DETERMINATION OF A PACKING NUMBER

Douglas Stinson

ABSTRACT

The packing number $D(2,4,19)$ is defined to be the maximum number of unordered quadruples of elements, chosen from a set of 19 elements, such that no pair of elements occurs in more than one quadruple. It is shown here that $D(2,4,19) = 25$.

1. *Introduction.*

A (t,k) packing of a v-set V is a family F of k-subsets (blocks) of V such that no t-subset of elements (varieties) occurs in more than one block of F. The maximum number of blocks in a (t,k) packing is often denoted by $D(t,k,v)$.

In this paper we consider $(2,4)$ packings, in particular, the $(2,4)$ packing on 19 varieties.

Denote $D(2,4,v)$ by $D(v)$. It is well-known that $D(v) \le B(v)$, where $B(v) = [\frac{v}{4}[\frac{v-1}{3}]]$, and $[x]$ denotes the integer part of x.

For $v \equiv 0 \pmod 3$, $v \neq 9$, Brouwer has shown [1] that $D(v) = B(v)$, while $D(9) = B(9) - 1 = 3$. For $v \equiv 2 \pmod 6$, $v \neq 8$, Brouwer has shown [1] that $D(v) = B(v)$, while $D(8) = B(8) - 2 = 2$. Also, Brouwer conjectures [1] that if $v \equiv 5 \pmod 6$, $v \neq 11$ or 17, then $D(v) = B(v)$, while $D(11) = B(11) - 2 = 6$, and $D(17) = B(17) - 1 = 20$.

The remaining cases are $v \equiv 1 \pmod 3$. For $v \equiv 1$ or $4 \pmod{12}$, Hanani has shown [3] that $D(v) = B(v)$. These packings are "perfect", in that every pair of varieties occurs in exactly one block, and hence are balanced incomplete block designs. For $v \equiv 7$ or $10 \pmod{12}$, Mullin has shown [5] that $D(v) \le B(v) - 1$, and that $D(v) = B(v) - 1$ for all

ARS COMBINATORIA, Vol. 3 (1977), pp. 89-114.

but a finite number of possible exceptions. For $v \equiv 7$ or $10 \pmod{12}$, $v \neq 10$ or 19, Brouwer has shown [1] that $D(v) = B(v) - 1$, while $D(10) = B(10) - 2 = 5$. The only remaining case is $v = 19$. We will prove here that $D(19) = B(19) - 3 = 25$.

Since $D(19) \leq B(19) - 1$, we have

THEOREM 1.1. *No (2,4) packing on 19 varieties contains more than 27 blocks.*

2. *Non-Occurrence Graphs.*

Let F be a (2,4) packing on 19 varieties. Construct a graph G, with vertices 1 to 19 representing varieties, so that an edge $ij \in G$ if and only if the pair ij does not occur in F. We say that G is the non-occurrence graph of F.

Let r_i be the number of times a variety i occurs in F. The number of pairs which involve i and do not occur in F is $18 - 3r_i$. Thus the valency of each vertex of G is a multiple of 3.

THEOREM 2.1. *No (2,4) packing on 19 varieties contains more than 26 blocks.*

Proof. This can be shown from the corresponding covering problem. A (t,k) covering of a v-set V is a family F of k-subsets (blocks) of V such that every t-subset of elements (varieties) occurs in at least one block of F. The minimum number of blocks in a (t,k) covering is often denoted by $N(t,k,v)$. Mills has shown [4] that $N(2,4,19) = 31$.

Now, assume $D(19) = 27$; thus we have a family F of 27 blocks, with no pairs repeated. There are $\binom{19}{2} = 171$ possible pairs of varieties, and $\binom{4}{2} \cdot 27 = 162$ pairs occur in F; so the pair-deficiency is 9. There are two possible non-isomorphic non-occurrence graphs, shown in Figures 1(a) and 1(b).

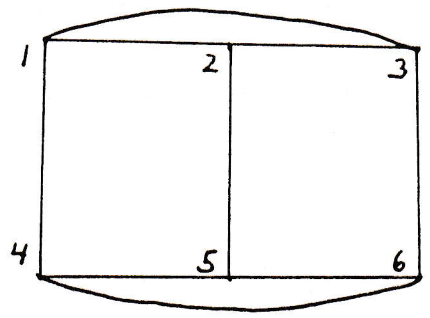


Figure 1(a)

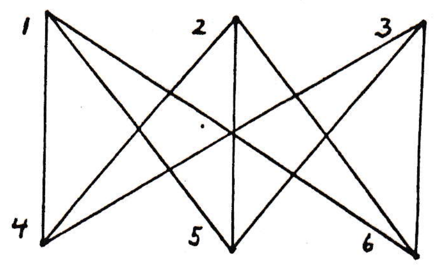


Figure 1(b)

CASE 1. To F, adjoin the blocks 1256, 2345, and 1346. Then we have a (2,4) covering with 30 blocks; this is impossible, since $N(2,4,19)=31$.

CASE 2. Here, adjoin the blocks 1235, 1345, and 1356; again, we have an impossibility.

We now consider the possibilities for a packing of 19 varieties into 26 blocks. For a variety i, let d_i, the deficiency of i, be $6 - r_i$.

LEMMA 2.2. *No variety in a packing with 26 blocks has a deficiency greater than 2.*

Proof. Suppose the contrary. Then there exists a variety i with $r_i \le 3$. Delete the blocks containing i from F. Then we have a packing of 18 varieties into 23 or more blocks. However $D(18) \le B(18) = 22$; this is a contradiction.

Let there be n_i varieties with deficiency i, for $0 \le i \le 2$. Since $\sum_{i=1}^{19} d_i = 19\cdot 6 - 26\cdot 4 = 10$, we have $n_1 + 2n_2 = 10$. Also,

$n_0 + n_1 + n_2 = 19$. We have the following solutions in non-negative integers.

	n_0	n_1	n_2
(1)	14	0	5
(2)	13	2	4
(3)	12	4	3
(4)	11	6	2
(5)	10	8	1
(6)	9	10	0

Each of the above cases will be considered in terms of possible non-occurrence graphs. We note that, if a variety has deficiency i, then the corresponding vertex in G has valency $3i$.

CASES (1),(2). Let 1 be a variety with deficiency 2. In G, vertex 1 has valency 6, but there are fewer than six other vertices with positive valency in G. Therefore cases (1) and (2) are impossible.

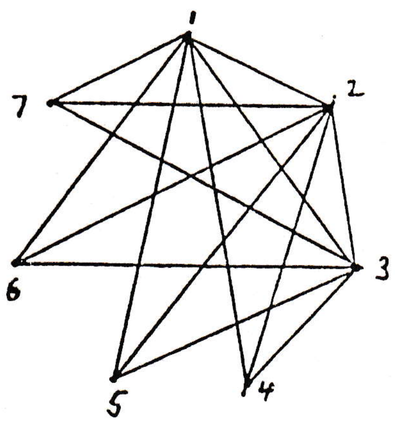
We will eliminate the non-occurrence graphs of cases (3) to (5) by either of the two following criteria.

(a) If G contains K_4 as a subgraph, then a block containing the four varieties corresponding to the vertices of K_4 may be added to F to produce a packing with 27 blocks; this is impossible by Theorem 2.1.

(b) In many cases, G has the property that four blocks may be adjoined to F to give a covering with 30 blocks; this is impossible (Theorem 2.1.).

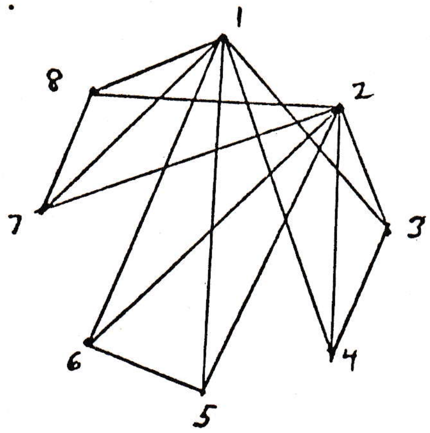
CASE (3). G is shown in Figure 2. We note that the subgraph on vertices 1, 2, 3 and 4 is a 4-clique.

Figure 2.

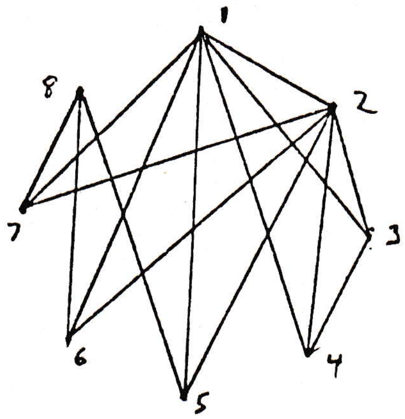


CASE (4). We have four non-isomorphic graphs G.

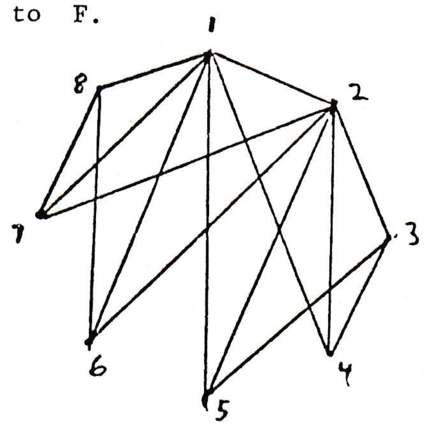
(a) Adjoin 1234, 1256, and 1278 to F.



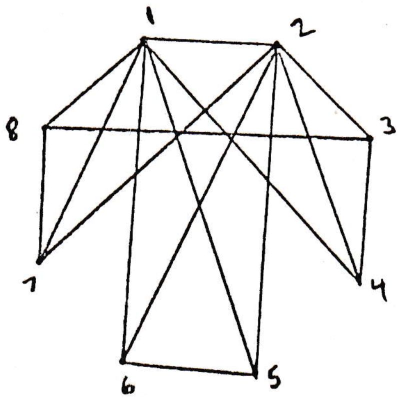
(b) The subgraph on vertices 1,2,3 and 4 is a 4-clique.



(c) Adjoin 1234, 1235, 1268 and 1278 to F.

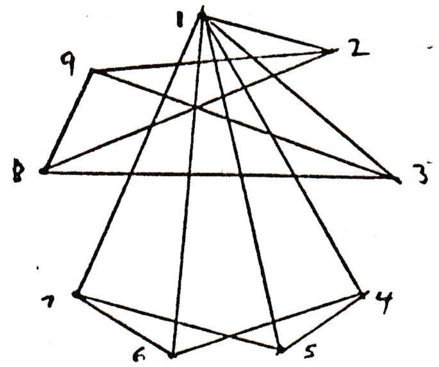


(d) The subgraph on vertices 1, 2, 5 and 6 is a 4-clique.

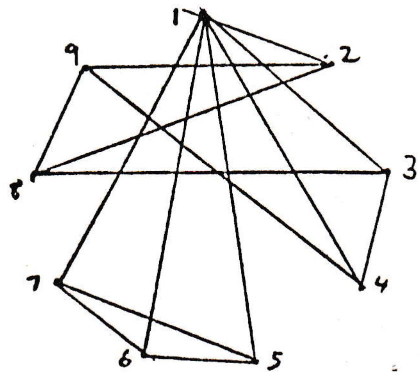


CASE (5).We have 12 non-isomorphic possibilities for G.

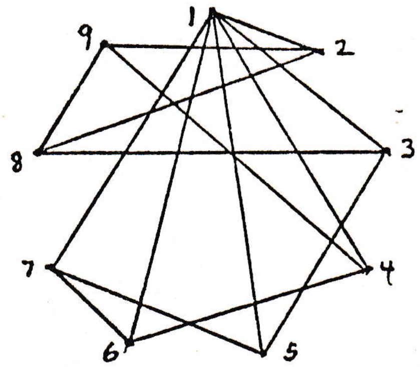
(a) Adjoin 2389, 4567, 1234 and 1567 to F.



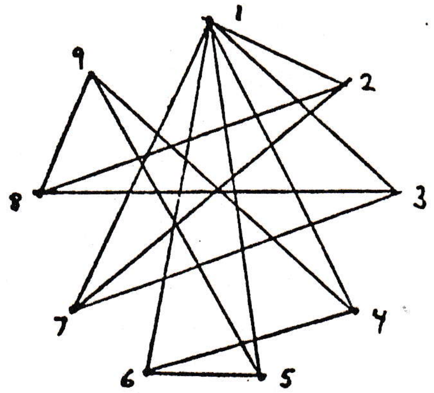
(b) The subgraph on vertices 1,5,6 and 7 is a 4-clique.



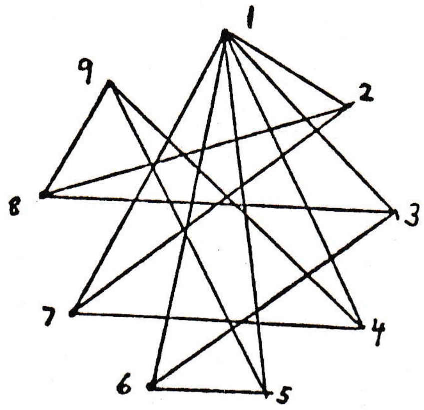
(c) Adjoin 1467, 1357, 3489 and 1289 to F.



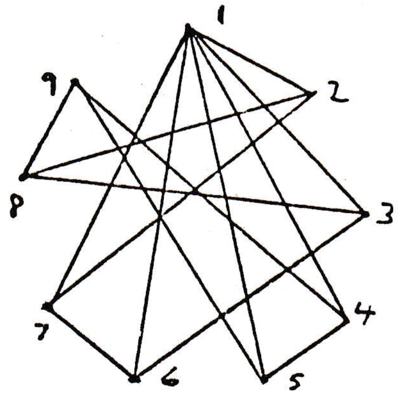
(d) Adjoin 4589, 1456, 1237 and 2389 to F.



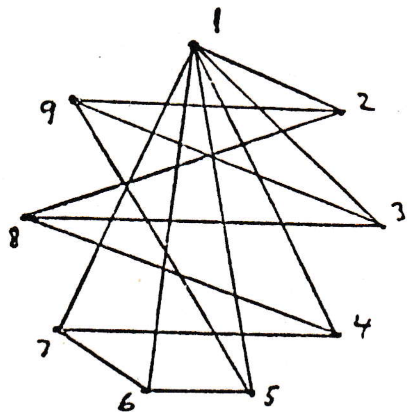
(e) Adjoin 1356, 1247, 2389, 4589 to F.



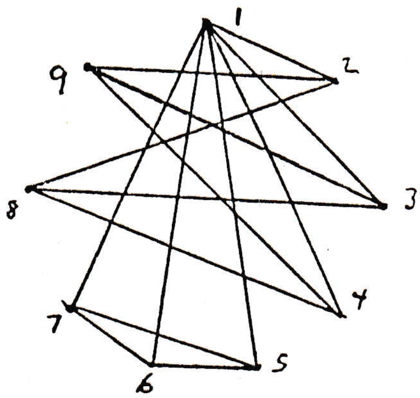
(f) Adjoin 1459, 2389, 1367, 1267 to F.



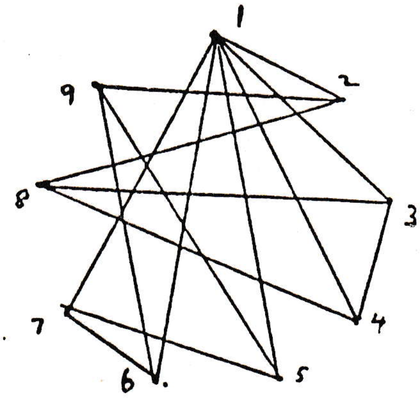
(g) Adjoin 1567, 1478, 2359, 1238 to F.



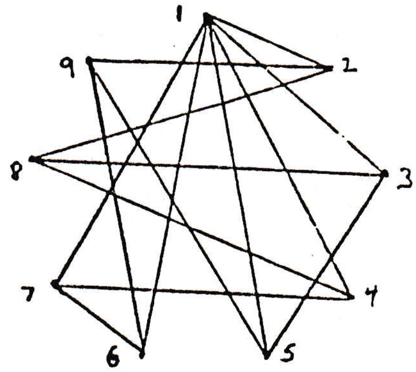
(h) The subgraph on vertices 1, 5, 6 and 7 is a 4-clique.



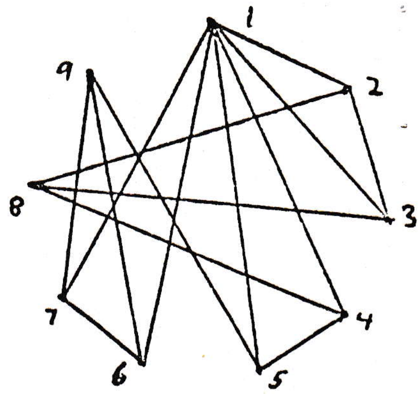
(i) Adjoin 1567, 2569, 1238 and 1348 to F.



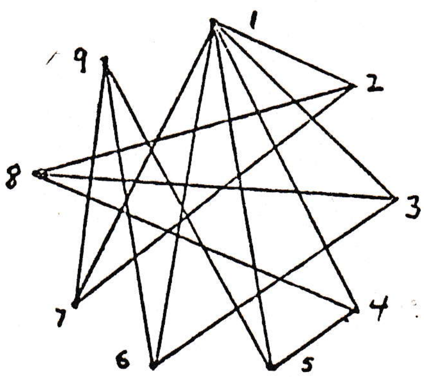
(j) Adjoin 1467, 1235, 2348 and 2569 to F.



(k) Adjoin 1238, 1679, 1459 and 1458 to F.



(ℓ) Adjoin 1278, 1368, 1458
and 5679 to F.



Case (6) is the only case remaining, and we have

THEOREM 2.3. *If a packing with 26 blocks exists, there are ten varieties with $r = 5$, and nine varieties with $r = 6$.*

3. *Notation and Plan of Attack.*

Set $A = \{1,2,\ldots,9\}$, $B = \{10,11,\ldots,19\}$. We will attempt to construct a (2,4) packing with 26 blocks, where $r_i = 6$ for $1 \le i \le 9$, and $r_i = 5$ for $10 \le i \le 19$.

Let a block of type i be a block containing i varieties from A. Let there be n_i blocks of type i in the packing. Then standard counting arguments yield

$$\sum_{i=0}^{4} n_i = 26, \quad \sum_{i=0}^{4} i n_i = 54, \quad \sum_{i=0}^{4} \binom{i}{2} n_i = 36.$$

We have the following solutions in non-negative integers:

	n_4	n_3	n_2	n_1	n_0
(1)	0	8	12	6	0
(2)	1	5	15	5	1
(3)	2	2	18	4	0
(4)	0	7	15	3	1
(5)	1	4	18	2	1
(6)	2	1	21	1	1
(7)	0	6	18	0	2

Each of cases (1) to (7) will be considered in the remainder of the paper.

For a variety $a \in A$, let a occur in a_i blocks of type i. Then

$$a_0 = 0, \quad \sum_{i=1}^{4} a_i = 6, \quad \sum_{i=1}^{4} (i-1) a_i = 9.$$

We have the following solutions in non-negative integers:

	a_4	a_3	a_2	a_1	a_0
(1)	2	1	0	3	0
(2)	2	0	2	2	0
(3)	1	2	1	2	0
(4)	1	1	3	1	0
(5)	1	0	5	0	0
(6)	0	4	0	2	0
(7)	0	3	2	1	0
(8)	0	2	4	0	0

If a variety $a \in A$ corresponds to solution (i) above, we say that a is a variety of type A_i.

Similarly, for $b \in B$, let b occur in b_i blocks of type i. Then

$$b_4 = 0, \quad \sum_{i=0}^{3} b_i = 5, \quad \sum_{i=0}^{3} (3-i)b_i = 6.$$

We have the following solutions in non-negative integers:

	b_4	b_3	b_2	b_1	b_0
(1)	0	3	0	0	2
(2)	0	2	1	1	1
(3)	0	1	3	0	1
(4)	0	2	0	3	0
(5)	0	1	2	2	0
(6)	0	0	4	1	0

If a variety $b \in B$ corresponds to solution (i) above, we say that b is a variety of type B_i.

Let there be x_i varieties of type A_i, y_i varieties of type B_i. The values of x_i and y_i will depend on which of cases (1) to (7) is being considered.

We will deal with many of the cases in the following manner. The varieties from A are placed in the design. Without loss of generality, we may place the varieties from B into the design one time each, so that each variety from B occurs with some particular variety, say 1, from A. Then a computer programme attempts to complete the design by adding each variety from B to the design 4 more times. The programme is basically an exhaustive search, with a few refinements. Each run requires only a few seconds of CPU time. Thus, the task is to determine all possible ways of placing the varieties from A into the design. However, all that is really necessary is to place the varieties from A into the blocks of type 3 and type 4. Then the pairs of varieties from A which do not occur in blocks of type 3 or type 4 can be adjoined as blocks of type 2, and the blocks of type 1 are also determined, since each variety from A occurs 6 times in the design. Other methods, such as considering the possible occurrences of the varieties from B in the blocks of type 0 and type 1 will also be used.

4. *Packings with 26 Blocks.*

LEMMA 4.1. *Case* (6) ($n_4 = 2$, $n_3 = 1$, $n_2 = 21$, $n_1 = 1$, $n_0 = 1$) *and Case* (7) ($n_4 = 0$, $n_3 = 6$, $n_2 = 18$, $n_1 = 0$, $n_0 = 2$) *are impossible.*

Proof. We note that there does not exist a variety $b \in B$ with $b_0 = b_1 = 0$. Now, there are no more than $3n_1 + 4n_0$ distinct varieties from B which occur in blocks of type 0 or type 1. In cases (7) and (7), $3n_1 + 4n_0 < 10$; thus, some variety from B has $b_0 = b_1 = 0$, which is not allowed.

CASE (5). Here $3n_1 + 4n_0 = 10$, so that every variety $b \in B$ occurs exactly once in a block of type 0 or type 1. We may let these blocks be

10	11	12	13
14	15	16	
17	18	19	

Then varieties 10 - 13 are of type B_3, and varieties 14 - 19 are of type B_6.

Consider the varieties from A which occur in the two blocks of type 1. We have two subcases: (1) some variety, say 1, occurs in both blocks of type 1; (2) two different varieties, say 1 and 2, occur in one block of type 1 each.

<u>Subcase (1)</u>. 1 is either (i) a variety of type A_6 or (ii) a variety of type A_3.

(i) We may let the block of type 4 be 2345. We note that varieties 10 - 13 occur in one block of type 1 each. Also, variety 1 occurs in each block of type 3. Then the blocks of type 0, 1, 3, 4 are

10	11	12	13
14	15	16	1
17	18	19	1
10	1		
11	1		
12	1		
13	1		
2	3	4	5

Then varieties 6 - 9 are varieties of type A_8, and each appears in at least two blocks of type 3. Since 1 occurs in each block of type 3, this is not allowed.

(ii) We may let the block of type 4 be 1234; we may let variety 1 occur with varieties 10 and 11 in the blocks of type 3. We have

10	11	12	13
14	15	16	1
17	18	19	1
10	1		
11	1		
12			
13			
1	2	3	4

Then varieties 2 - 4 are varieties of type A_5, and varieties 5 - 9 are varieties of type A_8.

The blocks of type 3 may be completed

10	1	5	6
11	1	7	8
12	5	7	9
13	6	8	9

The computer shows that the design cannot be completed.

Subcase (2). We have three possibilities.

(i) 1 and 2 are both type A_4 varieties,

(ii) 1 is a type A_4 variety, 2 is a type A_7 variety,

(iii) 1 and 2 are both type A_7 varieties.

(i): We may let the block of type 4 be 1234, and we may let 1 occur with 10, 2 occur with 11. Then the blocks of types 0, 1, 3, 4 may be completed

10	11	12	13	11	2	7	8
14	15	16	1	12	5	7	8
17	18	19	2	13	6	8	9
10	1	5	6	1	2	3	4

A computer search shows that the design cannot be completed.

(ii): We may let the block of type 4 be 1345. Then 3 - 5 are varieties of type A_5, 6 - 9 are varieties of type A_8. We may let 1 occur with 10; then there are two possibilities:

(a) 2 occurs with 10, 11, 12; or

(b) 2 occurs with 11, 12, 13.

(a) Here, 2 must occur with 5 more varieties in blocks of type 3. However, these varieties must be chosen from varieties 6 - 9; so some pair occurs twice. This is impossible.

(b) Here 2 must occur with six more varieties in blocks of type 3; this is impossible, since some pair would occur twice.

(iii): Here 1 and 2 each occur in three blocks of type 3. Since there are only four blocks of type 3, the pair 12 occurs twice. This is impossible.

Thus we have

LEMMA 4.2. *Case* (5) $(n_4 = 1, n_3 = 4, n_2 = 18, n_1 = 2, n_0 = 1)$ *is impossible.*

CASE (4). Here $x_i = 0$ for $1 \le i \le 6$. Also, we have $\sum_{i=1}^{8} x_i = 9$, and $2x_6 + x_7 = 3$. We have the following solutions in non-negative integers:

	x_6	x_7	x_8
(1)	1	1	7
(2)	0	3	6

Subcase (1): Let 1 be the variety of type A_6, 2 be the variety of type A_7, 3 - 9 be the varieties of type A_8. We may let the blocks of type 1 be 10 11 12 1, 13 14 15 1. Variety 1 also occurs in four blocks of type 3. Now, the only varieties from B with which 1 has not occurred are 16, 17, 18, 19; so each of 16, 17, 18, 19, occurs in a block of type 3. Hence, the block of type 0 must be 16 17 18 19, and we may let the remaining block of type 1 be 10 13 16 2.

Thus we have a skeleton

16	17	18	19
10	11	12	1
13	14	15	1
10	13	16	2
16	1		
17	1		
18	1		
19	1		
10			
13			
16			

Variety 2 occurs in three blocks of type 3. Since 2 has already occurred with 10, 13, and 16, 2 must occur with 17, 18 and 19; but then the pair 12 occurs three times. This is impossible.

Subcase (2); Let the varieties of type A_7 be 1 - 3; let the varieties of type A_8 be 4 - 9. We have the following possibilities:

(i) 1, 2, and 3 occur in some block of type 3;

(ii) Pairs 12, 13, 23, occur separately in three blocks of type 3;

(iii) Pairs 12, 13, occur separately in two blocks of type 3.

Since 1, 2, 3, appear a total of nine times in seven blocks of type 3, at least two pairs from these numbers occur, and the three cases listed are the only possibilities.

(i) We have three non-isomorphic ways of constructing the blocks of type 3, namely,

1	2	3	1	2	3	1	2	3
1	4	5	1	4	5	1	4	5
1	6	7	1	6	7	1	6	7
2	4	8	2	4	8	2	4	6
2	5	9	2	7	9	2	8	9
3	6	8	3	6	8	3	5	8
3	7	9	3	5	9	3	7	9

None of these partial designs can be completed.

(ii) We have the following six non-isomorphic ways of constructing the blocks of type 3:

1	2	4	1	2	4	1	2	4	1	2	4	1	2	4	1	2	4
1	3	5	1	3	5	1	3	5	1	3	5	1	3	5	1	3	5
2	3	6	2	3	6	2	3	6	2	3	6	2	3	6	2	3	6
1	6	7	1	6	8	1	6	9	1	6	8	1	6	9	1	7	8
2	5	8	2	5	9	2	5	8	2	7	8	2	7	8	2	7	9
3	4	9	3	7	8	3	7	8	3	7	9	3	7	9	3	8	9
7	8	9	4	7	9	4	7	9	4	5	9	4	5	8	4	5	6

None of these partial designs can be completed.

(iii) We have the following three non-isomorphic ways of constructing the blocks of type 3:

1	2	4	1	2	4	1	2	4
1	3	5	1	3	5	1	3	5
1	6	7	1	6	7	1	6	7
2	5	8	2	5	8	2	8	9
2	6	9	2	6	9	2	5	6
3	4	8	3	7	8	3	4	8
3	7	9	3	4	9	3	7	9

None of these partial designs can be completed.

Thus we have

LEMMA 4.3. *Case* (4) $(n_4 = 0, n_3 = 7, n_2 = 15, n_1 = 3, n_0 = 1)$ *is impossible.*

CASE (3). We have two possibilities for the blocks of type 4.

(1) 1 2 3 4 and 5 6 7 8 (2) 1 2 3 4 and 1 5 6 7.

Subcase (1): We may let the blocks of type 3 be 159 and 269. The partial design cannot be completed.

Subcase (2): Since 8, 9, do not appear in a block of type 4, they both must appear in at least two blocks of type 3. Since there are only two blocks of type 3, the pair 89 occurs twice; this is impossible.

Thus we have

LEMMA 4.4. *Case* (3) $(n_4 = 2, n_3 = 2, n_2 = 18, n_1 = 4, n_0 = 0)$ *is impossible.*

CASE (2). We may take the blocks of type 4 as 1 2 3 4; then we also have $x_1 = x_2 = 0$, $x_3 + x_4 + x_5 = 4$. We have the following solutions in non-negative integers:

	x_3	x_4	x_5		x_3	x_4	x_5
(1)	4	0	0	(9)	1	0	3
(2)	0	4	0	(10)	0	2	2
(3)	0	0	4	(11)	2	0	2
(4)	3	0	1	(12)	2	2	0
(5)	3	1	0	(13)	1	1	2
(6)	0	3	1	(14)	1	2	1
(7)	1	3	0	(15)	2	1	1
(8)	0	1	3				

Varieties 1 - 4 occur in $2x_3 + x_4$ blocks of type 1; so we must have $2x_3 + x_4 \leq 5$. This eliminates subcases (1), (4), (5), and (12). Also, varieties 1 - 4 occur in $x_3 + 3x_4 + 5x_5$ blocks of type 2. Since none of 1 - 4 can occur together again, we must have $x_3 + 3x_4 + 5x_5 \leq 15$. This eliminates subcases (3), (8), (9), and (10).

Subcase (2): Varieties 1 - 4 each occur in one block of type 1. Let 5 be the other variety from A occurring in a block of type 1. Then 5 is a variety of type A_7, 6 - 9 are varieties of type A_8. We have two possibilities for the blocks of type 3:

1	5	6		1	5	6
2	5	7		2	5	7
3	5	8		3	6	8
4	6	9		4	7	9
7	8	9		5	8	9

Neither partial design can be completed.

Subcase (6): Varieties 1 - 3 each occur in one block of type 1. We have two possibilities: (i) two varieties, say 5 and 6, each occur in one block of type 1, (ii) one variety, say 5, occurs in two blocks of type 1.

(i) Here 5 and 6 are varieties of type A_7, 6 - 9 are varieties of type A_8. The blocks of type A_3 are

1	5	6
2	5	7
3	6	8
5	8	9
6	7	9

This design cannot be completed.

(ii) Here 5 is a variety of type A_6, 6 - 9 are varieties of type A_8. Variety 5 occurs in four blocks of type 3 along with eight other varieties, which must be chosen from 1 - 3, 6 - 9. Then a pair occurs twice (impossible).

Subcase (7): Varieties 5 - 9 are varieties of type A_8. The blocks of type 3 are

1 5 6
1 7 8
2 5 9
3 7 9
4 6 8

The design cannot be completed.

Subcase (11): Varieties 1 and 2 each occur in two blocks of type 1. We may let the other variety occuring in a block of type 1 be 5. Then 5 is a variety of type A_7, 6 - 9 are varieties of type A_8. The blocks of type 3 are

1 5 6
1 7 8
2 5 7
2 6 9
5 8 9

The design cannot be completed.

Subcase (13): As in subcase (6), we have two possibilities: (i) two varieties, say 5 and 6, each occur in one block of type 1, (ii) one variety, say 5, occurs in two blocks of type 1.

(i) Here 5 and 6 are varieties of type A_7, 7 - 9 are varieties of type A_8. The blocks of type 3 are

1 5 7
1 6 8
2 5 6
5 8 9
6 7 9

The design cannot be completed.

(ii) This subcase can be eliminated in the same way as subcase (6) - (ii).

Subcase (14): Let 5 be the other variety from A which occurs in a block of type 1. Then 5 is a variety of type A_7, 6 - 9 are varieties of type A_8. We have three possibilities for the blocks of type 3:

1 5 6	1 5 6	1 6 7
1 7 9	1 7 8	1 8 9
2 5 7	2 5 7	2 5 7
3 5 8	3 6 9	3 5 9
6 8 9	5 8 9	5 6 8

None of the partial designs can be completed.

Subcase (15): Here, varieties 5 - 9 are varieties of type A_8. The blocks of type 3 are

1 5 6
1 7 8
2 5 9
2 6 7
3 8 9

The partial design cannot be completed.

Thus we have

LEMMA 4.5. *Case* (2) $(n_4 = 1, n_3 = 5, n_2 = 15, n_1 = 5, n_0 = 1)$ *is impossible.*

CASE (1). We have $x_i = 0$, for $1 \le i \le 5$. Also, $x_6 + x_7 + x_8 = 9$ and $2x_6 + x_7 = 6$. We have the following solutions in non-negative integers.

	x_6	x_7	x_8
(1)	3	0	6
(2)	2	2	5
(3)	1	4	4
(4)	0	6	3

Subcase (1): Let 1,2,3, be the varieties of type A_6. Each of 1,2,3, occurs in four blocks of type 3. Since there are eight blocks of type 3, there must be four pairs formed from these varieties. Since there are only three distinct pairs possible, some pair must be repeated (impossible).

Subcase (2): Let 1,2, be varieties of type A_6; 3,4, be varieties of type A_7. We have two possibilities: (i) 1 occurs with 2 in a block of type 3; (ii) 1 does not occur with 2 in a block of type 3.

(i) We have two non-isomorphic ways of constructing the blocks of type 3.

1 2 5	1 2 5
1 3 6	1 3 6
1 4 7	1 4 7
1 8 9	1 8 9
2 3 8	2 3 7
2 4 9	2 4 9
2 6 7	2 6 8
3 4 5	3 4 5

Neither of these partial designs can be completed.

(ii) Varieties 1 and 2 each occur in four of the eight blocks of type 3. Variety 3 occurs in three blocks of type 3, and so must occur twice with 1 or 2 (impossible).

Subcase (3): Let 1 be the variety of type A_6, 2 - 5 be varieties of type A_7, 6 - 9 be varieties of type A_8. There are three ways of distributing varieties 2 - 5 in the block of type 3 which contain variety 1 (1 occurs with all of varieties 2 - 9 in the blocks of type 3):

(i)		(ii)		(iii)	
	1 2 3		1 2 3		1 2
	1 4 5		1 4		1 3
	1		1 5		1 4
	1		1		1 5

(i) We have the type 3 blocks:

1 2 3, 1 4 5, 1 6 7, 1 8 9, 2 4 6, 2 5 7, 3 4 8, 3 5 9.

The design cannot be completed.

(ii) We have four non-isomorphic ways of constructing the blocks of type 3:

1 2 3	1 2 3	1 2 3	1 2 3
1 4 6	1 4 6	1 4 6	1 4 6
1 5 7	1 5 7	1 5 7	1 5 7
1 8 9	1 8 9	1 8 9	1 8 9
2 4 8	2 4 8	2 4 5	2 4 5
2 5 6	2 5 9	2 6 7	2 6 7
3 4 7	3 4 7	3 4 8	3 4 8
3 5 9	3 5 6	3 5 6	3 5 9

None of the designs can be completed.

(iii) We have four non-isomorphic ways of constructing the blocks of type 3:

1 2 6	1 2 6	1 2 6	1 2 6
1 3 7	1 3 7	1 3 7	1 3 7
1 4 8	1 4 8	1 4 8	1 4 8
1 5 9	1 5 9	1 5 9	1 5 9
2 4 7	2 4 9	2 4 5	2 4 5
2 5 8	2 5 7	2 7 8	2 7 9
3 4 9	3 4 6	3 4 9	3 4 6
3 5 6	3 5 8	3 5 6	3 5 8

None of these designs can be completed.

<u>Subcase (4)</u>: Let 1 - 6 be varieties of type A_7, 7 - 9 be varieties of type A_8.

We now define a block of type i' to be a block of type 3 which contains i varieties of type A_7. Let there be n_i' type i' blocks, for $0 \le i \le 3$. Then $\sum_{i=0}^{3} n_i' = 8$, and $\sum_{i=0}^{3} i\, n_i' = 18$. We have the following solutions in non-negative integers:

	n_0'	n_1'	n_2'	n_3'
(i)	2	0	0	6
(ii)	1	1	1	5
(iii)	0	3	0	5
(iv)	1	0	3	4
(v)	0	2	2	4
(vi)	0	1	4	3
(vii)	0	0	6	2

Since there are only $\binom{6}{2} = 15$ pairs possible from varieties 1 - 6, we must have $n_2' + 3n_3' \leq 15$. Hence, (i) and (ii) are not possible.

(iii): Here $n_2' + 3n_2' = 15$; so there are no missing pairs formed from varieties of type A_7. However, any variety which occurs in a block of type 1' cannot occur with the five other varieties of type A_7; so not every pair can occur (impossible).

(iv): Here we have a block of type 0', and it must be 789. Then each of 7,8,9 occurs four times in a block of type 4, no two in the same block. Thus each of the twelve blocks of type 2 contains 7, 8 or 9. Now there are only eight blocks of type 3; so there exists a variety $b \in B$ which does not occur in a block of type 3. Then b is a variety of type B_6, and b occurs in four blocks of type 2. But then b occurs twice with one of 7, 8, or 9 (impossible).

(v): We have twelve varieties of type A_7 in the four blocks of type 3'. No variety may occur three times; so each variety of type A_7 must occur twice in blocks of type 3'. Then the blocks of type 3 are:

1 2 3 , 1 4 5 , 2 4 6 , 3 5 6 , 1 6 7 , 2 5 8 , 3 7 9 , 4 8 9.

The design cannot be completed.

(vi): We have three blocks of type 3', and they contain nine varieties of type A_2. No more than three varieties can occur twice in blocks of type 3'. If fewer than three varieties occur twice in blocks of type 3', then there are no more than eight varieties of type A_7 in the blocks of type 3' (contradiction). Thus we have three varieties of type A_7, say 1, 2, 3, each of which occurs in two blocks of type 3', and three varieties of type A_7, say 4, 5, 6, each of which occurs in one block of type 3'. We have four non-isomorphic ways of constructing the blocks of type 3:

1 2 4	1 2 4	1 2 4	1 2 4
1 3 5	1 3 5	1 3 5	1 3 5
2 3 6	2 3 6	2 3 6	2 3 6
1 7 8	1 7 8	1 6 7	1 6 7
2 5 7	2 5 9	2 5 7	2 5 8
3 4 9	3 4 9	3 4 8	3 4 9
4 6 8	4 6 7	4 5 9	4 5 7
5 6 9	5 6 8	6 8 9	6 8 9

None of the designs can be completed.

(vii): We have two possibilities. The two blocks of type 3' contain either (a) six distinct varieties of type A_7 or

(b) five distinct varieties of type A_7.

(a) The blocks of type 3 are:

1 2 3, 4 5 6, 1 4 7, 1 5 8, 2 4 8, 2 6 9, 3 5 9, 3 6 7.

The design cannot be completed.

(b) We have three non-isomorphic ways of constructing the blocks of type 3:

1 2 3	1 2 3	1 2 3
1 4 5	1 4 5	1 4 5
1 6 9	1 6 9	1 6 9
2 4 7	2 4 8	2 4 9
2 5 9	2 5 7	2 5 7
3 4 8	3 4 9	3 4 8
3 6 7	3 6 7	3 6 7
5 6 8	5 6 8	5 6 8

None of the designs can be completed.

Thus we have

LEMMA 4.6. *Case* (1) ($n_4 = 0$, $n_3 = 8$, $n_2 = 12$, $n_1 = 6$, $n_0 = 0$) *is impossible.*

Then, combining Theorem 2.3 and Lemmas 4.1 - 4.6, we have

THEOREM 4.7. *No (2,4) packing on 19 varieties contains more than 25 blocks.*

5. *Conclusion.*

We have the following packing with 25 blocks, which was obtained from one of the partial designs of case (2) - (1):

1 2 3 4	2 6 11 15	6 7 13 16
1 5 6 10	2 8 12 16	6 9 14 17
2 5 7 17	3 5 13 19	7 8 10 19
3 6 8 18	3 7 14 15	1 17 18 19
4 7 9 18	3 9 10 12	2 10 14 18
5 8 9 11	4 5 14 16	3 11 16 17
1 7 11 12	4 6 12 19	4 10 11 13
1 8 13 14	4 8 15 17	5 12 15 18
1 9 15 16		

This packing, with Theorem 4.7, gives

THEOREM 5.1. *The maximum number of blocks in a (2,4) packing on 19 varieties is 25, and thus the packing number* $D(2, 4, 19) = 25$.

REFERENCES

[1] A.E. Brouwer, *Optimal Packings of* K_4*'s into a* K_n. *- The Case* $n \not\equiv 2 \pmod 3$, Math. Centr. report ZW 76(1976).

[2] ______________, *A Group-Divisible Design* $GD(4,1,2;n)$ *Exists iff* $n \equiv 2 \pmod 6$, $n \neq 8$ *(or: the packing of cocktail party graphs with* K_4*'s)*, Math. Cent. report ZW 64(1976).

[3] H. Hanani, *The Existence and Construction of Balanced Incomplete Block Designs*, Ann. Math. Stat. 32(1961), 381-386.

[4] W.H. Mills, *On the Covering of Pairs by Quadruples II*, J. Comb. Theory A. 15(1973), 138-166.

[5] R.C. Mullin, *Packings of (6,4)-Vectors in Spaces of Dimension 3s and 3s + 1*, Lecture Notes, Second MANIWAT Summer Research Workshop (1976).

University of Waterloo
Waterloo, Ontario.