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NEAR-SELF-COMPLEMENTARY DESIGNS AND A METHOD OF MIXED SUMS

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ABSTRACT. An important class of BIBDs is that of (strongly) self-complementary-designs, designs which are invariant under complementation. Their parameter sets satisfy the relation $v = 2k$, yet for k odd, there are an infinity of parameter sets which cannot be realized as self-complementary designs. For these parameters the idea of near-self-complementary designs is introduced. These designs have many aspects similar to self-complementary designs.

An extension of Bose's method of mixed differences is introduced and is applied to show the residuality of certain near-self-complementary designs.

1. INTRODUCTION.

A *balanced incomplete block design* BIBD (v,b,r,k,λ) is a pair (V,F) where V is a v-set of objects called varieties, F is a family of k-subsets of V, b in number, which has the property that each variety occurs in precisely r of these subsets and each pair of distinct varieties occurs in precisely λ of these subsets. An important subclass of these designs is the *self-complementary or strongly self-complementary* designs, which are invariant under complementation. (Some authors refer to self-complementary designs as those isomorphic to their complements, for this reason, the option "strongly self-complementary design" is given as an alternative; for sake of simplicity, we use self-complementary or SCD). An SCD (v,b,r,k,λ) is *simple* if $(b,r,\lambda) = 1$. Simple self-complementary designs enjoy the following properties.

(i) Simple self-complementary designs are affine resolvable, that is, any block meets all blocks except itself and its complement in precisely $k/2$ varieties [5].

(ii) Simple self-complementary designs are residual designs (cf. [3]).

(iii) Simple self-complementary designs are 3-designs, that is, every triple of varieties occurs in λ_3 blocks, where λ_3 is independent of the triple chosen. (This property is valid for any SCD design).

(iv) There exists an SCD $(4t+4,\ 8t+6,\ 4t+3,\ 2t+2,\ 2t+1)$ if and only if there exists an Hadamard matrix H_{4t+4} [4].

Clearly a necessary condition for the existence of an SCD (v,b,r,k,λ) is that $v = 2k$. This yields parameters of the form $(2x+2,\ t(4x+2),\ t(2x+1),\ x+1,\ tx)$.

However, if k is odd and $t = (b,r,\lambda)$ is also odd, then it is known [5] that no SCD exists. Because of this deficiency, the following definition is given. A BIBD D is *near-self-complementary* (NSC) if there exists an involutory mapping ϕ defined on the blocks of D such that (i) $|B \cap \phi B| = 1$ and (ii) $|B \cup \phi B| = v-1$, for all blocks B of D. ϕB is the *near-complement* of B. An NSC (v,b,r,k,λ) is simple if $(b,r,\lambda) = 1$.

It is evident from the definition that in any NSC (v,b,r,k,λ) the relation $v = 2k$ holds, hence the set of parameters again has the form $(2x+2, t(4x+2), t(2x+1), x+1, tx)$. Since $(t(4x+2, t(2x+1), tx) = t$, for simple NSC designs the parameters have the form $(2x+2, 4x+2, 2x+1, x+1, x)$. Not surprisingly, the designs have different properties in the cases of x even and x odd. In either case the designs are quasi-residual [3], and some aspects of residuality are discussed in a later section.

2. PROPERTIES OF SIMPLE NSC DESIGNS WITH ODD BLOCK SIZE

The most interesting of the cases is that in which x is even, or k is odd, since no SCD can exist in this case. Letting $x = 2s$, the parameters become $(4s+2, 8s+2, 4s+1, 2s+1, 2s)$.

THEOREM 2.1. In an NSC $(4s+2, 8s+2, 4s+1, 2s+1, 2s)$ any block meets any block other than itself or its near-complement in either s or $s+1$ elements.

PROOF. The result follows from a standard argument involving intersection numbers. (See, for example [5]). □

A variety of an NSC D is said to be an *infinite element* if for all $B \in D$, ∞ is in $B \cup \phi B$. Since there are r pairs $\{B,\phi B\}$, each infinite element occurs in precisely one of $\{B,\phi B\}$.

THEOREM 2.2. In an NSC $(4s+2, 8s+2, 4s+1, 2s+1, 2s)$ there are either one or two infinite elements.

PROOF. Since there are $4s+2$ varieties and only $4s+1$ pairs of blocks, there is at least one infinite element. Let us now assume that there are at least three infinite elements $\infty_1, \infty_2, \infty_3$. Let F_i denote the set of blocks containing ∞_i for $i = 1,2,3$. Let $|F_1 \cap F_2 \cap F_3| = \alpha$. Then $|F_1 \cap F_2 \cap \bar{F}_3| = |F_1 \cap \bar{F}_2 \cap F_3| = |\bar{F}_1 \cap F_2 \cap F_3|$ where $\bar{F}_i$ is the complement of F_i in the block set of the design. Let β denote the common value of these cardinalities. Further let $\gamma = |F_1 \cap (F_2 \cup F_3)|$. Then $|F_1| = \alpha + 2\beta + \gamma = 4s + 1$, and $\alpha + \gamma = |F_2 \cap F_3| = \lambda = 2$ hence $2\beta = 2s + 1$, which is clearly impossible. □

Regrettably the NSC designs do not share the balance property with respect to triples that the SC designs possess. However the number of blocks containing a fixed triple cannot vary greatly within such a design, as is shown below.

THEOREM 2.3. In an NSC $(4s+2, 8s+2, 4s+1, 2s+1, 2s)$ with one infinite element every triple of distinct varieties occurs in either $s-2$, $s-1$, s, or $s+1$ blocks.

PROOF. If A is a subset of varieties, then let S_A denote the set of blocks of the design D which contain A. Let u, v, w denote three distinct varieties of D. Usually if a block contains u and not v or w, then its near-complement will contain v and w but not u.

This fails only if one of u, v or w is repeated or omitted from the pair $\{B,\phi B\}$. It is easily established that in any NSC with one infinite element, every non-infinite variety is contained in the intersection of precisely one near-complementary pair and is omitted from the union of precisely one near-complementary pair. Let $\delta_1(x,y)$ denote the number of near-complementary block pairs which contain u in one block, x in the other block, and omit y from both, $\delta_2(x,y)$ denote the number of complementary block pairs which contain u in both blocks and which are such that uxy occur together in one of the blocks, $\delta_3(x,y)$ denote the number of near-complementary block pairs such that ux is in one block and xy is in the other, and $\delta_4(x,y)$ be the number of block pairs in which u occurs in neither but x and y occur together in one of the blocks.

Let $\delta = \delta_1(v,w) + \delta_1(w,v) + \delta_2(v,w) - \delta_3(v,w) - \delta_3(w,v) - \delta_4(v,w)$. Then

$$|S_u - \{S_{uv} \cup S_{uw}\}| = |S_{vw} - S_u| + \delta(v,w).$$

Since each element is repeated at most once and omitted at most once $0 \le \delta_i \le 1$ for $i = 1,2,3,4$. Hence $-3 \le \delta \le 3$.

Now $|S_u - \{S_{uv} \cup S_{uw}\}| = |S_u| - |S_{uv}| - |S_{uw}| + |S_{uvw}| = r - 2\lambda + |S_{uvw}|$ and $|S_{vw} - S_u| = |S_{vw}| - |S_{uvw}| = \lambda - |S_{uvw}|$. This yields $|S_{uvw}| = t + (\delta-1)/2$. □

COROLLARY. If T is a triple of varieties which contains the infinite element, then T occurs in either $s-1$ or s blocks.

This follows from the fact that if $u = \infty$, then $\delta_2 = \delta_4 = 0$. □

In the case of two infinite elements it can again be shown that any triple of varieties again occurs in $s-2$, $s-1$, s or $s+1$ blocks, and a triple containing both infinite elements can only occur in $s-1$ or s blocks. The proof is similar to the above.

The authors know of only one such design with two infinite elements. It is listed below

$$\begin{array}{llll} B_1 : \infty_1\infty_2\ 1 & & B_6 : & 1\ 2\ 3 \\ B_2 : \infty_1\infty_2\ 2 & & B_7 : & 1\ 2\ 4 \\ B_3 : \infty_1\ 1\ 3 & & B_8 : & \infty_2\ 1\ 4 \\ B_4 : \infty_1\ 3\ 4 & & B_9 : & \infty_2\ 2\ 3 \\ B_5 : \infty_1\ 2\ 4 & & B_{10} : & \infty_2\ 3\ 4 \end{array}$$

This design can also be re-partitioned to yield a design with just one infinite element as follows. (B_1,B_9) (B_2,B_7) (B_3,B_8) (B_4,B_6) (B_5,B_{10}).

For larger values of the parameters, no repartitioning of any such NSC design is possible in view of theorem 2.1.

3. PROPERTIES OF SIMPLE NSC DESIGNS WITH EVEN BLOCK SIZE

Nearly self-complementary designs with even block size are of less interest since their parameters coincide with those of self-complementary designs and the latter are known to exist for all possible parameter sets provided that Hadamard matrices H_{4n} exist for all positive n. Moreover the properties of NSC designs are weaker for these parameter sets.

These properties are listed below (without proofs, since these are analogous to those of the previous section).

THEOREM 3.1. In any NSC $(4s+4,\ 8s+6,\ 4s+3,\ 2s+2,\ 2s+1)$ there are either one, two or three infinite elements.

THEOREM 3.2. In any NSC $(4s+4,\ 8s+6,\ 4s+3,\ 2s+2,\ 2s+1)$ with either one or two infinite elements, every triple of distinct varieties occurs in either $s-1$, s or $s+1$ blocks. In any NSC $(4s+4,\ 8s+6,\ 4s+3,\ 2s+2,\ 2s+1)$ with three infinite elements every triple of distinct varieties occurs in either $s-2$, $s-1$, s, $s+1$ or $s+2$ blocks.

4. CYCLIC NSC DESIGNS

In this section a standard method [1] for obtaining certain NSC designs based on cyclic groups (cyclic designs) is discussed. As usual Z_n denotes the cyclic group of order n, and Z_n^* denotes the non-zero elements of Z_n.

THEOREM 4.1. Let $G = Z_n$ where $n = 2s+1$. If one can find a pair of blocks of the form $A^* = \{\infty,0\} \cup A$ and $B^* = \{0\} \cup B$ where A and B are $s-1$ and s-subsets of G^* such that $A \cap B = \phi$ and the differences of $\{0\} \cup A$ and $\{0\} \cup B$ are symmetrically repeated (each occurring s times), then the translates of A^* and B^* form an NSC $(2s+2,\ 4s+2,\ 2s+1,\ s+1,\ s)$.

PROOF. That the configuration is a BIBD with the required parameters is immediate from the method of differences. The appropriate block pairs are $\{A^*+\theta, B^*+\theta\}$, $\theta \in G$. □

EXAMPLES.

CASE 1 (s EVEN)

∞ 0 2	0 3 4	mod 5,
∞ 0 2 4 5	0 3 6 7 8	mod 9,
∞ 0 2 3 4 8 11	0 5 6 7 9 10 12	mod 13.

CASE 2 (s ODD)

∞ 0	0 2	mod 3,
∞ 0 2 6	0 3 4 5	mod 7.

Note that the general parameters $(2s+2,\ 4s+2,\ 2s+1,\ s+1,\ s)$ are parameters of the designs derived from the symmetric designs $(4s+3,\ 4s+3,\ 2s+1,\ 2s+1,\ s)$ which exist if and only if there exists an Hadamard matrix H_{4s+4} (see, for example, [4]). It is well known (see, for example, [2, p.256] that a quasi-residual design ("design with the parameters of a residual design") is not necessarily a residual design. However we shall show that every cyclic NSC design is residual, hence the existence of such a design implies the existence of an Hadamard matrix. In particular, if k is odd, say $k = 2s+1$, then the corresponding Hadamard matrix has order $8s+4$. For this reason, the existence of cyclic NSC designs with odd values for k could prove useful in the theory of Hadamard matrices. To prove the cited result, we will use a new approach to the method of mixed differences [1].

5. A GENERALIZATION OF THE METHOD OF MIXED DIFFERENCES

It will be assumed here (as it was above) that the reader is familiar with the contents of [1]. The method will be extended here in terms of rings.

Let R be a finite ring of order n, $R = \{0,r,s,\ldots,t\}$. Consider m "copies" of R,

$$\begin{aligned} R_1 &= \{0_1,r_1,s_1,\ldots,t_1\} \\ R_2 &= \{0_2,r_2,s_2,\ldots,t_2\} \\ &\vdots \\ R_m &= \{0_m,r_m,s_m,\ldots,t_m\}. \end{aligned}$$

Given two elements x_i and y_i from the *ith* copy, we define the *pure difference* $x_i - y_i$ as the element $(x-y)_i$. Thus pure differences are the natural differences operating within the *ith* copy. Now to each copy R_k of R assign an invertible element $w(k)$ of R, called the *weight* of k. By the *exterior combination* $x_i \ominus y_j$ of x_i and y_j (where $x_i \in R_i$ and $y_j \in R_j$, $i \neq j$) we mean the quantity $(w(i)x - w(j)y)_{ij}$. Let $S = \{s_i, t_j, \ldots, v_\ell\}$ be a subset of $V = \bigcup_{i=1}^{m} R_i$. If one forms the set of blocks $S \oplus \theta$ for $\theta \in R$ where

$$S \oplus \theta = \{(s + (w(i))^{-1}\theta)_i,\ (t + (w(j))^{-1}\theta)_j,\ \ldots,\ (v + (w(\ell))^{-1}\theta)_\ell\},$$

then S is said to be developed through R.

THEOREM 5.1. Let m copies $R_1, R_2, \ldots, R_m$ of a ring R be given. Let $w(1), w(2), \ldots, w(m)$ be a set of corresponding invertible weights, also be given. If one can find a set of t blocks $B_1, B_2, \ldots, B_t$ each of size k with elements in $V = \bigcup_{i=1}^{m} R_i$ such that

(i) the non-zero pure differences are symmetrically repeated, each occurring λ times, and

(ii) the exterior combinations x_{ij} are symmetrically repeated, each occurring λ times,

then the blocks $B_1, B_2, \ldots, B_t$ when developed through R, form BIBD (mn, ms, r, k, λ) for an appropriate value of r.

PROOF. Since the pure differences are symmetrically repeated λ times, each pair $\{x_i, y_i\}$, $x \neq y$ occurs in precisely λ blocks. Suppose that x_i and y_j are given with $i \neq j$. Let d_{ij} denote the exterior combination, so that $d = w(i)x - w(j)y$. Now d_{ij} is represented as an exterior sum λ times in the set of blocks $B_1, B_2, \ldots, B_t$. Let $u_i \ominus v_j$ be such a representation in B_ℓ. Then

$$w(i)u - w(j)v = w(i)x - w(j)y,$$
$$y = v + (w(j))^{-1} w(i)(x-u).$$

However there exists a unique $\theta \in R$ such that $u + (w(i))^{-1}\theta = x$, namely $\theta = w(i)(x-u)$. Then $B_\ell + \theta$ contains (corresponding to $\{u_i, v_j\}$ the pair $\{x_i,\ (v + (w(j))^{-1}w(i)(x-u))_j\} = \{x_i, y_j\}$. Hence the pair $\{x_i, y_j\}$ occurs in precisely λ blocks of the developed set. □

If all weights $w(i) = 1$ in the above, then the result is the standard method of mixed differences. The case of interest here is that of two copies of Z_n (for appropriate n) and weights $1, -1$. Thus we look at mixed sums. Examples of this case are given below. (The blocks here can also be employed for mixed differences if minor modifications are made).

EXAMPLE 1. BIBD (10,30,9,3,2)

Initial blocks $(0_1,1_2,4_2)$ $(0_1,2_2,3_2)$
$(0_1,0_2,1_2)$ $(0_1,2_2,4_2)$ $(0_1,3_1,0_2)$
$(0_1,1_1,2_1)$ modulo 5

EXAMPLE 2. BIBD (16,80,15,3,2)

Initial blocks $(0_1,1_2;7_2)$ $(0_1,2_2,6_2)$ $(0_1,3_2,5_2)$
$(0_1,0_2,1_2)$ $(0_1,2_2,7_2)$ $(0_1,3_2,6_2)$ $(0_1,4_2,5_2)$
$(0_1,4_1,4_2)$ $(0_1,1_1,3_1)$ $(0_1,1_1,3_1)$ modulo 8

EXAMPLE 3. BIBD (22,154,21,3,2)

Initial blocks $(0_1,1_2,10_2)$ $(0_1,2_2,9_2)$ $(0_1,3_2,8_2)$
$(0_1,4_2,7_2)$ $(0_1,5_2,6_2)$ $(0_1,0_2,1_2)$ $(0_1,2_2,10_2)$
$(0_1,3_2,9_2)$ $(0_1,4_2,8_2)$ $(0_1,5_2,7_2)$ $(0_1,6_1,0_2)$
$(0_1,1_1,2_1)$ $(0_1,2_1,5_1)$ $(0_1,3_1,7_1)$ modulo 11.

It is also easy to construct such BIBD's modulo 14, 17, 20, 23, 26, 29 and probably for any modulus $\equiv 2 \pmod 3$.

6. THE RESIDUALITY OF CYCLIC NSC DESIGNS

As mentioned in section 4, cyclic NSC designs are generated by initial blocks of the form $A^* = \{\infty,0\} \cup A$ and $B^* = \{0\} \cup B$ where $A \cap B = \phi$.

THEOREM 6.1. Let D be a cyclic NSC $(2x+2,\ 4x+2,\ 2x+1,\ x+1,\ x)$ design. Then D is the residual of a symmetric $(4x+3,\ 4x+3,\ 2x+1,\ 2x+1,\ x)$ design.

PROOF. D is defined by the cyclic group $G = Z_{2x+1}$. View G as the additive group of the ring R of integers modulo $2x+1$. Take two copies R_1 and R_2 of R, and weights $w(1) = 1$ and $w(2) = -1$. Using the notation employed above, let A' denote $\{0\} \cup A$. Let B^c denote the complement of B^* in R. We adopt the convention that if S is a subset of R, then S_i is the corresponding set in R_i, $i = 1,2$. It is readily verified that in A' and B^c the differences are symmetrically repeated, each occurring $x-1$ times. Consider the set of blocks

$$\{\alpha = A_1^* \cup B_2^c,\ \beta = B_1^* \cup A_2'\}.$$

Let $S_i \oplus S_j$ denote the multiset of exterior sums formed between S_i and S_j. Since addition is commutative,

$$B_1^* \oplus A_2' = A_1' \oplus B_2^*.$$

Moreover it is clear that

$$(X \oplus Y) \mathbin{\&} (X \oplus W) = X \oplus (Y \cup W)$$

for disjoint sets Y and W, where $\&$ denotes multiset union.

Hence

$$\begin{aligned}(A_1' \oplus B_2^c) \mathbin{\&} (B_1^* \oplus A_2') &= (A_1' \oplus B_2^c) \mathbin{\&} (A_1' \oplus B_2^*)\\ &= A_1' \oplus (B_2^c \cup B_2^*)\\ &= A_1' \oplus R_2,\end{aligned}$$

and the mixed sums of α and β are symmetrically repeated, each occurring $|A_1'| = x$ times. Hence α and β developed through R, together with B, a block consisting of the elements of R_2 form the required design. □

As an example of the previous theorem, the design

$$\infty\ 0\ 2 \qquad 2\ 3\ 4 \qquad \bmod 5$$

is embedded in

$$\begin{array}{l}
\qquad\quad 0_2\ 1_2\ 2_2\ 3_2\ 4_2\\
\infty\ 0_1\ 2_1\ 0_2\ 1_2 \qquad 2_1\ 3_1\ 4_1\ 0_2\ 2_2\\
\infty\ 1_1\ 3_1\ 4_2\ 0_2 \qquad 3_1\ 4_1\ 0_1\ 4_2\ 1_2\\
\infty\ 2_1\ 4_1\ 3_2\ 4_2 \qquad 4_1\ 0_1\ 1_1\ 3_2\ 0_2\\
\infty\ 3_1\ 0_1\ 2_2\ 3_2 \qquad 0_1\ 1_1\ 2_1\ 2_2\ 4_2\\
\infty\ 4_1\ 1_1\ 1_2\ 2_2 \qquad 1_1\ 2_1\ 3_1\ 1_2\ 3_2
\end{array}$$

as a residual design.

In conclusion, the authors reiterate the fact that NSC designs are in a position to contribute to the theory of Hadamard matrices. We ask if all simple NSC designs are residual.

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