A Perfect One-factorization for $K_{40}$

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Abstract

In this paper, we construct an even starter-induced perfect one-factorization for the complete graph on 40 vertices.

1. Introduction

A one-factorization of a complete graph $K_{2n}$ is a partition of the edge-set of $K_{2n}$ into $2n - 1$ one-factors, each of which contains $n$ edges that partition the vertex set of $K_{2n}$. A perfect one-factorization (P1F) is a one-factorization in which every pair of distinct one-factors forms a Hamiltonian cycle of the graph.

It has been known for some time that P1Fs on $K_{2n}$ exist when $n$ is prime, when $2n - 1$ is prime, and when $2n \in \{16, 28, 244, 344\}$ (see [1]). In fact, these were the only known examples until recently. A P1F of $K_{36}$ was found by the authors [8]; a non-isomorphic example was also constructed by Kobayashi, Awoki, Nakazaki and Nakamura [5]. A P1F of $K_{40}$ was found by the authors and E. Ihrig [4]. P1Fs of $K_{1332}$ and $K_{4860}$ were found by Kobayashi and Kiyasu-Zen'iti [6]. Also, Dinitz and Stinson used quotient starters to construct seven new P1Fs, of $K_{126}$, $K_{170}$, $K_{170}$, $K_{1370}$, $K_{1830}$, $K_{2198}$, and $K_{3126}$ [3].

In this paper, we use a combination of hill-climbing and backtracking algorithms to generate even starter-induced one-factorizations, and discover a P1F for $K_{40}$. The smallest unknown case is now $K_{52}$.

1 Research supported by NSERC grant OGP0036870.
2 Research supported by NSERC grant A9287.

CONGRESSUS NUMERANTIIUM 68(1989), pp.211-214
2. Even starters

An even starter in $\mathbb{Z}_{2n}$ is a set of $(n - 1)$ pairs of elements $E = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})\}$ such that

1. every non-zero element of $\mathbb{Z}_{2n}$ except one, denoted $m$, occurs as an element in some pair of $E$, and
2. every non-zero element of $\mathbb{Z}_{2n}$ except $n$ occurs as a difference of some pair of $E$.

Define $E^* = E \cup \{(0, \infty_1)\} \cup \{(m, \infty_2)\}$, and $g + \infty_i = \infty_i + g = \infty_i$ for $g \in \mathbb{Z}_{2n}$ and $i = 1, 2$. Also define $Q^* = \{(g, g+n) : g \in \mathbb{Z}_{2n}\} \cup \{(\infty_1, \infty_2)\}$. Then $F = \{E^* + g : g \in \mathbb{Z}_{2n}\} \cup \{Q^*\}$ is a one-factorization of $K_{2n+2}$.

Most known examples of PIF arise from starters (for a definition, see [2] or [8]) or even starters. In [1], Anderson enumerates all PIFs in $K_{2n}$ arising from starters and even starters, for $2n \leq 22$. These empirical results suggest that there exist both a starter-induced PIF and an even starter-induced PIF in $K_{2n}$ for all $2n \geq 12$. Hence, it seems likely that starters or even starters might provide new examples of PIF for larger values of $2n$.

We use a modification of the hill-climbing algorithm in [2] to generate in a random manner the first six pairs of elements (having differences one through six) of an even starter. For each such partial even starter, we then use a backtracking algorithm to generate all (complete) even starters extending the given partial even starter, and test the induced one-factorizations for perfection. For those who are interested in hill-climbing algorithms, we suggest [7].

3. A PIF for $K_{40}$

We implemented the algorithms above using Pascal VS on the University of Manitoba Amdahl 580 computer. After 150 hours of CPU time, we found the following even starter in $\mathbb{Z}_{38}$ which induces a PIF of $K_{40}$:

$$
\{(1, 2), (3, 5), (29, 32), (23, 27), (14, 19), (7, 13), (30, 37),
(28, 36), (8, 17), (15, 25), (20, 31), (4, 16), (9, 22), (21, 35),
(11, 26), (18, 24), (33, 12), (6, 24)\}.
$$

We note that the omitted element $m = 10$. The automorphism group of the induced PIF is $Z_38$. 

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References


