{COQ}^\star = \{v$: there exist ICOQ($v$)$\}$.

The following result was essentially given in [6, Theorem 6 and Corollary 7], but contained a couple of minor typographical errors. We correct them here.

Theorem 5.1. *If v is a prime power, $v \neq 2, 3, 4, 5$, or 8, then there exist ICOQ(v). Further, there exist COQ(v) for $v = 4, 5$, and 8.*

Proof: Let $Q = GF(v)$. For any $\lambda \in Q$, $\lambda \neq 0, 1$, define a quasigroup $(Q, \otimes_\lambda)$ by $q \otimes_\lambda r = \lambda q + (1-\lambda)r$. It is easy to see that $(Q, \otimes_\lambda)$ is idempotent. It is also easy to see that $(Q, \otimes_\lambda)$ and $(Q, \otimes_\kappa)$ are orthogonal if $\kappa \neq \lambda$. Also, we observe that any conjugate of $(Q, \otimes_\lambda)$ is a $(Q, \otimes_{\lambda'})$ for some $\lambda' \neq 0$ or 1. Since $|\langle(Q, \otimes_\lambda)\rangle| \leq 6$ for any λ, it follows that there exist ICOQ(v) if $12 \leq v - 2$, i.e. if $v > 13$. We now consider $v = 13, 11, 9$, and 7. If $v = 13$ or 11, then $|\langle(Q, \otimes_2)\rangle| = 3$, and $v - 2 - 3 \geq 6$, so there exist ICOQ(13) and ICOQ(11). If $v = 9$, then $|\langle(Q, \otimes_2)\rangle| = 1$, and $v - 2 - 1 \geq 6$, so there exist ICOQ(9). Finally, if $v = 7$, then $|\langle(Q, \otimes_2)\rangle| = 3$, and $|\langle(Q, \otimes_3)\rangle| = 2$, so there exists ICOQ(7).

For $v = 4$ or 8, $|\langle(Q, \otimes_\lambda)\rangle| = v - 2$ for any $\lambda \neq 0, 1$. Define $(Q, \oplus)$ by $q \oplus r = q + r$. Then, it is easy to verify that $|\langle(Q, \oplus)\rangle| = 1$, and that $(Q, \otimes_\lambda)$ and $(Q, \oplus)$ are COQ(v). Finally, for $v = 5$, define $(Q, \otimes)$ by $q \otimes r = q + 2r$, and define $(Q, \oplus)$ by $q \oplus r = 4q + 4r$. Then, we can check that $|\langle(Q, \otimes)\rangle| = 6$, $|\langle(Q, \oplus)\rangle| = 1$, and that $(Q, \otimes)$ and $(Q, \oplus)$ are COQ(5). ∎

We now mention some recursive constructions for COQ(v) and ICOQ(v). First, we state direct product and singular direct product constructions without proof (see, for example, [16]).

Lemma 5.2.
(Direct Product) If there exist COQ(u) and COQ(v), then there exist COQ(uv). If there exist ICOQ(u) and ICOQ(v), then there exist ICOQ(uv).

(Singular Direct Product) If there exist ICOQ(u), COQ(v) containing sub-COQ(w) as a subdesign, and COQ($v-w$), then there exist COQ($u(v-w)+w$). If there exist ICOQ(u), ICOQ(v) containing sub-ICOQ(w) as a subdesign, and COQ($v - w$), then there exist ICOQ($u(v - w) + w$).

Using the direct product construction, the following corollary is immediate.

Corollary 5.3 [6, Corollary 8]. *If v has prime power factorization $v = 2^{a_2}3^{a_3}5^{a_5}$ ..., where $a_2 \neq 1$ and $a_3 \neq 1$, then there exist COQ(v).*

We can also use PBD and GDDs to construct conjugate orthogonal quasigroups recursively. We state the following GDD-construction without proof.

Lemma 5.4. *Suppose $(X, \mathcal{G}, \mathcal{A})$ is a K-GDD, where $K \subseteq \mathcal{COQ}^\star$ and $|G| \in \mathcal{COQ}$ for all $G \in \mathcal{G}$. Then there exist COQ($|X|$). Further, if $|G| \in \mathcal{COQ}^\star$ for all $G \in \mathcal{G}$, then there exist ICOQ($|X|$).*

Proof: This is simply the Bose-Shrikhande-Parker construction (see [2]). ∎

Corollary 5.5. *The set $\mathcal{COQ}^\star$ is PBD-closed.*

Proof: A PBD can be thought of as a GDD where every group has size 1. Apply Lemma 5.4. ∎

Hence, we can obtain some results on existence of ICOQ(v) by using known classes of PBDs. In [12], the set $\boldsymbol{B}(P_7)$ is investigated, where $P_7 = \{v \geq 7 : v$ is an odd prime power$\}$. Since $P_7 \subseteq \mathcal{COQ}^\star$ and $\mathcal{COQ}^\star$ is PBD-closed, we have that $B(P_7) \subseteq \mathcal{COQ}^\star$. It is proved in [12] that $v \in \boldsymbol{B}(P_7)$ if v is odd and $v \geq 2129$, and that there are at most 103 odd values of v, $5 < v < 2129$, which are not members of $\boldsymbol{B}(P_7)$. These 103 possible exceptions are those elements in the set

$E(P_7) = \{15, 21, 33, 35, 39, 45, 51, 55, 65, 69, 75, 87, 93, 95, 105, 111, 115, 123, 129, 135, 141, 155, 159, 165, 183, 185, 195, 201, 205, 213, 215, 219, 231, 235, 237, 245, 249, 255, 265, 267, 285, 291, 295, 303, 305, 309, 315, 321, 327, 335, 339, 345, 355, 363, 365, 375, 381, 395, 415, 445, 447, 453, 455, 465, 471, 483, 485, 501, 507, 519, 525, 543, 573, 579, 597, 605, 615, 651, 655, 699, 717, 735, 805, 843, 845, 861, 903, 921, 933, 945, 951, 957, 1047, 1077, 1119, 1227, 1315, 1383, 1515, 1595, 1623, 1795, 2127\}$.

So, we have proved

Theorem 5.6. *If $v > 5$ is odd and $v \notin E(P_7)$, then there is an ICOQ(v).*

We can show that most of the integers in $E(P_7)$ are in $\mathcal{COQ}^\star$ or $\mathcal{COQ}$. First, we eliminate several values by starting with a TD(17, m), deleting some points from one group, and applying Lemma 5.4.

Lemma 5.7. *Suppose there is a TD(17, m), and let $0 \leq u \leq m$. If there exist ICOQ(m) and COQ(u), then there exist COQ($16m + u$). If there exist ICOQ(m) and ICOQ(u), then there exist ICOQ($16m + u$).*

We give several applications of Lemma 5.7 in Table 5.1.

Table 5.1
Applications of Lemma 5.7

265 = 16.16 + 9	445 = 16.27 + 13	615 = 16.37 + 23	1077 = 16.64 + 53
267 = 16.16 + 11	471 = 16.29 + 7	699 = 16.43 + 11	1119 = 16.67 + 47
285 = 16.17 + 13	483 = 16.29 + 19	717 = 16.43 + 29	1227 = 16.73 + 59
305 = 16.19 + 1	507 = 16.31 + 11	861 = 16.53 + 13	1315 = 16.81 + 19
315 = 16.19 + 11	519 = 16.32 + 7	945 = 16.59 + 1	1595 = 16.97 + 43
321 = 16.19 + 17	525 = 16.32 + 13	951 = 16.59 + 7	1623 = 16.101 + 7
375 = 16.23 + 7	543 = 16.32 + 31	957 = 16.59 + 13	1795 = 16.107 + 83
381 = 16.23 + 13	605 = 16.37 + 13	1047 = 16.64 + 23	2127 = 16.131 + 31

Next, we give several applications of the direct product construction to members of $E(P_7)$.

Lemma 5.8. *There exist COQ(v) if* $v \in \{35, 45, 55, 65, 95, 115, 135, 155, 205, 215, 235, 245, 295, 335, 355, 365, 395, 415, 445, 455, 485, 655, 735, 805\}$.

Proof: Apply Corollary 5.3. ∎

Finally, we present some further applications of the product constructions in Table 5.2.

Table 5.2
Product constructions for COQ

equation	$\in COQ^*$?	equation	$\in COQ^*$?
$69 = 17(5-1)+1$		$345 = 5.69$	
$93 = 23(5-1)+1$		$453 = 113(5-1)+1$	
$105 = 13(9-1)+1$	yes	$465 = 5.93$	
$129 = 32(5-1)+1$		$501 = 125(5-1)+1$	
$165 = 41(5-1)+1$		$573 = 143(5-1)+1$	
$185 = 23(9-1)+1$	yes	$597 = 149(5-1)+1$	
$213 = 53(5-1)+1$		$651 = 7.93$	
$237 = 59(5-1)+1$		$845 = 211(5-1)+1$	
$249 = 31(9-1)+1$	yes	$933 = 233(5-1)+1$	
$309 = 77(5-1)+1$			

Combining the above results, we have the following.

Theorem 5.9. *Suppose* $v > 1$ *is odd. Then there exist COQ(v) if* $v \notin \{3, 15, 21, 33, 39, 51, 75, 87, 111, 123, 141, 159, 183, 195, 201, 219, 231, 255, 291, 303, 327, 339, 363, 447, 579, 843, 903, 921, 1383, 1515\}$. *Further, there exist ICOQ(v) if* $v \notin \{3, 15, 21, 33, 35, 39, 45, 51, 55, 65, 69, 75, 87, 93, 95, 111, 115, 123, 129, 135, 141, 155, 159, 165, 183, 195, 201, 205, 213, 215, 219, 231, 235, 237, 245, 255, 291, 295, 303, 309, 327, 335, 339, 345, 355, 363, 365, 395, 415, 447, 453, 455, 465, 485, 501, 573, 579, 597, 651, 655, 735, 805, 843, 845, 903, 921, 933, 1383, 1515\}$.

For even integers, we only obtain a preliminary bound beyond which COQ(v) always exist. First, we construct one representative in each congruence class modulo 32, using the product constructions.

Table 5.3

equation	modulo 32	equation	modulo 32
$162 = 23(8-1)+1$	2	4	4
$134 = 19(8-1)+1$	6	8	8
$330 = 47(8-1)+1$	10	$44 = 4.11$	12
$302 = 43(8-1)+1$	14	16	16
$50 = 7(8-1)+1$	18	$20 = 4.5$	20
$470 = 67(8-1)+1$	22	$56 = 8.7$	24
$218 = 31(8-1)+1$	26	$28 = 4.7$	28
$190 = 27(8-1)+1$	30	32	0

We can now prove the following.

Theorem 5.10. *Suppose $v \geq 58150$ is even. Then there exist COQ(v).*

Proof: Observe that $58150 = 16.3605+470$. Then, if $v \geq 58150$ is even, we can write v in the form $v = 16m+u$, where m is odd, $m \geq 3605$, and $u \in \{4, 8, 16, 20, 28, 32, 44, 50, 56, 134, 162, 190, 218, 302, 330, 470\}$. We apply Lemma 5.7, noting that a TD$(17, m)$ exists for all odd $m \geq 3605$. ∎

6. Orthogonal Steiner triple systems.

Using PBD constructions, we have proved in Section 4 that there is an $(n, P_{1,6} \cup \{15, 27\})$-PBD for all $n \equiv 3$ modulo 6, $n > 3$, with 1039 possible exceptions. For the same values of n, there exist OSTS(n) since $B(P_{1,6} \cup \{15, 27\}) \subseteq \mathcal{OSTS}$. In this section, we construct OSTS(n) for 121 of the above exceptions.

Our main tool is a generalization of orthogonal Steiner triple systems which we refer to as *orthogonal group-divisible designs*. Let $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ be two 3-GDDs having the same groups. We say that they are *orthogonal* if the following properties are satisfied:

1) if $\{u, v, s\} \in \mathcal{A}$ and $\{u, v, t\} \in \mathcal{B}$, then s and t belong to different groups.
2) if $\{u, v, w\}$ and $\{x, y, w\} \in \mathcal{A}$, and $\{u, v, s\}$ and $\{x, y, t\} \in \mathcal{B}$, then $s \neq t$.

We shall use the abbreviation OGDD to denote orthogonal 3-GDDs. It is easy to see that OSTS(n) are equivalent to OGDD of type 1^n, since condition 1) implies that $\mathcal{A} \cap \mathcal{B} = \emptyset$.

We now give several constructions involving OSTS, OGDD and COQ.

Theorem 6.1. *Suppose there is a K-GDD of type T, where $K \subseteq \mathcal{OSTS}$. Then there exist OGDD of type T.*

Proof: Let $(X, \mathcal{G}, \mathcal{A})$ be the hypothesized GDD. For every block $A \in \mathcal{A}$, let $(A, \mathcal{B}_1(A))$ and $(A, \mathcal{B}_2(A))$ be OSTS($|A|$). Define $\mathcal{B}_i = \cup_{A \in \mathcal{A}} \mathcal{B}_i(A)$, for

$i = 1, 2$. We will show that $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$ are OGDD. Suppose $\{u, v, s\} \in \mathcal{B}_1$ and $\{u, v, t\} \in \mathcal{B}_2$. Then $\{u, v, s\} \in \mathcal{B}_1(A)$ and $\{u, v, t\} \in \mathcal{B}_2(A)$, for some block A. Since $\mathcal{B}_1(A)$ and $\mathcal{B}_2(A)$ are OSTS($|A|$), $s \neq t$. Since $\{s, t\} \subseteq A$ and $\mathcal{A}$ is a GDD, s and t belong to different groups. This proves 1). Now, suppose $\{u, v, w\}$ and $\{x, y, w\} \in \mathcal{B}_1$ and $\{u, v, s,\}$ and $\{x, y, s\} \in \mathcal{B}_2$. Let $\{u, v\} \subseteq A \in \mathcal{A}$ and $\{x, y\} \subseteq A' \in \mathcal{A}$. If $A \neq A'$, then $w \in A \cap A'$ and $s \in A \cap A'$, which implies that $w = s$. Then $\{u, v, w\} \in \mathcal{B}_1(A) \cap \mathcal{B}_2(A)$, a contradiction, since they are OSTS. Hence, $A = A'$. Then $\{u, v, w\}$ and $\{x, y, w\} \in \mathcal{B}_1(A)$ and $\{u, v, s\}$ and $\{x, y, s\} \in \mathcal{B}_2(A)$. Again, this contradicts the orthogonality of these OSTS. This proves 2). ∎

Corollary 6.2. *There exist OGDD of type 3^{15}.*

Proof: There exists a 7-GDD of type 3^{15} (see [1]). Apply Theorem 6.1. ∎

Theorem 6.3. *Suppose there exist OGDD of type T, and suppose there exist COQ(m). Then, there exist OGDD of type $mT = \{mt: t \in T\}$.*

Proof: Suppose $(Q, \otimes_1)$ and $(Q, \otimes_2)$ are COQ(m) and that $(X, \mathcal{G}, \mathcal{A}_1)$ and $(X, \mathcal{G}, \mathcal{A}_2)$ are OGDD of type T. We will construct OGDD on point set $X \times Q$, having groups $\mathcal{H} = \{G \times Q: G \in \mathcal{G}\}$.

Arbitrarily impose an ordering on the points in X. For every block $A \in \mathcal{A}_i$, $(i = 1, 2)$ say $A = \{x, y, z\}$ where $x < y < z$, construct the m^2 blocks

$$\mathcal{B}_i(A) = \{\{(x, a), (y, b), (z, a \otimes_i b)\}: a, b \in Q\}.$$

Define $\mathcal{B}_i = \cup_{A \in \mathcal{A}} \mathcal{B}_i(A)$, for $i = 1, 2$. We will show that $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$ are OGDD. First, suppose $\{(x, a), (y, b), (z, c)\} \in \mathcal{B}_1$ and $\{(x, a), (y, b), (w, d)\} \in \mathcal{B}_2$. Then $\{x, y, z\} \in \mathcal{B}_1(A)$ and $\{x, y, w\} \in \mathcal{B}_2(A)$. Since $\mathcal{B}_1(A)$ and $\mathcal{B}_2(A)$ are OGDD, z and w belong to different groups of $\mathcal{G}$, and hence (z, c) and (w, d) belong to different groups of $\mathcal{H}$. This proves 1). Next, suppose that $\{(u, a), (v, b), (w, c)\}$ and $\{(x, d), (y, e), (w, c)\} \in \mathcal{B}_1$ are distinct blocks and that $\{(u, a), (v, b), (t, f)\}$ and $\{(x, d), (y, e), (t, f)\} \in \mathcal{B}_2$ are distinct blocks. Then $\{u, v, w\}$ and $\{x, y, w\} \in \mathcal{A}_1$ and $\{u, v, t\}$ and $\{x, y, t\} \in \mathcal{A}_2$. By the orthogonality of $\mathcal{A}_1$ and $\mathcal{A}_2$, it follows that $\{u, v\} = \{x, y\}$ and $w \neq t$. Without loss of generality, suppose $(u, v) = (x, y)$. Hence, our blocks are $\{(u, a), (v, b), (w, c)\}$ and $\{(u, d), (v, e), (w, c)\} \in \mathcal{B}_1$ and $\{(u, a), (v, b), (t, f)\}$ and $\{(u, d), (v, e), (t, f)\} \in \mathcal{B}_2$. Now, $c = a \oplus_1 b$ for some $(Q, \oplus_1) \in \langle (Q, \otimes_1) \rangle$ and $f = a \oplus_2 b$ for some $(Q, \oplus_2) \in \langle (Q, \otimes_2) \rangle$. Then, $c = d \oplus_1 e$ and $f = d \oplus_2 e$. Hence, $(a \oplus_1 b, a \oplus_2 b) = (d \oplus_1 e, d \oplus_2 e)$. Since $(Q, \otimes_1)$ and $(Q, \otimes_2)$ are COQ, $(a, b) = (d, e)$. But then the blocks $\{(u, a), (v, b), (w, c)\}$ and $\{(x, d), (y, e), (w, c)\}$ are identical, a contradiction. This proves 2). ∎

Corollary 6.4. *Suppose there exist OSTS(u) and COQ(v). Then there exist OGDD of type v^u.*

Proof: OSTS(u) are equivalent to OGDD of type 1^u. Apply Theorem 6.3. ∎

Lemma 6.5. *Suppose there exist OGDD of type v^u, and OSTS(v). Then there exist OSTS(uv) and OGDD of type $1^{v(u-1)}v^1$.*

Proof: This is a standard "filling in groups" construction. ∎

Theorem 6.6. *Suppose there exist OSTS(u) and OSTS(v), and COQ(v). Then there exist OSTS(uv) and OGDD of type $1^{v(u-1)}v^1$.*

Proof: This is an immediate consequence of Corollary 6.4 and Lemma 6.5. ∎

Lemma 6.7. *Suppose there exist OGDD of types m^u and $1^m w^1$. Then there exist OGDD of type $1^{mu}w^1$. If, further, there exist OSTS(w), then there exist OSTS$(mu + w)$.*

Proof: This is a standard "filling in groups" construction. ∎

The following construction can be thought of as a singular direct product construction for OSTS. It was first presented in [6, Theorem 4].

Theorem 6.8. *Suppose there exist OSTS(u) and OSTS(w), COQ$(v - w)$, and OGDD of type $1^{v-w}w^1$. Then there exist OSTS$(u(v - w) + w)$.*

Proof: From Theorem 6.3, there exist OGDD of type $(v - w)^u$. Then from Lemma 6.4, we get OGDD of type $1^{u(v-w)}w^1$. Since there are OSTS(w), there exist OSTS$(u(v - w) + w)$. ∎

Corollary 6.9. *Suppose there exist OSTS(u) and OSTS(v), and COQ$(v - 1)$. Then there exist OSTS$(u(v - 1) + 1)$.*

We now give several applications of the above constructions.

Lemma 6.10. *There exist OSTS(105) and OSTS(195).*

Proof: Apply Theorem 6.6 with $u = 15$ and $v = 7, 13$. ∎

Lemma 6.11. *There exist OSTS(225) and OSTS(2925).*

Proof: Start with the OGDD of type 3^{15} (Corollary 6.2), give every point weight 5, and apply Theorem 6.3, producing OGDD of type 15^{15}. Using Lemma 6.5, we get OSTS(225). If we give every point of the OGDD of type 3^{15} weight 65, then we have OGDD of type 195^{15}, and we produce OSTS(2925). ∎

Lemma 6.12. *There exist OSTS(n) for $n = 1275, 1365, 1575, 3375, 7755, 9555$, and 25389.*

Proof: These are all applications of Theorem 6.6, writing $n = uv$ as follows: $1275 = 15.85$, $1365 = 195.7$, $1575 = 225.7$, $3375 = 15.225$, $7755 = 15.517$, $9555 = 1365.7$, and $25389 = 1953.13$. ∎

Lemma 6.13. *There exist OSTS(n) for n =* 1353, 1569, 1977, 2913, 3573, 4551, 5253, 5397, 5601, 6033, 7101, 8817, 9549, 10509, 10977, 11337, 11601, 12801, 19041, *and* 25377.

Proof: These are all applications of Corollary 6.11, writing $n = u(v-1)+1$ as given in Table 6.1. In each case, we also justify the existence of COQ($v-1$) by means of a factorization of $v-1$. Except for 350, all the factorizations are prime power factorizations, whence Corollary 5.3 can be applied. 350 = 7.50, so Lemma 5.2 can be applied here since there exist COQ(50) (Table 5.3). ∎

Table 6.1

$n = u(v-1)+1$	COQ($v-1$)	$n = u(v-1)+1$	COQ($v-1$)
1353 = 13(105 − 1) + 1	104 = 8.13	7101 = 25(285 − 1) + 1	284 = 4.71
1569 = 7(225 − 1) + 1	224 = 32.7	8817 = 19(465 − 1) + 1	464 = 16.29
1977 = 19(105 − 1) + 1	104 = 8.13	9549 = 7(1365 − 1) + 1	1364 = 4.11.31
2913 = 13(225 − 1) + 1	224 = 32.7	10509 = 37(285 − 1) + 1	284 = 4.71
3573 = 19(189 − 1) + 1	188 = 4.47	10977 = 7(1569 − 1) + 1	1568 = 32.49
4551 = 13(351 − 1) + 1	350 = 7.50	11337 = 109(105 − 1) + 1	104 = 8.13
5253 = 13(405 − 1) + 1	404 = 4.101	11601 = 25(465 − 1) + 1	464 = 16.29
5397 = 19(285 − 1) + 1	284 = 4.71	12801 = 25(513 − 1) + 1	512 = 512
5601 = 25(225 − 1) + 1	224 = 32.7	19041 = 85(225 − 1) + 1	224 = 32.7
6033 = 13(465 − 1) + 1	464 = 16.29	25377 = 13(1953 − 1) + 1	1952 = 32.61

Lemma 6.14. *There exist OSTS(693) and OSTS(4845).*

Proof: Since there exist OSTS(7), OSTS(15) and COQ(7), there exist OGDD of type $1^{98}7^1$ by Theorem 6.6. Since there exist OSTS(7) and a TD(7,98), there exist OGDD of type 98^7 by Theorem 6.1. Applying Lemma 6.7, we obtain OSTS(693). Then, since OSTS(7) and COQ(692) exist, we can apply Corollary 6.9 to construct OSTS(4845). ∎

Lemma 6.15. *There exist OSTS(1485) and OSTS(10389).*

Proof: As in the proof of Lemma 6.11, there exist OGDD of type 15^{15} by Theorem 6.3. Since there exist OSTS(15), there exist OGDD of type $1^{210}15^1$ by Lemma 6.5. Since there exist OSTS(7) and a TD(7,210), there exist OGDD of type 210^7 by Theorem 6.1. Applying Lemma 6.7, we obtain OSTS(1485). Then, since OSTS(7) and COQ(1484) exist, we can apply Corollary 6.9 to construct OSTS(10389). ∎

Lemma 6.16. *There exist OSTS(1359), OSTS(8919) and OSTS(9507).*

Proof: There exist TD(7,194), TD(7,1274) and TD(7,1358), so we have from Theorem 6.1 OGDD of types 194^7, 1274^7 and 1358^7. There exist OSTS(195)

(Lemma 6.10) and OSTS(1275) (Lemma 6.12). Hence, applying Lemma 6.7 with $w = 1$, there exist OSTS(1359) and OSTS(8919). Having constructed OSTS(1359), we again apply Lemma 6.7 to produce OSTS(9507). ∎

Lemma 6.17. *There exist OSTS(7203).*

Proof: Since there exist ICOQ(7), COQ(79) and COQ(80), there exist COQ(554) by Lemma 5.2. Then, thc existencc of OSTS(13) and OSTS(555) imply the existence of OSTS(7203) by Corollary 6.9. ∎

We also construct two previously unknown OSTS(n), $n \equiv 1$ modulo 6.

Lemma 6.18. *There exist OSTS(253) and OSTS(685).*

Proof: There exist OSTS(7), OSTS(37) and COQ(36), so OSTS(253) exist by Corollary 6.9. Similarly, OSTS(19), OSTS(37) and COQ(36) give rise to OSTS(685). ∎

Next, we have several applications of the indirect product construction, using block sizes from the set $\mathcal{OSTS}$. These are presented in Appendix 4.

We now have our main existence results.

Theorem 6.20. *For any $n > 27363$, $n \equiv 3$ modulo 6, there exist a pair of orthogonal Steiner triple systems of order n. Further, a pair of orthogonal Steiner triple systems of order n exist for all $n \equiv 3$ modulo 6, $3 < n \leq 27363$, with at most 918 possible exceptions, which are the values in Appendix 3 which are not underlined.*

Theorem 6.21. *For any $n > 1921$, $n \equiv 1$ modulo 6, there exist a pair of orthogonal Steiner triple systems of order n. Further, a pair of orthogonal Steiner triple systems of order n exist for all $n \equiv 1$ modulo 6, $7 \leq n \leq 1921$, with at most 29 possible exceptions, namely the elements in the set* {55, 115, 145, 205, 235, 265, 295, 319, 355, 391, 415, 445, 451, 493, 649, 655, 667, 697, 745, 781, 799, 805, 1243, 1255, 1315, 1585, 1795, 1819, 1921}.

7. Summary.

Clearly, there remains considerable work to be done before the spectrum of orthogonal Steiner triple systems is completely determined. It is striking, however, that existence can be proved for all $n > 27363$ when $n \equiv 3$ modulo 6 when only two small examples are known (namely, $n = 15$ and 27).

Of course, many of the remaining exceptions could be handled if one or more other small examples of OSTS could be constructed. Also, if we had more examples of conjugate orthogonal quasigroups of small orders, some exceptions could be eliminated.

The spectrum of conjugate orthogonal quasigroups seems to be an interesting problem in its own right. For even orders especially, not many small examples are known. Consequently, the bound of Theorem 5.10 is quite large.

Acknowledgement.
We would like to thank Frank Bennett for helpful discussions.

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Appendix 1

Some intervals obtained from the singular indirect product construction

v	w	7(v-a)+a		7(w-a)+a		v	w	7(v-a)+a		7(w-a)+a	
4065	271	27345	27351	787	793	1959	97	13533	13677	499	643
4065	271	27309	27333	751	775	1935	73	13491	13521	457	487
4065	271	27261	27297	703	739	1935	73	13455	13473	421	439
4065	271	27249	27249	691	691	1935	73	13431	13443	397	409
4065	271	27231	27237	673	679	1935	73	13395	13419	361	385
4065	271	27219	27219	661	661	1935	73	13359	13383	325	349
4065	271	27057	27201	499	643	1935	73	13335	13347	301	313
3867	241	26973	27051	1591	1669	1905	127	13257	13329	811	883
4065	271	26955	26967	397	409	1935	73	13245	13251	211	217
3855	229	26703	26949	1321	1567	1905	127	13233	13239	787	793
3813	187	26643	26691	1261	1309	1935	73	13185	13227	151	193
3807	181	26631	26631	1249	1249	1905	127	13149	13179	703	733
3807	181	26193	26619	811	1237	1935	73	13107	13143	73	109
3753	139	26109	26187	811	889	1905	127	12945	13089	499	643
3765	139	26085	26103	703	721	1905	127	12903	12933	457	487
3753	139	26049	26073	751	775	1905	127	12867	12885	421	439
3765	139	26043	26043	661	661	1905	127	12843	12855	397	409
3753	139	26001	26037	703	739	1905	127	12807	12831	361	385
3765	139	25881	25995	499	613	1905	127	12771	12795	325	349
3753	139	25797	25875	499	577	1905	127	12747	12759	301	313
3765	139	25779	25791	397	409	1905	127	12717	12735	271	289
3753	139	25755	25773	457	475	1815	121	12669	12705	811	847
3765	139	25743	25749	361	367	1809	67	12651	12663	457	469
3753	139	25719	25737	421	439	1815	121	12645	12645	787	787
3765	139	25707	25713	325	331	1905	127	12597	12639	151	193
3753	139	25695	25701	397	403	1809	67	12591	12591	397	397
3765	139	25683	25689	301	307	1815	121	12561	12585	703	727
3753	139	25659	25677	361	379	1809	67	12555	12555	361	361
3765	139	25653	25653	271	271	1815	121	12549	12549	691	691
3753	139	25623	25647	325	349	1809	67	12519	12543	325	349
3753	139	25599	25611	301	313	1809	67	12495	12507	301	313
3765	139	25593	25593	211	211	1815	121	12357	12489	499	631
3753	139	25569	25587	271	289	1809	67	12345	12351	151	157
3765	139	25533	25563	151	181	1815	121	12315	12339	457	481
3753	139	25509	25527	211	229	1809	67	12261	12303	67	109
3753	139	25449	25497	151	199	1815	121	12255	12255	397	397
3753	139	25437	25437	139	139	1815	121	12219	12243	361	385
3615	241	25209	25305	1591	1687	1815	121	12183	12207	325	349
3615	241	24939	25197	1321	1579	1815	121	12159	12171	301	313
3615	241	24879	24927	1261	1309	1815	121	12129	12147	271	289
3615	241	24867	24867	1249	1249	1815	121	12117	12117	259	259
3615	241	24429	24855	811	1237	1815	121	12099	12105	241	247
3615	241	24405	24411	787	793	1815	121	12069	12087	211	229
3615	241	24369	24393	751	775	1815	121	12009	12057	151	199
3615	241	24321	24357	703	739	1815	121	11979	11997	121	139
3615	241	24309	24309	691	691	1647	61	11523	11529	421	427
3615	241	24291	24297	673	679	1647	61	11499	11511	397	409
3615	241	24279	24279	661	661	1647	61	11463	11487	361	385

3615	241	24117	24261	499	643
3615	241	24075	24105	457	487
3615	241	24039	24057	421	439
3435	229	24033	24033	1591	1591
3615	241	24015	24027	397	409
3435	229	23763	24009	1321	1567
3429	127	23613	23757	499	643
3435	229	23253	23607	811	1165
3345	223	23175	23247	1321	1393
3435	229	23145	23169	703	727
3345	223	23115	23139	1261	1285
3345	223	23103	23103	1249	1249
3345	223	22665	23091	811	1237
3237	199	22587	22659	1321	1393
3267	121	22521	22581	499	559
3219	181	22515	22515	1249	1249
3267	121	22479	22509	457	487
3213	175	22077	22473	811	1207
3165	211	21999	22071	1321	1393
3159	121	21969	21993	703	727
3165	211	21939	21963	1261	1285
3159	121	21927	21927	661	661
3165	211	21489	21915	811	1237
3159	121	21477	21483	211	217
3165	211	21465	21471	787	793
3159	121	21417	21459	151	193
3165	211	21381	21411	703	733
3165	211	21369	21369	691	691
3165	211	21351	21357	673	679
3165	211	21339	21339	661	661
3165	211	21177	21321	499	643
3165	211	21135	21165	457	487
3165	211	21099	21117	421	439
3165	211	21075	21087	397	409
3165	211	21039	21063	361	385
3165	211	21003	21027	325	349
3165	211	20979	20991	301	313
3165	211	20949	20967	271	289
3165	211	20937	20937	259	259
3165	211	20919	20925	241	247
3165	211	20889	20907	211	229
2985	199	20823	20883	1321	1381
2985	199	20763	20811	1261	1309
2985	199	20751	20751	1249	1249
2985	199	20313	20739	811	1237
2943	109	20295	20307	457	469
2985	199	20289	20289	787	787
2943	109	20259	20277	421	439
2895	193	20235	20253	1321	1339
2985	199	20205	20229	703	727
2895	193	20175	20199	1261	1285
2943	109	20163	20169	325	331
2895	193	19725	20151	811	1237
2985	199	19713	19719	211	217

1647	61	11427	11451	325	349
1635	109	11385	11421	703	739
1647	61	11373	11379	271	277
1635	109	11355	11361	673	679
1647	61	11343	11349	241	247
1647	61	11313	11331	211	229
1635	109	11181	11307	499	625
1647	61	11163	11175	61	73
1635	109	11139	11157	457	475
1635	109	11103	11121	421	439
1635	109	11079	11091	397	409
1635	109	11043	11067	361	385
1635	109	11007	11031	325	349
1635	109	10983	10995	301	313
1635	109	10953	10971	271	289
1635	109	10941	10941	259	259
1635	109	10923	10929	241	247
1635	109	10893	10911	211	229
1635	109	10833	10881	151	199
1635	109	10803	10821	121	139
1545	103	10797	10797	703	703
1635	109	10791	10791	109	109
1545	103	10785	10785	691	691
1545	103	10767	10773	673	679
1545	103	10755	10755	661	661
1545	103	10593	10737	499	643
1545	103	10551	10581	457	487
1545	103	10515	10533	421	439
1545	103	10491	10503	397	409
1545	103	10455	10479	361	385
1545	103	10419	10443	325	349
1545	103	10395	10407	301	313
1545	103	10365	10383	271	289
1545	103	10353	10353	259	259
1545	103	10335	10341	241	247
1545	103	10305	10323	211	229
1545	103	10245	10293	151	199
1545	103	10215	10233	121	139
1545	103	10197	10203	103	109
1455	97	10179	10185	673	679
1455	97	10167	10167	661	661
1455	97	10005	10149	499	643
1455	97	9963	9993	457	487
1455	97	9927	9945	421	439
1455	97	9903	9915	397	409
1455	97	9867	9891	361	385
1455	97	9831	9855	325	349
1455	97	9807	9819	301	313
1455	97	9777	9795	271	289
1455	97	9765	9765	259	259
1455	97	9747	9753	241	247
1455	97	9717	9735	211	229
1455	97	9657	9705	151	199
1455	97	9627	9645	121	139

2895	193	19701	19707	787	793
2895	193	19665	19689	751	775
2895	193	19617	19653	703	739
2895	193	19605	19605	691	691
2895	193	19587	19593	673	679
2895	193	19575	19575	661	661
2895	193	19413	19557	499	643
2781	103	19407	19407	661	661
2895	193	19371	19401	457	487
2781	103	19245	19365	499	619
2895	193	19239	19239	325	325
2781	103	19203	19233	457	487
2895	193	19185	19197	271	283
2781	103	19167	19179	421	433
2895	193	19155	19161	241	247
2781	103	19143	19149	397	403
2895	193	19125	19137	211	223
2781	103	19107	19119	361	373
2781	103	19071	19095	325	349
2781	103	19047	19059	301	313
2781	103	19017	19035	271	289
2715	181	18999	19005	1261	1267
2781	103	18987	18993	241	247
2715	181	18549	18975	811	1237
2715	181	18525	18531	787	793
2715	181	18489	18513	751	775
2715	181	18441	18477	703	739
2715	181	18429	18429	691	691
2715	181	18411	18417	673	679
2715	181	18399	18399	661	661
2715	181	18237	18381	499	643
2607	157	17961	18231	811	1081
2715	181	17949	17955	211	217
2571	121	17937	17943	787	793
2715	181	17919	17931	181	193
2559	109	17901	17913	751	763
2559	109	17853	17889	703	739
2553	103	17841	17841	691	691
2547	97	17823	17829	673	679
2547	97	17811	17811	661	661
2547	97	17649	17793	499	643
2535	169	17373	17643	811	1081
2547	97	17361	17367	211	217
2535	169	17349	17355	787	793
2547	97	17301	17343	151	193
2535	169	17265	17295	703	733
2547	97	17247	17259	97	109
2535	169	17235	17241	673	679
2535	169	17223	17223	661	661
2535	169	17061	17205	499	643
2445	163	16785	17055	811	1081
2535	169	16773	16779	211	217
2445	163	16761	16767	787	793
2535	169	16731	16755	169	193
1455	97	9603	9615	97	109
1323	49	9243	9261	325	343
1323	49	9219	9231	301	313
1323	49	9189	9207	271	289
1323	49	9177	9177	259	259
1323	49	9159	9165	241	247
1323	49	9129	9147	211	229
1323	49	9069	9117	151	199
1323	49	9039	9057	121	139
1323	49	8979	9027	61	109
1323	49	8967	8967	49	49
1185	79	8241	8295	499	553
1185	79	8199	8229	457	487
1185	79	8163	8181	421	439
1185	79	8139	8151	397	409
1161	43	8127	8127	301	301
1185	79	8103	8121	361	379
1161	43	8097	8097	271	271
1185	79	8067	8091	325	349
1161	43	8037	8055	211	229
1185	79	8013	8031	271	289
1161	43	7977	8007	151	181
1185	79	7953	7971	211	229
1161	43	7947	7947	121	121
1185	79	7893	7941	151	199
1161	43	7887	7887	61	61
1185	79	7863	7881	121	139
1185	79	7821	7851	79	109
1095	73	7653	7665	499	511
1095	73	7611	7641	457	487
1095	73	7575	7593	421	439
1095	73	7551	7563	397	409
1095	73	7515	7539	361	385
1095	73	7479	7503	325	349
1095	73	7455	7467	301	313
1095	73	7425	7443	271	289
1095	73	7413	7413	259	259
1095	73	7395	7401	241	247
1095	73	7365	7383	211	229
1095	73	7305	7353	151	199
1095	73	7275	7293	121	139
1095	73	7227	7263	73	109
1005	67	7023	7035	457	469
1005	67	6987	7005	421	439
999	37	6975	6981	241	247
1005	67	6963	6969	397	403
999	37	6945	6957	211	223
1005	67	6927	6939	361	373
999	37	6885	6921	151	187
1005	67	6867	6879	301	313
999	37	6855	6861	121	127
1005	67	6837	6849	271	283
999	37	6795	6831	61	97
1005	67	6777	6789	211	223

2445	163	16725	16725	751	751
2445	163	16677	16713	703	739
2445	163	16665	16665	691	691
2445	163	16647	16653	673	679
2445	163	16635	16635	661	661
2445	163	16473	16617	499	643
2355	157	16197	16467	811	1081
2445	163	16185	16191	211	217
2355	157	16173	16179	787	793
2445	163	16137	16167	163	193
2355	157	16089	16125	703	739
2355	157	16077	16077	691	691
2355	157	16059	16065	673	679
2355	157	16047	16047	661	661
2355	157	15885	16029	499	643
2355	157	15843	15873	457	487
2265	151	15609	15837	811	1039
2355	157	15597	15603	211	217
2265	151	15585	15591	787	793
2355	157	15543	15579	157	193
2265	151	15501	15537	703	739
2265	151	15489	15489	691	691
2265	151	15471	15477	673	679
2265	151	15459	15459	661	661
2265	151	15297	15441	499	643
2265	151	15255	15285	457	487
2265	151	15219	15237	421	439
2265	151	15195	15207	397	409
2265	151	15159	15183	361	385
2265	151	15123	15147	325	349
2265	151	15099	15111	301	313
2265	151	15069	15087	271	289
2265	151	15057	15057	259	259
2265	151	15039	15045	241	247
2265	151	15009	15027	211	229
2265	151	14949	14997	151	199
2133	79	14877	14931	499	553
2133	79	14835	14865	457	487
2133	79	14799	14817	421	439
2133	79	14775	14787	397	409
2133	79	14739	14763	361	385
2133	79	14703	14727	325	349
2133	79	14679	14691	301	313
2133	79	14649	14667	271	289
2133	79	14637	14637	259	259
2133	79	14619	14625	241	247
2133	79	14589	14607	211	229
2085	139	14433	14583	811	961
2085	139	14409	14415	787	793
2085	139	14373	14397	751	775
2085	139	14325	14361	703	739
2085	139	14313	14313	691	691
2085	139	14295	14301	673	679
2085	139	14283	14283	661	661

999	37	6771	6771	37	37
1005	67	6717	6765	151	199
1005	67	6687	6705	121	139
1005	67	6633	6675	67	109
915	61	6399	6405	421	427
915	61	6375	6387	397	409
915	61	6339	6363	361	385
915	61	6303	6327	325	349
915	61	6279	6291	301	313
915	61	6249	6267	271	289
915	61	6237	6237	259	259
915	61	6219	6225	241	247
915	61	6189	6207	211	229
915	61	6129	6177	151	199
915	61	6099	6117	121	139
915	61	6039	6087	61	109
837	31	5853	5859	211	217
837	31	5793	5841	151	199
837	31	5763	5781	121	139
837	31	5703	5751	61	109
837	31	5673	5691	31	49
735	49	5127	5145	325	343
735	49	5103	5115	301	313
735	49	5073	5091	271	289
735	49	5061	5061	259	259
735	49	5043	5049	241	247
735	49	5013	5031	211	229
735	49	4953	5001	151	199
735	49	4923	4941	121	139
735	49	4863	4911	61	109
735	49	4851	4851	49	49
675	25	4701	4725	151	175
675	25	4671	4689	121	139
675	25	4611	4659	61	109
675	25	4575	4599	25	49
645	43	4515	4515	301	301
645	43	4485	4503	271	289
645	43	4473	4473	259	259
645	43	4455	4461	241	247
645	43	4425	4443	211	229
645	43	4365	4413	151	199
645	43	4335	4353	121	139
645	43	4275	4323	61	109
645	43	4257	4263	43	49
555	37	3885	3885	259	259
555	37	3867	3873	241	247
555	37	3837	3855	211	229
555	37	3777	3825	151	199
555	37	3747	3765	121	139
555	37	3687	3735	61	109
555	37	3663	3675	37	49
513	19	3579	3591	121	133
513	19	3519	3567	61	109
513	19	3477	3507	19	49

2085	139	14121	14265	499	643	465	31	3249	3255	211	217
2085	139	14079	14109	457	487	465	31	3189	3237	151	199
2085	139	14043	14061	421	439	465	31	3159	3177	121	139
2085	139	14019	14031	397	409	465	31	3099	3147	61	109
2085	139	13983	14007	361	385	465	31	3069	3087	31	49
2085	139	13947	13971	325	349	375	25	2601	2625	151	175
2085	139	13923	13935	301	313	375	25	2571	2589	121	139
1989	127	13845	13917	811	883	375	25	2511	2559	61	109
2085	139	13833	13839	211	217	375	25	2475	2499	25	49
1983	121	13821	13827	787	793	351	13	2427	2457	61	91
2085	139	13773	13815	151	193	351	13	2379	2415	13	49
1971	109	13737	13767	703	733	285	19	1983	1995	121	133
1965	103	13725	13725	691	691	285	19	1923	1971	61	109
1959	97	13707	13713	673	679	285	19	1881	1911	19	49
1959	97	13695	13695	661	661	189	7	1281	1323	7	49

Appendix 2

Some applications of the singular direct and indirect product constructions

1161 = 7(189 - 27) + 27, w = 27	13353 = 7(1911 - 4) + 4, w = 7
1917 = 7(285 - 13) + 13, w = 15	13389 = 7(1917 - 5) + 5, w = 7
2295 = 7(351 - 27) + 27, w = 27	13425 = 7(1923 - 6) + 6, w = 7
2673 = 7(405 - 27) + 27, w = 27	13449 = 7(1923 - 2) + 2, w = 7
2745 = 7(405 - 15) + 15, w = 15	13479 = 7(1929 - 4) + 4, w = 7
2757 = 7(405 - 13) + 13, w = 15	13485 = 7(1929 - 3) + 3, w = 7
2793 = 7(405 - 7) + 7, w = 7	13527 = 7(1935 - 3) + 3, w = 7
2799 = 7(405 - 6) + 6, w = 7	13683 = 7(1959 - 5) + 5, w = 7
2805 = 7(405 - 5) + 5, w = 7	13689 = 7(1959 - 4) + 4, w = 7
2811 = 7(405 - 4) + 4, w = 7	13701 = 7(1959 - 2) + 2, w = 7
2817 = 7(405 - 3) + 3, w = 7	13719 = 7(1965 - 6) + 6, w = 7
2823 = 7(405 - 2) + 2, w = 7	13731 = 7(1965 - 4) + 4, w = 7
2829 = 7(405 - 1) + 1, w = 1	13941 = 7(1995 - 4) + 4, w = 285
3429 = 7(513 - 27) + 27, w = 27	14013 = 7(2295 - 342) + 342, w = 351
3879 = 7(555 - 1) + 1, w = 1	14073 = 7(2295 - 332) + 332, w = 351
4509 = 7(645 - 1) + 1, w = 1	14613 = 7(2295 - 242) + 242, w = 351
4563 = 7(675 - 27) + 27, w = 27	14769 = 7(2133 - 27) + 27, w = 27
5055 = 7(735 - 15) + 15, w = 15	14793 = 7(2295 - 212) + 212, w = 351
5067 = 7(735 - 13) + 13, w = 15	14871 = 7(2379 - 297) + 297, w = 351
5097 = 7(729 - 1) + 1, w = 1	15033 = 7(2379 - 270) + 270, w = 351
5697 = 7(837 - 27) + 27, w = 27	15051 = 7(2451 - 351) + 351, w = 351
7089 = 7(1161 - 173) + 173, w = 189	15063 = 7(2295 - 167) + 167, w = 351
7155 = 7(1161 - 162) + 162, w = 189	15093 = 7(2457 - 351) + 351, w = 351
7179 = 7(1161 - 158) + 158, w = 189	15117 = 7(2457 - 347) + 347, w = 351
7209 = 7(1161 - 153) + 153, w = 189	15153 = 7(2295 - 152) + 152, w = 351
7269 = 7(1161 - 143) + 143, w = 189	15213 = 7(2451 - 324) + 324, w = 351
7359 = 7(1161 - 128) + 128, w = 189	15243 = 7(2295 - 137) + 137, w = 351
7449 = 7(1161 - 113) + 113, w = 189	15291 = 7(2379 - 227) + 227, w = 351
7671 = 7(1107 - 13) + 13, w = 15	15483 = 7(2379 - 195) + 195, w = 351
7677 = 7(1107 - 12) + 12, w = 13	15495 = 7(2295 - 95) + 95, w = 351
7683 = 7(1107 - 11) + 11, w = 13	16035 = 7(2295 - 5) + 5, w = 351
7689 = 7(1107 - 10) + 10, w = 13	16041 = 7(2295 - 4) + 4, w = 351
7695 = 7(1107 - 9) + 9, w = 13	16053 = 7(2295 - 2) + 2, w = 351
7701 = 7(1107 - 8) + 8, w = 13	16083 = 7(2379 - 95) + 95, w = 351
7707 = 7(1107 - 7) + 7, w = 13	16131 = 7(2379 - 87) + 87, w = 351
7713 = 7(1107 - 6) + 6, w = 7	16623 = 7(2379 - 5) + 5, w = 351
7719 = 7(1107 - 5) + 5, w = 13	16629 = 7(2379 - 4) + 4, w = 351
7725 = 7(1107 - 4) + 4, w = 13	16641 = 7(2379 - 2) + 2, w = 351
7731 = 7(1107 - 3) + 3, w = 13	16659 = 7(2385 - 6) + 6, w = 19
7737 = 7(1107 - 2) + 2, w = 13	16671 = 7(2385 - 4) + 4, w = 19
7743 = 7(1107 - 1) + 1, w = 1	16719 = 7(2391 - 3) + 3, w = 25
7803 = 7(1161 - 54) + 54, w = 189	17211 = 7(2475 - 19) + 19, w = 375
7809 = 7(1161 - 53) + 53, w = 189	17217 = 7(2475 - 18) + 18, w = 375
8157 = 7(1281 - 135) + 135, w = 189	17229 = 7(2475 - 16) + 16, w = 375
8301 = 7(1197 - 13) + 13, w = 15	17799 = 7(2547 - 5) + 5, w = 7
8307 = 7(1197 - 12) + 12, w = 13	17805 = 7(2547 - 4) + 4, w = 7
8313 = 7(1197 - 11) + 11, w = 13	17817 = 7(2547 - 2) + 2, w = 7
8319 = 7(1197 - 10) + 10, w = 13	17835 = 7(2553 - 6) + 6, w = 7
8325 = 7(1197 - 9) + 9, w = 13	17847 = 7(2553 - 4) + 4, w = 7
8331 = 7(1197 - 8) + 8, w = 13	17895 = 7(2559 - 3) + 3, w = 7
8337 = 7(1197 - 7) + 7, w = 13	18387 = 7(2673 - 54) + 54, w = 405
8343 = 7(1197 - 6) + 6, w = 7	18393 = 7(2673 - 53) + 53, w = 405
8349 = 7(1197 - 5) + 5, w = 13	18405 = 7(2673 - 51) + 51, w = 405
8355 = 7(1197 - 4) + 4, w = 13	18423 = 7(2673 - 48) + 48, w = 405

8361 = 7(1197 - 3) + 3, w = 13	18435 = 7(2673 - 46) + 46, w = 405
8367 = 7(1197 - 2) + 2, w = 13	18483 = 7(2673 - 38) + 38, w = 405
8373 = 7(1197 - 1) + 1, w = 1	18981 = 7(2745 - 39) + 39, w = 405
8379 = 7(1281 - 98) + 98, w = 189	19011 = 7(2793 - 90) + 90, w = 405
8403 = 7(1323 - 143) + 143, w = 189	19065 = 7(3069 - 403) + 403, w = 465
8409 = 7(1317 - 135) + 135, w = 189	19101 = 7(2793 - 75) + 75, w = 405
8451 = 7(1317 - 128) + 128, w = 189	19563 = 7(2799 - 5) + 5, w = 13
8481 = 7(1281 - 81) + 81, w = 189	19569 = 7(2799 - 4) + 4, w = 13
8493 = 7(1323 - 128) + 128, w = 189	19581 = 7(2799 - 2) + 2, w = 13
8541 = 7(1317 - 113) + 113, w = 189	19599 = 7(2805 - 6) + 6, w = 19
8559 = 7(1281 - 68) + 68, w = 189	19611 = 7(2805 - 4) + 4, w = 19
8571 = 7(1317 - 108) + 108, w = 189	19659 = 7(2811 - 3) + 3, w = 25
8583 = 7(1323 - 113) + 113, w = 189	19695 = 7(2817 - 4) + 4, w = 31
8613 = 7(1323 - 108) + 108, w = 189	20157 = 7(3069 - 221) + 221, w = 465
8625 = 7(1317 - 99) + 99, w = 189	20745 = 7(3069 - 123) + 123, w = 465
8631 = 7(1317 - 98) + 98, w = 189	20757 = 7(3069 - 121) + 121, w = 465
8643 = 7(1281 - 54) + 54, w = 189	20817 = 7(2985 - 13) + 13, w = 15
8649 = 7(1281 - 53) + 53, w = 189	20943 = 7(3069 - 90) + 90, w = 465
8667 = 7(1323 - 99) + 99, w = 189	20973 = 7(3069 - 85) + 85, w = 465
8673 = 7(1323 - 98) + 98, w = 189	21033 = 7(3069 - 75) + 75, w = 465
8733 = 7(1317 - 81) + 81, w = 189	21123 = 7(3069 - 60) + 60, w = 465
8739 = 7(1281 - 38) + 38, w = 189	21171 = 7(3069 - 52) + 52, w = 465
8751 = 7(1281 - 36) + 36, w = 189	21327 = 7(3069 - 26) + 26, w = 465
8775 = 7(1323 - 81) + 81, w = 189	21333 = 7(3069 - 25) + 25, w = 465
8805 = 7(1281 - 27) + 27, w = 189	21345 = 7(3069 - 23) + 23, w = 465
8811 = 7(1317 - 68) + 68, w = 189	21363 = 7(3069 - 20) + 20, w = 465
8829 = 7(1281 - 23) + 23, w = 189	21375 = 7(3069 - 18) + 18, w = 465
8841 = 7(1281 - 21) + 21, w = 189	21921 = 7(3135 - 4) + 4, w = 7
8853 = 7(1323 - 68) + 68, w = 189	21933 = 7(3135 - 2) + 2, w = 7
8895 = 7(1317 - 54) + 54, w = 189	24063 = 7(3585 - 172) + 172, w = 513
8901 = 7(1317 - 53) + 53, w = 189	24069 = 7(3591 - 178) + 178, w = 513
8925 = 7(1281 - 7) + 7, w = 189	24111 = 7(3585 - 164) + 164, w = 513
8931 = 7(1281 - 6) + 6, w = 189	24267 = 7(3477 - 12) + 12, w = 513
8937 = 7(1281 - 5) + 5, w = 189	24273 = 7(3477 - 11) + 11, w = 513
8943 = 7(1281 - 4) + 4, w = 189	24285 = 7(3477 - 9) + 9, w = 513
8949 = 7(1281 - 3) + 3, w = 189	24303 = 7(3477 - 6) + 6, w = 513
8955 = 7(1281 - 2) + 2, w = 189	24315 = 7(3477 - 4) + 4, w = 513
8961 = 7(1281 - 1) + 1, w = 1	24363 = 7(3483 - 3) + 3, w = 25
8973 = 7(1287 - 6) + 6, w = 7	24399 = 7(3489 - 4) + 4, w = 31
9033 = 7(1293 - 3) + 3, w = 7	24417 = 7(3489 - 1) + 1, w = 1
9063 = 7(1299 - 5) + 5, w = 7	24423 = 7(3495 - 7) + 7, w = 37
9123 = 7(1305 - 2) + 2, w = 7	24861 = 7(3555 - 4) + 4, w = 7
9153 = 7(1311 - 4) + 4, w = 7	24873 = 7(3555 - 2) + 2, w = 7
9171 = 7(1311 - 1) + 1, w = 1	24933 = 7(3567 - 6) + 6, w = 109
9183 = 7(1317 - 6) + 6, w = 189	25203 = 7(3795 - 227) + 227, w = 555
9213 = 7(1317 - 1) + 1, w = 1	25311 = 7(3663 - 55) + 55, w = 555
9237 = 7(1323 - 4) + 4, w = 189	25317 = 7(3663 - 54) + 54, w = 555
11367 = 7(1635 - 13) + 13, w = 15	25323 = 7(3663 - 53) + 53, w = 555
11457 = 7(1881 - 285) + 285, w = 285	25335 = 7(3663 - 51) + 51, w = 555
11547 = 7(1881 - 270) + 270, w = 285	25341 = 7(3663 - 50) + 50, w = 555
11577 = 7(1881 - 265) + 265, w = 285	25347 = 7(3663 - 49) + 49, w = 555
11625 = 7(1905 - 285) + 285, w = 285	25353 = 7(3795 - 202) + 202, w = 555
11637 = 7(1881 - 255) + 255, w = 285	25371 = 7(3663 - 45) + 45, w = 555
11685 = 7(1881 - 247) + 247, w = 285	25395 = 7(3795 - 195) + 195, w = 555
11691 = 7(1905 - 274) + 274, w = 285	25401 = 7(3879 - 292) + 292, w = 555
11715 = 7(1905 - 270) + 270, w = 285	25419 = 7(3663 - 37) + 37, w = 555
11727 = 7(1881 - 240) + 240, w = 285	25425 = 7(3663 - 36) + 36, w = 555
11745 = 7(1905 - 265) + 265, w = 285	25431 = 7(3663 - 35) + 35, w = 555
11805 = 7(1905 - 255) + 255, w = 285	25443 = 7(3663 - 33) + 33, w = 555
11817 = 7(1881 - 225) + 225, w = 285	25503 = 7(3663 - 23) + 23, w = 555
11847 = 7(1881 - 220) + 220, w = 285	25617 = 7(3663 - 4) + 4, w = 7
11853 = 7(1905 - 247) + 247, w = 285	26079 = 7(3729 - 4) + 4, w = 7
11895 = 7(1905 - 240) + 240, w = 285	26625 = 7(3807 - 4) + 4, w = 7
11901 = 7(1881 - 211) + 211, w = 285	26637 = 7(3807 - 2) + 2, w = 7
11907 = 7(1881 - 210) + 210, w = 285	26697 = 7(3819 - 6) + 6, w = 7
12177 = 7(1881 - 165) + 165, w = 285	27207 = 7(4257 - 432) + 432, w = 645
12213 = 7(1989 - 285) + 285, w = 285	27213 = 7(4257 - 431) + 431, w = 645
12765 = 7(1989 - 193) + 193, w = 285	27225 = 7(4257 - 429) + 429, w = 645
13095 = 7(1881 - 12) + 12, w = 285	27243 = 7(4257 - 426) + 426, w = 645
13101 = 7(1881 - 11) + 11, w = 285	27255 = 7(4257 - 424) + 424, w = 645

Appendix 3

1039 values of n for which an $(n, P_{1,6} \cup \{15, 27\})$-PBD is unknown for underlined values, OSTS(n) are known to exist

9	21	33	39	45	51	57	63	69	75	81	87
93	99	105	111	117	123	129	135	141	147	153	159
165	171	177	183	195	201	207	213	219	225	231	237
243	249	255	261	267	273	279	291	297	303	309	315
321	327	333	339	345	357	363	369	381	387	393	399
411	417	423	429	435	441	447	453	459	471	477	483
489	495	501	507	519	525	531	537	543	549	561	567
573	579	585	591	597	603	609	615	621	627	633	639
651	657	663	669	681	687	693	699	705	711	717	723
741	747	753	759	765	771	777	783	789	795	801	807
813	819	825	831	843	849	855	861	867	873	879	885
891	897	903	909	921	927	933	939	945	951	957	963
969	975	981	987	993	1011	1017	1023	1029	1035	1041	1047
1053	1059	1065	1071	1077	1083	1089	1101	1113	1119	1125	1131
1137	1143	1149	1155	1167	1173	1179	1191	1203	1209	1215	1221
1227	1233	1239	1245	1251	1257	1263	1269	1275	1329	1335	1341
1347	1353	1359	1365	1371	1377	1383	1389	1395	1401	1407	1413
1419	1425	1431	1437	1443	1449	1461	1467	1473	1479	1485	1491
1497	1503	1509	1515	1521	1527	1533	1539	1551	1557	1563	1569
1575	1581	1587	1593	1599	1605	1611	1617	1623	1629	1641	1653
1659	1665	1671	1677	1683	1689	1695	1701	1707	1713	1719	1725
1731	1737	1743	1749	1755	1761	1767	1773	1779	1785	1791	1797
1803	1821	1827	1833	1839	1845	1851	1857	1863	1869	1875	1977
2001	2007	2013	2019	2025	2031	2037	2043	2049	2055	2061	2067
2073	2079	2091	2097	2103	2109	2115	2121	2127	2139	2145	2151
2157	2163	2169	2175	2181	2187	2193	2199	2205	2211	2217	2223
2229	2235	2241	2247	2253	2259	2271	2277	2283	2289	2301	2307
2313	2319	2325	2331	2337	2343	2349	2361	2373	2421	2463	2469
2505	2565	2595	2631	2637	2643	2649	2655	2661	2667	2679	2685
2691	2697	2703	2709	2721	2727	2733	2739	2751	2763	2769	2775
2787	2841	2847	2853	2859	2865	2871	2877	2883	2889	2901	2907
2913	2919	2925	2931	2937	2949	2955	2961	2967	2973	2979	2991
2997	3003	3009	3015	3021	3027	3033	3039	3045	3051	3057	3063
3093	3153	3183	3243	3261	3273	3279	3285	3291	3297	3303	3309
3315	3321	3327	3333	3339	3351	3357	3363	3369	3375	3381	3387
3393	3399	3405	3411	3417	3423	3441	3447	3453	3459	3465	3471
3513	3573	3597	3603	3609	3621	3627	3633	3639	3645	3651	3657
3681	3741	3771	3831	3861	3891	3897	3903	3909	3915	3921	3927
3933	3939	3945	3951	3957	3963	3969	3975	3981	3987	3993	3999
4005	4011	4017	4023	4029	4035	4041	4047	4053	4059	4071	4083
4089	4095	4101	4107	4113	4119	4125	4131	4137	4143	4149	4161
4167	4173	4179	4185	4191	4197	4203	4209	4215	4221	4227	4233
4251	4269	4329	4359	4419	4449	4467	4479	4521	4527	4533	4539
4545	4551	4557	4569	4665	4731	4737	4743	4749	4755	4761	4767
4773	4779	4785	4791	4797	4803	4809	4815	4821	4827	4833	4839
4845	4857	4917	4947	5007	5037	5121	5151	5157	5163	5169	5175
5181	5187	5193	5199	5205	5217	5223	5229	5241	5247	5253	5259
5271	5277	5283	5289	5295	5301	5307	5313	5319	5325	5331	5337
5343	5349	5355	5361	5367	5379	5385	5391	5397	5403	5409	5421
5427	5433	5439	5445	5451	5457	5463	5469	5475	5481	5487	5493
5499	5511	5517	5523	5529	5535	5541	5547	5553	5559	5565	5571

5577	5583	5589	5601	5607	5613	5619	5631	5637	5643	5649	5655
5661	5667	5757	5787	5847	5871	5883	5889	5895	5901	5907	5913
5919	5925	5931	5937	5943	5949	5961	5967	5973	5979	5985	5991
5997	6003	6009	6015	6027	6033	6093	6123	6213	6231	6243	6273
6297	6333	6369	6393	6411	6417	6423	6429	6435	6441	6447	6453
6459	6465	6471	6477	6483	6489	6501	6513	6519	6525	6531	6537
6543	6549	6555	6561	6567	6573	6579	6591	6597	6603	6609	6615
6621	6627	6681	6711	7011	7017	7041	7053	7059	7065	7071	7077
7083	7095	7101	7107	7113	7119	7131	7137	7143	7149	7161	7167
7173	7185	7191	7197	7203	7221	7299	7389	7407	7419	7473	7509
7545	7569	7599	7605	7647	7755	7761	7767	7773	7779	7785	7791
7797	7815	7857	8061	8133	8187	8193	8235	8385	8391	8397	8415
8421	8427	8433	8439	8445	8457	8463	8469	8475	8487	8499	8505
8511	8517	8523	8529	8535	8547	8553	8577	8589	8595	8601	8607
8619	8637	8661	8679	8685	8691	8697	8703	8709	8715	8721	8727
8745	8757	8763	8769	8781	8787	8793	8799	8817	8823	8847	8859
8865	8871	8877	8883	8889	8907	8913	8919	9267	9273	9279	9291
9297	9303	9309	9315	9321	9327	9333	9339	9345	9351	9357	9363
9369	9381	9387	9393	9399	9405	9411	9417	9429	9435	9441	9447
9453	9459	9471	9477	9483	9489	9495	9501	9507	9513	9519	9525
9531	9537	9543	9549	9555	9561	9567	9573	9579	9585	9591	9597
9621	9651	9711	9741	9759	9771	9801	9825	9861	9897	9921	9951
9957	9999	10155	10161	10173	10191	10209	10239	10299	10329	10347	10359
10389	10413	10449	10485	10509	10539	10587	10743	10749	10761	10779	10827
10887	10917	10947	10977	11001	11037	11073	11097	11127	11133	11337	11493
11517	11541	11553	11559	11565	11571	11583	11589	11595	11601	11607	11613
11619	11631	11643	11649	11655	11661	11667	11673	11679	11697	11703	11709
11721	11733	11739	11751	11757	11763	11769	11775	11781	11787	11793	11799
11811	11823	11829	11835	11841	11859	11865	11871	11877	11883	11889	11913
11919	11925	11931	11937	11943	11949	11955	11961	11967	11973	12003	12063
12093	12111	12123	12153	12249	12309	12513	12711	12741	12801	12837	12861
12891	12939	13977	14037	14115	14271	14277	14289	14307	14367	14403	14421
14427	14631	14643	14673	14697	14733	14823	14829	14937	14943	15003	15189
15249	15447	15453	15465	15879	16071	18519	18537	18543	19041	20283	20913
20931	20997	21069	21093	21129	23097	23109	25329	25359	25365	25377	25383
25389	25407	25413	27303	27339	27357	27363					

Appendix 4

Applications of the singular indirect product, using block sizes from $\mathcal{OSTS}$

n = 7(v - a) + a	w	PBD with flat	7(w - a) + a
1239 = 7(189 - 14) + 14	27	189 = 7·27	105
2373 = 7 (351 - 14) + 14	27	351 = 13·27	105
2751 = 7(405 - 14) + 14	27	405 = 15·27	105
4467 = 7(645 - 8) + 8	43	645 = 15·43	253
4557 = 7(729 - 91) + 91	105	729 = 7(105 - 1) + 1	189
4743 = 7(729 - 60) + 60	105	729 = 7(105 - 1) + 1	375
4761 = 7(735 - 64) + 64	105	735 = 7·105	351
4773 = 7(729 - 55) + 55	105	729 = 7(105 - 1) + 1	405
4785 = 7(735 - 60) + 60	105	735 = 7·105	375
4815 = 7(735 - 55) + 55	105	735 = 7·105	405
4833 = 7(729 - 45) + 45	105	729 = 7(105 - 1) + 1	465
5121 = 7(735 - 4) + 4	7	735 = 7·105	25
6231 = 7(915 - 29) + 29	61	915 = 15·61	253
7407 = 7(1095 - 43) + 43	73	1095 = 15·73	253
8133 = 7(1239 - 90) + 90	105	see above	195
8385 = 7(1365 - 195) + 195	195	1365 = 7·195	195
8415 = 7(1365 - 190) + 190	195	1365 = 7·195	225
8433 = 7(1359 - 180) + 180	195	1359 = 7(195 - 1) + 1	285
8475 = 7(1365 - 180) + 180	195	1365 = 7·195	285
8499 = 7(1359 - 169) + 169	195	1359 = 7(195 - 1) + 1	351
8523 = 7(1359 - 165) + 165	195	1359 = 7(195 - 1) + 1	375
8553 = 7(1359 - 160) + 160	195	1359 = 7(195 - 1) + 1	405
8589 = 7(1317 - 105) + 105	189	1317 = 7·(189 - 1) + 1	693
8595 = 7(1365 - 160) + 160	195	1365 = 7·195	405
8637 = 7(1239 - 6) + 6	7	see above	13
8661 = 7(1239 - 2) + 2	7	see above	37
8703 = 7(1359 - 135) + 135	195	1359 = 7(195 - 1) + 1	555
8745 = 7(1365 - 135) + 135	195	1365 = 7·195	555
8793 = 7(1359 - 120) + 120	195	1359 = 7(195 - 1) + 1	645
8823 = 7(1359 - 115) + 115	195	1359 = 7(195 - 1) + 1	675
8865 = 7(1365 - 115) + 115	195	1365 = 7·195	675
8877 = 7(1359 - 106) + 106	195	1359 = 7(195 - 1) + 1	729
8883 = 7(1281 - 14) + 14	189	1281 = 9(189 - 7) + 7	1239
9297 = 7 (1365 - 43) + 43	195	1365 = 7·195	1107
9309 = 7(1359 - 34) + 34	195	1359 = 7(195 - 1) + 1	1161
9333 = 7(1359 - 30) + 30	195	1359 = 7(195 - 1) + 1	1185
9345 = 7 (1359 - 28) + 28	195	1359 = 7(195 - 1) + 1	1197
9351 = 7(1365 - 34) + 34	195	1365 = 7·195	1161
9387 = 7(1359 - 21) + 21	195	1359 = 7(195 - 1) + 1	1239
9429 = 7(1359 - 14) + 14	195	1359 = 7(195 - 1) + 1	1281
9435 = 7(1359 - 13) + 13	195	1359 = 7(195 - 1) + 1	1287
9441 = 7(1359 - 12) + 12	195	1359 = 7(195 - 1) + 1	1293
9447 = 7(1359 - 11) + 11	195	1359 = 7(195 - 1) + 1	1299
9453 = 7(1359 - 10) + 10	195	1359 = 7(195 - 1) + 1	1305
9459 = 7(1359 - 9) + 9	195	1359 = 7(195 - 1) + 1	1311
9471 = 7(1359 - 7) + 7	195	1359 = 7(195 - 1) + 1	1323
9477 = 7(1359 - 6) + 6	7	1359 = 7(195 - 1) + 1	13
9483 = 7(1359 - 5) + 5	7	1359 = 7(195 - 1) + 1	19
9489 = 7(1359 - 4) + 4	7	1359 = 7(195 - 1) + 1	25

9495 = 7(1359 - 3) + 3	7	1359 = 7(195 - 1) + 1	31
9501 = 7(1359 - 2) + 2	195	1359 = 7(195 - 1) + 1	1353
9513 = 7(1365 - 7) + 7	195	1365 = 7·195	1323
9519 = 7(1365 - 6) + 6	7	1365 = 7·195	13
9525 = 7(1365 - 5) + 5	7	1365 = 7·195	19
9531 = 7(1365 - 4) + 4	7	1365 = 7·195	25
9537 = 7(1365 - 3) + 3	7	1365 = 7·195	31
9543 = 7(1365 - 2) + 2	195	1365 = 7·195	1353
9759 = 7(1455 - 71) + 71	97	1455 = 15·97	253
9801 = 7(1575 - 204) + 204	225	1575 = 7·225	351
9825 = 7(1575 - 200) + 200	225	1575 = 7·225	375
9921 = 7 (1569 - 177) + 177	225	1569 = 7(225 - 1) + 1	513
10347 = 7(1545 - 78) + 78	103	1545 = 15·103	253
10413 = 7(1569 - 95) + 95	225	1569 = 7(225 - 1) + 1	1005
10449 = 7(1575 - 96) + 96	225	1575 = 7·225	999
10743 = 7(1575 - 47) + 47	225	1575 = 7·225	1293
10749 = 7(1575 - 46) + 46	225	1575 = 7·225	1299
10761 = 7 (1569 - 37) + 37	225	1569 = 7(225 - 1) + 1	1353
10779 = 7(1545 - 6) + 6	103	1545 = 15·103	685
10947 = 7(1569 - 6) + 6	7	1569 = 7(225 - 1) + 1	13
11001 = 7(1575 - 4) + 4	7	1575 = 7·225	25
11865 = 7(1881 - 217) + 217	285	1881 = 7(285 - 19) + 19	693
12111 = 7(1815 - 99) + 99	121	1575 = 15·121	253
12741 = 7(1881 - 71) + 71	285	1881 = 7(285 - 19) + 19	1569
14307 = 7(2085 - 48) + 48	139	2085 = 15·139	685
14631 = 7(2133 - 50) + 50	79	2133 = 27·79	253
16071 = 7(2355 - 69) + 69	157	2355 = 15·157	685
18537 = 7(2829 - 211) + 211	405	2829 = 7(405 - 1) + 1	1569
18543 = 7(2829 - 210) + 210	405	2829 = 7(405 - 1) + 1	1575
20931 = 7(3165 - 204) + 204	211	3165 = 15·211	253
23109 = 7(3663 - 422) + 422	555	3663 = 7(555 - 37) + 37 555 = 15·43	1353
25329 = 7(3663 - 52) + 52	555	3663 = 7(555 - 37) + 37 555 = 15·43	3573
27303 = 7(4473 - 668) + 668	729	4473 = 7(729 - 105) + 105 729 = 7(105 -1) + 1	1095