SOME LARGE CRITICAL SETS

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1. Introduction

A partial transversal design (of order n) is a triple $(X, \mathcal{G}, \mathcal{A})$, where $|X| = 3n$ (elements of X are called points) $\mathcal{G} = \{G_1, G_2, G_3\}$ is a partition of X into three groups of size n, and $\mathcal{A}$ is a set of subsets of X (called blocks) such that (1) a block and a group contain precisely one common point, (2) no pair of points is contained in more than one block. $|\mathcal{A}|$ is the size of the partial transversal design. Clearly $|\mathcal{A}| \leq n^2$; if $|\mathcal{A}| = n^2$ we say that $(X, \mathcal{G}, \mathcal{A})$ is a transversal design of order n. In a transversal design, every pair of points not in the same group occurs in a unique block. We will abbreviate partial transversal design to PTD and transversal design to TD.

If $T_1 = (X_1, \mathcal{G}_1, \mathcal{A}_1)$ and $T_2 = (X_2, \mathcal{G}_2, \mathcal{A}_2)$ are PTDs, we say that $T_1 \subseteq T_2$ provided $X_1 = X_2$, $\mathcal{G}_1 = \mathcal{G}_2$, and $\mathcal{A}_1 \subseteq \mathcal{A}_2$. A PTD T is said to be completable if there is a TD T' with $T \subseteq T'$. (We say that T completes to T'). If $T = (X, \mathcal{G}, \mathcal{A})$ and $T' = (X, \mathcal{G}, \mathcal{A}')$ are PTDs, we define $T \cap T' = (X, \mathcal{G}, \mathcal{A} \cap \mathcal{A}')$. If $T \subseteq T'$, define $T' - T = (X, \mathcal{G}, \mathcal{A}' \backslash \mathcal{A})$ If $A \in \mathcal{A}'$, define $T - A = (X, \mathcal{G}, \mathcal{A}' \backslash \{A\})$. Finally, if $T \subseteq T'$ and $A \in \mathcal{A}' - \mathcal{A}$, define $T + A = (X, \mathcal{G}, \mathcal{A} \cup \{A\})$.

We now define a closure operation: for a PTD T, let $cl(T) = \bigcap_{T \subseteq T'} T'$. Thus, if T completes to T_1, then $cl(T) \subseteq T_1$ and $cl(T)$ is the smallest PTD with this property.

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We now define critical sets. A PTD, T, is said to be uniquely completable (or: T has (UC)) if cl(T) is a TD. T is said to be essential provided that, for all $T' \subsetneq T$, $cl(T') \subsetneq cl(T)$. (That is, if $T = (X, G, A)$, then for all $A \in A$, $T - A$ does not have (UC)). T is a critical set if it is both uniquely completable and essential. Thus a critical set is a PTD which can be completed to precisely one TD, and is minimal with respect to this property.

It is well known that a TD of order n is equivalent to a Latin square of order n. We will use the terminology interchangably, in a particular situation using that which seems easiest. We will pictorially represent critical sets as being subsets of Latin squares. However, we find that most proofs and definitions are more easily presented in terms of TDs.

Critical sets were first investigated by Curran and van Rees [1]. They introduced the functions lcs(n) and scs(n), which denote, respectively, the cardinality of the largest and smallest critical sets in any TD of order n. Their main result is the following.

Theorem 1.1 $scs(n) \leq n^2/4$, and if n is even, there exists a critical set of size $n^2/4$.

The above theorem seems to be quite good, inasmuch as we are unable to improve it, even for a single value of n. We ask if

$$\lim_{n\to\infty} \frac{scs(n)}{n^2} = 1/4 .$$

In this paper, we investigate large critical sets. For C a critical set in a TD of order n, define $\delta(C) = |C|/n^2$. (We say that $\delta(C)$ is the density of C). Also define $\delta(n) = \max\{\delta(C) : C$ is a critical set in a TD of order $n\}$. For small orders at least, it seems difficult to find critical sets of density substantially exceeding 1/2. However, using recursive techniques, we show that there exist critical sets of density arbitrarily close to 1. In particular, $\Delta(2^k) \geq 1-(3/4)^k$ for all positive integers k. Thus $\limsup_{n\to\infty} \Delta(n) = 1$; we conjecture that $\lim_{n\to\infty} (n) = 1$.

Also, using a variety of predominantly ad hoc constructions, we produce a list of lower bounds of $lcs(n)$, for several small values of n.

2. A Doubling Construction

In this section, we shall describe a doubling construction for certain critical sets. First, we establish some preliminary lemmata concerning critical sets in general.

Lemma 2.1 Let $E = (X, \mathcal{G}, \mathcal{A})$ and $D = (X, \mathcal{G}, \mathcal{A}')$ be PTDs with $E \subseteq D$. Suppose that (1) D has (UC) and (2) for all $A \in \mathcal{A}$, $D - A$ does not have (UC). Then there is a critical set C with $E \subseteq C \subseteq D$.

Proof: By induction on $|\mathcal{A}'| - |\mathcal{A}|$. If $|\mathcal{A}| = |\mathcal{A}'|$, then D is a critical set by definition, so assume $|\mathcal{A}'| - |\mathcal{A}| > 0$.

If for all $A \in \mathcal{A}' \setminus \mathcal{A}$, $D - A$ does not have (UC), then D is critical. If not, then there is a $A \in \mathcal{A}'$ such that $D - A$ has (UC). Delete A from D. Then $|\mathcal{A}'| - |\mathcal{A}|$ is decreased by 1 and induction can be applied.□

Corollary 2.2. If D has (UC), then there is $C \subseteq D$ which is a critical set.

Proof. Let E be the PTD which has no blocks, and apply Lemma 2.1. □

Note that Lemma 2.1 does not state that every essential set is contained in a critical set. In fact we shall later construct a counterexample.

A sub-TD of a TD $T = (X,\mathcal{G},\mathcal{A})$ is a TD $(X_1,\mathcal{G}_1,\mathcal{A}_1)$ where $X_1 \subseteq X$, $\mathcal{A}_1 \subseteq \mathcal{A}$, and $\mathcal{G}_1 = \{G \cap X_1 : G \in \mathcal{G}\}$. A sub-TD corresponds precisely to a subsquare of a Latin square. The following is immediate.

Lemma 2.3. Suppose C completes uniquely to a TD T. Then for any sub-TD U of T, $C \cap U$ completes uniquely to U.

We also have the following obvious, but very useful criterion for showing that at PTD is essential.

Lemma 2.4. Suppose $E = (X,\mathcal{G},\mathcal{A})$ can be completed to T. Also, suppose that, for every $A \in \mathcal{A}$, there is a sub-TD $U = (X_1,\mathcal{G}_1,\mathcal{A}_1)$ of T where $U \neq T$, such that $E \cap U$ is essential in U. Then E is essential.

Let $n_1,\ldots n_\ell$ be distinct positive integers. We say that a PTD, E, is $n_1,\ldots,n_\ell$-essential if it has sub-TD's U, as described in lemma 2.4, with orders chosen from $n_1,\ldots,n_\ell$. We say that E is $n_1,\ldots,n_\ell$-critical if it is both $n_1,\ldots,n_\ell$-essential and has (UC).

Lemma 2.5. Suppose E = (X,G,A) is an essential PTD, E ⊆ T (T = (X,G,A') is a TD) and U = (X_1, G_1, A_1) is a sub-TD of T of order 2 disjoint from E. Further suppose that the following property holds:
(*) for each A ∈ A_1, there is an f(A) ∈ A such that f(A) is a block of cl(E + A − f(A)). Then there is no critical set C (of T) containing E.

Proof. Suppose C is a critical set of T containing E. By Lemma 2.3, C must contain a block A of U. Then f(A) is a block of cl(E + A − C(A)) ⊆ cl(C − f(A). But cl(C) = T; hence cl(C − f(A)) = T and C is not essential. Thus C cannot be critical.

Example 2.6. An essential set E (of a Latin square L) contained in no critical set of L.

	2	3	4	5	6	7	8
		4	3	6	7	8	5
			6	7	8	5	2
				8	5	2	7
	6	7	8		2	3	4
		8	5			4	3
			2				6

E

1	2	3	4	5	6	7	8
2	1	4	3	6	7	8	5
3	4	1	6	7	8	5	2
4	3	6	1	8	5	2	7
5	6	7	8	1	2	3	4
6	7	8	5	2	1	4	3
7	8	5	2	3	4	1	6
8	5	2	7	4	3	6	1

L

Proof. It can be checked that E is 2,3-essential. T corresponds to the indicated 2-by-2 subsquare M, and for each cell C of M, f(C) is the cell of L obtained by reflection in the main diogonal. Then property (*) of lemma 2.5 is satisfied, and the result follows. □

We now define a doubling operation. This is most easily described in terms of Latin squares. Suppose L is a Latin square of order n. The double of L (denoted by 2xL) will be the Latin square

L_1	L_2
L_2	L_1

where L_i $(i = 1,2)$ is a copy of L with every symbol replaced by x_i.

Now suppose C is a critical set of L. We define two partial Latin squares 2 o C and 2 * C, both of which complete to 2 x L. Let

2 * C =

L_1	C_2
C_2	C_1

and let

2 o C =

$(L-C)_1$	C_2
C_2	C_1

with notation as before.

Lemma 2.7. Let C be a critical set in a Latin square L. Then there is a critical set A in 2 x L, with $2 \circ C \subseteq A \subseteq 2 * C$.

Proof. We will apply lemma 2.1. First we show that $2 * C$ has (UC). Let K be any completion of $2 * C$. $2 * C$ has the subsquare L_1 of order n in the top left-hand corner, so $K = \begin{array}{|c|c|} \hline L_1 & M_2 \\ \hline N_2 & O_1 \\ \hline \end{array}$,

where M, N, and O are Latin squares of order n. But $C \subseteq M$ and C has (UC); thus $M = L$. Similarly $N = O = L$, and $K = 2 \times L$.

Now, we show that deleting any cell of $2 \circ C$ from $2 * C$ yields a partial Latin square which does not have (UC). Any cell of $2 \circ C$ is either in a set C_i which is critical in a subsquare L_i, or in a two-by-two subsquare of the form $\begin{array}{|c|c|} \hline X_1 & X_2 \\ \hline X_2 & X_1 \\ \hline \end{array}$ where X occurs in a cell of $L - C$. Lemma 2.1 implies the result. □

Under certain circumstances we can show that $2 * C$ is a critical set.

Theorem 2.8. Let C be 2-critical in a Latin square L. Then $2 * C$ is 2-critical in $2 \times L$.

Proof. In view of Lemma 2.7 and its proof, it suffices to show that removing a cell of C_1 (with contents X_1, say) from the top left-hand corner of $2 \times L$, yields a partial Latin square that does not have (UC). In L, there is a two-by-two subsquare, of the form $\begin{array}{|c|c|} \hline x & y \\ \hline y & x \\ \hline \end{array}$

which intersects C in the given cell X. Then, in $2 \times L$

x_1	y_2
y_2	x_1

is a similar such subsquare. □

Define $lcs_2(n)$ to be the largest 2-critical set in a Latin square of order n. Also let $\Delta_2(n) = lcs_2(n)/n^2$.

The following is easily proved by induction.

Corollary 2.10 For all integers $\ell \geq 0$, and all positive integers n, $\Delta_2(2^\ell n) \geq 1-(\frac{3}{4})^\ell (1-\Delta_2(n))$.

Curran and van Rees [1] have shown

Lemma 2.11 For all positive integers K, $\Delta_2(2K) \geq \frac{1}{4}$.

Corollary 2.12 If $n \equiv 0$ modulo 2^ℓ, then $\Delta_2(n) \geq 1-(\frac{3}{4})^\ell$.

Corollary 2.13 $\limsup_{n \to \infty} \Delta(n)=1$.

3. Examples

In this section we construct critical sets in some Latin squares of low orders, and give a list of lower bounds for $lcs(n)$.

Lemma 3.1 $lcs(2) = lcs_2(2) = 1$; and $lcs_2(3) = 0$, $lcs(3) = 3$.

Proof. See [1]. □

Lemma 3.2 $lcs_2(4) \geq 7$, $lcs_2(8) \geq 37$, and $lcs(16) \geq 175$.

Proof. Corollary 2.12. □

Lemma 3.3 $lcs(5) \geq 10$.

Proof: It can be verified that C is critical in L. (see the figure following). □

5				
4	5			
3	4	5		
2	3	4	5	

C

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

L

Lemma 3.4 $lcs_2(6) \geq 15$; $lcs(6) \geq 18$.

Proof: Consider C_1, L_1, C_2, and L_2.

	2		4	5	6
			6	3	5
3				6	2
		5		2	
					4
			3		

C_1

1	2	3	4	5	6
2	1	4	6	3	5
3	4	1	5	6	2
4	6	5	1	2	3
5	3	6	2	1	4
6	5	2	3	4	1

L_1

Note that L_1 is symmetric, with constant main diagonal (i.e. unipotent). No pair of cells of L_1, symmetric with respect to the diagonal, are both in C_1. Thus C_1 is 2-essential. It is also easy to check that C_1 has (UC); thus C_1 is 2-critical in L_1.

3			6		
2	3		5	6	
			1	3	2
5			2	1	3
6	5		3	2	1

C_2

1	2	3	4	5	6
3	1	2	6	4	5
2	3	1	5	6	4
4	6	5	1	3	2
5	4	6	2	1	3
6	5	4	3	2	1

L_2

It can also be verified that C_2 is 2,3-critical in L_2. □

<u>Lemma 3.5</u> $lcs_2(7) \geq 19$; $lcs(7) \geq 24$.

<u>Proof.</u> C_1 is 2-critical in L_1, and C_2 is critical in L_2.

	3	6		5		
3		4	0			
			5	1		0
1				6	2	
	2				0	
4		3				1
	5		4			

C_1

0	3	6	1	5	4	2
3	1	4	0	2	6	5
6	4	2	5	1	3	0
1	0	5	3	6	2	4
5	2	1	6	4	0	3
4	6	3	2	0	5	1
2	5	0	4	3	1	6

L_1

1	0		6		4	2
0	2					1
		3	0			4
6		0	4		1	3
4			1		6	
2	1	4	3			0

C_2

1	0	5	6	3	4	2
0	2	6	5	4	3	1
5	6	3	0	1	2	4
6	5	0	4	2	1	3
3	4	1	2	5	0	6
4	3	2	1	0	6	5
2	1	4	3	6	5	0

L_2

□

Lemma 3.6   lcs(9) ≥ 39 .   C is 3-critical in L.

1		3		5	6	7	8	
	3	1		6	4	8		
3	1	2	6	4	5			
		6	7	8		1	2	
5	6	4	8			2		
6	4	5						
7	8		1	2		4	5	
8			2			5		

C

1	2	3	4	5	6	7	8	9
2	3	1	5	6	4	8	9	7
3	1	2	6	4	5	9	7	8
4	5	6	7	8	9	1	2	3
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
9	7	8	3	1	2	6	4	5

L

□

Lemma 3.7 $lcs_2(10) \geq 45$, $lcs(10) \geq 55$

Proof: C_1 is 2-critical in L_1 (similar to the example of order 6 in Lemma 3.4), and C_2 is 2;5-critical in L_2.

	2	3	4		6				
		4	8	10		6			
			2	8	5		7		
				9	7	3		10	
5					2	4	3		6
	9					8	10	4	
7		10					9	5	2
8	5	6						2	4
9	3	6		7					8
10	7	9	5		3				

C_1

1	2	3	4	5	6	7	8	9	10
2	1	4	8	10	9	6	5	3	7
3	4	1	2	8	5	10	7	6	9
4	8	2	1	9	7	3	6	10	5
5	10	8	9	1	2	4	3	7	6
6	9	5	7	2	1	8	10	4	3
7	6	10	3	4	8	1	9	5	2
8	5	7	6	3	10	9	1	2	4
9	3	6	10	7	4	5	2	1	8
10	7	9	5	6	3	2	4	8	1

L_1

5					0				
4	5				9	0			
3	4	5			8	9	0		
2	3	4	5		7	8	9	0	
					1	5	4	3	2
7					2	1	5	4	3
8	7				3	2	1	5	4
9	8	7			4	3	2	1	5
0	9	8	7		5	4	3	2	1

C_2

1	2	3	4	5	6	7	8	9	0
5	1	2	3	4	0	6	7	8	9
4	5	1	2	3	9	0	6	7	8
3	4	5	1	2	8	9	0	6	7
2	3	4	5	1	7	8	9	0	6
6	0	9	8	7	1	5	4	3	2
7	6	0	9	8	2	1	5	4	3
8	7	6	0	9	3	2	1	5	4
9	8	7	6	0	4	3	2	1	5
0	9	8	7	6	5	4	3	2	1

L_2

□

<u>Lemma 3.8</u> $\mathrm{lcs}_2(12) \geq 81$, $\mathrm{lcs}_2(14) \geq 106$, $\mathrm{lcs}_2(20) \geq 235$, $\mathrm{lcs}_2(24) \geq 387$, and $\mathrm{lcs}_2(28) \geq 514$.

<u>Proof</u>: Corollary 2.9. □

We conclude with a list of lower bounds.

Table 1

lower bounds for $lcs(n)$ and $lcs_2(n)$, $n \le 10$

n	lcs(n)	$lcs_2(n)$
2	1	1
3	2	0
4	7	7
5	10	0
6	18	15
7	24	19
8	37	37
9	39	0
10	55	45

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Reference

[1] D. Curran and G.H.J. van Rees, Critical sets in Latin squares, Proc. Eighth Manitoba Conference on Numerical Mathematics and Computing, 1978, 165-168.