A Generalization of Howell Designs

D. R. Stinson

University of Waterloo

1. Introduction

A Howell design of *side* s and *order* $2n$, or, more briefly, an $H(s,2n)$, is an s by s array H, in which each cell either is empty or contains an unordered pair of elements (called *symbols*), chosen from some set S of size $2n$, which satisfies:

(1) every symbol occurs in exactly one cell of each row and column of H (i.e. each row and column is *Latin*).

(2) no unordered pair of symbols occurs in more than one cell of H. The spectrum of Howell designs has recently been determined.

Theorem 1.1. An $H(s,2n)$ exists if and only if $n \leq s \leq 2n-1$ and $(s,2n) \neq (2,4)$, $(3,4)$, $(5,6)$ or $(5,8)$.

Proof. For s odd the result was established by Stinson [3]; for s even, by Anderson, Schellenberg, and Stinson [1]. □

Property (2) may be rephrased as "every unordered pair of symbols occurs in either zero or one cell of H". This suggests the following more general definition for Howell designs: replace property (2) by

(2') every unordered pair of symbols occurs in either λ or $\lambda+1$ cells of H, for some non-negative integer λ.

We refer to such an array as a Howell design (of *index* λ). We shall see that λ is determined by the values of s and $2n$. If we wish to emphasize the value of λ we will use the notation $H(s,2n;\lambda)$.

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A symbol occurs s times in an $H(s,2n)$, and it occurs with every other symbol either λ or $\lambda+1$ times. Thus we obtain

$$\lambda(2n-1) \leq s \leq (\lambda+1)(2n-1).$$

If $\lambda = 0$ we have the additional constraint $n \leq s$, since at most $2s$ symbols can occur in a row of H.

In the boundary cases $s = \lambda(2n-1)$, every pair occurs exactly λ times. Such a design may be denoted either $H(s,2n;\lambda-1)$ or $H(s,2n;\lambda)$.

In this paper we establish precisely for which ordered pairs $(s,2n)$ a Howell design $(s,2n)$ exists (of the appropriate index λ).

2. Constructions

Our main recursive construction is a simple "direct sum" construction. Let H_i, for $i = 1,2$, be an $H(s_i,2n;\lambda_i)$ on symbol set $I_{2n} = \{1,\ldots,2n\}$. The direct sum $H = H_1 \oplus H_2$ (of H_1 and H_2) will denote the array

H_1	
	H_2

Under certain circumstances $H_1 \oplus H_2$ will be a Howell design. It is clear that this array is Latin, and that every pair of sumbols occurs in either $\lambda_1+\lambda_2$, $\lambda_1+\lambda_2+1$, or $\lambda_1+\lambda_2+2$ cells.

For any $H(s,2n;\lambda)$ on symbol set I_{2n}, let $G = G(H,\lambda)$ be the graph defined on vertex set I_{2n}, by joining two vertices i and j by an edge if and only if $\{i,j\}$ occurs λ times in H. (We say that G is the λ-graph of H). Clearly, $G(H, \lambda+1)$ is the complement of $G(H, \lambda)$.

There are two ways in which $H_1 \oplus H_2$ can be a Howell design: no pairs occur $\lambda_1+\lambda_2$ times, or no pairs occur $\lambda_1+\lambda_2+2$ times. The λ- and $(\lambda+1)$-graphs of H_1 and H_2 determine when these situations can arise. We have the following obvious result.

Lemma 2.1. For $i = 1,2$, let H_i be an $H(s_i, 2n; \lambda_i)$.

(1) If $G(H_1, \lambda_1)$ and $G(H_2, \lambda_2)$ contain no common edge, then $H_1 \oplus H_2$ is an $H(s_1 + s_2, 2n; \lambda_1+\lambda_2+ 1)$.

(2) If $G(H_1, \lambda_1+1)$ and $G(H_2, \lambda_2+1)$ contain no common edge, then $H_1 \oplus H_2$ is an $H(s_1+s_2, 2n; \lambda_1+\lambda_2)$.

Corollary 2.2. If H_1 is an $H(t(2n-1), 2n)$ for some $t \geq 1$, and H_2 is an $H(s, 2n; \lambda)$ then $H_1 \oplus H_2$ is an $H(s+t(2n-1), 2n; \lambda+t)$.

Proof. H_1 is an $H(t(2n-1), 2n; t)$, and $G(H_1, t+1)$ contains no edges. □

Lemma 2.3. If there exists an $H(t(2n-1), 2n)$ for some $t \geq 1$, and an $H(s, 2n)$, then there exists an $H(s+tj(2n-1), 2n)$ for all $j \geq 0$.

Proof. Take the direct sum of an $H(s, 2n)$ and j copies of an $H(t(2n-1), 2n)$. □

Our second recursive construction uses the idea of "projections". Let H be an $H(s, 2n; \lambda)$. A transversal of H is a set T of n cells of H, no two in the same row or column, such that

(1) every symbol occurs in exactly one cell of T, and

(2) a pair of symbols in any cell of T occurs exactly λ times in H.

We project T as follows. Index the rows and columns of H by I_s, and then construct H_1, with rows and columns indexed by I_{s+1}, by defining

$$H_1(i,j) = \begin{cases} H(i,j) & \text{if } (i,j) \notin T \\ H(k,j) & \text{if } i = s+1 \text{ and } (k,j) \in T \\ H(i,k) & \text{if } j = s+1 \text{ and } (i,k) \in T \\ \text{empty, otherwise} & \end{cases}$$

Lemma 2.4. If H is an $H(s,2n;\lambda)$ and T is a transversal, then H_1, described above, is an $H(s+1,2n;\lambda)$.

Proof. The properties of T are precisely those that ensure that H_1 will be a Howell design. □

Example 2.5. An $H(6,6;1)$ is shown in Figure 1 below. The cells containing {4,6}, {2,5}, and {1,3} form a transversal T. The $H(7,6;1)$ obtained by projecting T is shown in Figure 2.

Figure 1. An H(6,6;1).

12	34	56			
			12	34	56
35	16	24			
46				15	23
	25		36		14
		13	45	26	

Figure 2. An H(7,6;1)

12	34	56				
			12	34	56	
35	16	24				
				15	23	46
			36		14	25
			45	26		13
46	25	13				

Two transversals T and T' of an $H(s,2n;\lambda)$ are said to be <u>disjoint</u> provided there does not exist a cell C, of T, and a cell C', of T', such that C and C' contain the same pair (in particular, $C \neq C'$). Several transversals are disjoint provided each pair is.

<u>Lemma 2.6</u>. If there exists an $H(s,2n;\lambda)$ containing t disjoint transversals, then there exists an $H(s+i,2n;\lambda)$ for $0 \le i \le t$.

<u>Proof</u>. The t transversals may be projected one by one. □

3. <u>The spectrum</u>

<u>Lemma 3.1</u>. An $H(s,2)$ exists for all $s \ge 1$.

<u>Proof</u>.

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is an $H(1,2)$. Apply Lemma 2.3 with $s = t = n = 1$. □

<u>Lemma 3.2</u>. An $H(s,4)$ exists if and only if $s \ge 6$.

<u>Proof</u>. There are only four symbols, say {1,2,3,4}, so if {1,2}, say, occurs in a cell of some $H(s,4)$, then {3,4} occurs in both the row and the column containing {1,2}. It follows that every pair occurs either not at all or at least twice; thus $s \ge 6$.

An $H(6,4)$ is shown in Figure 3. Replace

12	34
34	12

by

12	34	
34		12
	12	34

to construct an $H(7,4)$. A similar operation on the blocks

13	24
24	13

, and then

14	23
23	14

produces $H(8,4)$ and $H(9,4)$. The $H(9,4;3)$ thus produced has two disjoint transversals (it has three, but we only need two of them). Thus $H(10,4)$

and H(11,4) can be produced, so we have H(s,4) for $6 \le s \le 11$.

Now apply Lemma 2.3 with $t = 2$, $n = 2$ for each s, $6 \le s \le 11$, to obtain all H(s,4) with $s \ge 6$. □

Figure 3. An H(6,4)

12	34				
34	12				
		13	24		
		24	13		
				14	23
				23	14

Lemma 3.3. An H(s,6) exists if and only if $s \ge 3$, $s \ne 5$.

Proof. H(3,6) and H(4,6) exist, and H(5,6) does not exist, by Theorem 1.1. An H(6,6) is given in Example 2.5. This H(6,6) has four disjoint transversals, formed by the cells containing {4,6}, {2,5} and {1,3}; {3,5}, {1,4}, and {2,6}; {1,6}, {2,3}, and {4,5}; and {2,4}, {1,5}, and {3,6}. Thus we may construct H(s, 6) for $6 \le s \le 10$.

We need three more small H(s,6): H(11,6), H(12,6), and H(15,6). We construct these by direct sum. An H(3,6) is given by

14	25	36
26	34	15
35	16	24

and its 0-graph is two disjoint triangles.

Any H(4,6) has an 0-graph which consists of three disjoint edges (i.e. a 1-factor of K_6). It is easily seen that K_6 may be partitioned into two triangles and three 1-factors. Applying the direct sum construction (and suitably relabelling designs), we construct H(s,6) for s = 11, 12, and 15 (note: 11 = 3+4+4, 12 = 4+4+4, and 15 = 3+4+4+4).

Now apply Lemma 2.3 with $t = 2$, $n = 3$, and $s = 3,4,6,7,8,9,10,11,12$ and 15, to construct the desired $H(s,6)$. □

In order to show the existence of $H(s,2n)$ with $n \geq 4$, we make essential use of Room cubes. A <u>Room cube</u> of side s is a three-dimensional array of side s, each cell of which is either empty or contains an unordered pair of symbols, such that each two-dimensional projection is an $H(s,s+1)$. The following is established in Dinitz and Stinson [2].

<u>Lemma 3.4</u>. There exists a Room cube of side s if and only if s is an odd positive integer other than 3 or 5.

Room cubes are of use in constructing Howell designs, as we now demonstrate.

<u>Lemma 3.5</u>. There exists a Room cube of side s if and only if there exists an $H(s,s+1)$ containing s disjoint transversals.

<u>Proof</u>. Take a two-dimensional projection of a Room cube of side s, to obtain an $H(s,s+1;1)$. The filled cells in any "level" of the Room cube become a transversal of the resulting $H(s,s+1)$, and the s transversals resulting from the s levels of the Room cube are disjoint. The process can be reversed. □

<u>Lemma 3.6</u>. Let $n \geq 4$. Then there exists an $H(s,2n)$ if and only if $s \geq n$, $(s,2n) \neq (5,8)$.

<u>Proof</u>. For $s \leq 2n-1$, the result is obtained from Theorem 1.1, so assume $s \geq 2n-1$. By lemmata 3.4 and 3.5, we have an $H(2n-1,2n)$ with $2n-1$ disjoint transversals. Using lemma 2.6, we can construct $H(s,2n)$ for $2n-1 \leq s \leq 4n-2$. Now apply lemma 2.3 with $t = 1$ and $2n-1 \leq s \leq 4n-2$, to construct the remaining designs. □

Combining lemmata 3.1, 3.2, 3.3, and 3.6, we have our main result.

Theorem 3.7. Let $s \geq n$. Then an $H(s,2n)$ exists if and only if $(s,2n) \neq (2,4), (3,4), (4,4), (5,4), (5,6)$ or $(5,8)$.

References

[1] B.A. Anderson, P.J. Schellenberg and D.R. Stinson, The existence of Howell designs of even side, Journal of Combinatorial Theory, Series A, submitted.

[2] J.H. Dinitz and D.R. Stinson, The spectrum of Room cubes, European Journal of Combinatorics,(to appear).

[3] D.R. Stinson, The existence of Howell designs of odd side, Journal of Combinatorial Theory, Series A, (to appear).