THE CONSTRUCTION OF NESTED CYCLE SYSTEMS*

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Abstract. In this paper, we prove for any integer $m \geq 3$ that there exists a nested m-cycle system of order n if and only if $n \equiv 1 \bmod 2m$, with at most 13 possible exceptions (for each value of m).

1. Introduction. Let G be a graph, and let $m \geq 3$ be an integer. An *m-cycle decomposition* of g is an edge-decomposition of G into cycles of size m. We will write the m-cycle decomposition as a pair $(G, \mathcal{C})$, where $\mathcal{C}$ is the set of cycles in the edge-decomposition. An m-cycle decomposition of K_n will be called an *m-cycle system* of *order* n. Of course, a 3-cycle system is a *Steiner triple system*; these designs exist for all orders $n \equiv 1$ or 3 modulo 6.

We will say that an m-cycle decomposition, $(G, \mathcal{C})$, can be *nested* if we can associate with each cycle $C \in \mathcal{C}$ a vertex of G, which we denote $f(C)$, such that $f(C) \notin C$, and such that the edges in $\{\{x, f(C)\} : x \in C, C \in \mathcal{C}\}$ form an edge-decomposition of G. Alternatively, we can view a nested m-cycle decomposition as an edge-decomposition of the multigraph $2G$ into wheels with m spokes, where every edge occurs in one wheel of the decomposition as a spoke and in one wheel on the rim.

It is easy to see that a necessary condition for the existence of a nested m-cycle system of order n is that $n \equiv 1 \bmod 2m$. The first examples of nested m-cycle systems to be studied in the literature were nested 3-cycle systems (i.e., nested Steiner triple systems). It was proved by Stinson [5] that there exists a nested Steiner triple system of order n if and only if $n \equiv 1$ modulo 6. In the smallest even-cycle case, $m = 4$, it has been shown by Stinson [6] that the necessary condition $n = 1 \bmod 8$ is sufficient for existence, with the possible exceptions $n = 57, 65, 97, 113, 185$ and 265. More recently, Lindner, Rodger and Stinson [3] showed for each odd $m \geq 3$ that there exists a nested m-cycle system of order n if and only if $n \equiv 1 \bmod 2m$, with at most 13 possible exceptions. Then, Lindner and Stinson [4] proved for any even $m \geq 4$ that there exists a nested m-cycle system of order n if and only if $n \equiv 1 \bmod 2m$, with at most 13 possible exceptions.

In this paper, we give a condensed proof of these existence results.

2. Some constructions. In this section, we present a small number of direct and recursive constructions for nested cycle decompositions that will enable us to prove our existence results in Section 3. Many of these constructions involve nested cycle decompositions of complete multipartite graphs. We refer to the parts of a complete multipartite graph as *holes*. The *type* of a complete multipartite

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graph is defined to be the multiset consisting of the sizes of the holes. We usually use an "exponential" notation to describe types: a type $t_1^{u_1}t_2^{u_2}\dots t_k^{u_k}$ denotes u_i occurrences of $t_i, 1 \le i \le k$. If T is the type $t_1^{u_1}t_2^{u_2}\dots t_k^{u_k}$ and m is an integer, then mT is defined to be the type $(mt_1)^{u_1}(mt_2)^{u_2}\dots(mt_k)^{u_k}$. Also, we will denote the complete multipartite graph having type T by $K(T)$.

First, we give a multiplication construction for nested cycle decompositions of complete multipartite graphs.

Multiplication construction. Suppose there is a nested m-cycle decomposition of a complete multigraph $K(T)$. Let $k \ge 1$. Then there is a nested (km)-cycle decomposition of $K(kT)$.

Proof. Replace every vertex v of $K(T)$ by k independent vertices, (named $v_i, 1 \le i \le k$), thereby constructing $K(kT)$. Let $(K(T), \mathcal{C})$ be an m-cycle decomposition, and let f be a nesting of $\mathcal{C}$. Each cycle $C \in \mathcal{C}$ corresponds to a subgraph of $K(kT)$ isomorphic to the Cartesian product $C \otimes (K_k)^c$ (each vertex of C is replaced by k independent vertices, and each edge is replaced by k^2 edges forming a complete bipartite graph $K_{k,k}$). It is well-known that the graph $C \otimes (K_k)^c$ has an (mk)-cycle decomposition (this is a decomposition into Hamiltonian cycles; see [1] or [2]). The number of (mk)-cycles in this decomposition is k. Suppose these cycles are named $C_i, 1 \le i \le k$. We define a nesting by associating with each C_i the vertex $f(C)_i$. If we do this for every cycle C, we obtain the desired nesting. □

Let S be a set, and let $\{S_1, \dots, S_n\}$ be a partition of S. An $\{S_1, \dots S_n\}$-*Room frame* is an $|S|$ by $|S|$ array, F, indexed by S, which satisfies the following properties:

1) every cell of F either is empty or contains an unordered pair of symbols of S,
2) the subarrays $S_i \times S_i$ are empty, for $1 \le i \le n$ (these subarrays are referred to as *holes*),
3) each symbol of $S \backslash S_i$ occurs once in row (or column) s, for any $s \in S_i$,
4) the pairs occurring in F are those $\{s,t\}$, where $(s,t) \in (S \times S) \backslash \bigcup_{i=1}^{n} (S_i \times S_i)$.

We shall say that F is *skew* if, for any pair of cells (s,t) and (t,s), where $(s,t) \in (S \times S) \backslash \cup (S_i \times S_i)$, precisely one is empty. The *type* of F is defined to be the multiset $\{|S_i| : 1 \le i \le n\}$. As before, we use an "exponential" notation to describe types.

The next construction produces a nested cycle decomposition of a complete multigraph from a skew Room frame.

Skew Room frame construction. [3, Theorem 3.1] Suppose there is a skew Room frame of type T. Let $m \ge 3$ be an integer. Then there is a nested m-cycle decomposition of the complete multipartite graph $K(mT)$.

Proof. Let $r = \lfloor \frac{m}{2} \rfloor$. For $0 \le i \le r$, define $d_i = (-1)^{i+1} \lfloor \frac{i+1}{2} \rfloor$. Let F be a skew Room frame of type T based on symbol set X. We shall define our complete multipartite graph $K(mT)$ on vertex set $X \times \mathbf{Z}_m$. The holes of $K(mT)$ will be $S_i \times \mathbf{Z}_m$, for every hole S_i of the frame F.

For any $x, y, z \in X$, define $C(\{x,y\},z;0)$ to be the cycle

$$(x,d_0)(y,d_1)(x,d_2),\ldots,(x,d_{r-1})(z,d_r)(y,d_{r-1}),\ldots,(x,d_1)(y,d_0)(x,d_0) \text{ if } r \text{ is odd}$$
$$(x,d_0)(y,d_1)(x,d_2),\ldots,(y,d_{r-1})(z,d_r)(x,d_{r-1}),\ldots,(x,d_1)(y,d_0)(x,d_0) \text{ if } r \text{ is even.}$$

For any $x, y, z \in X$ and $i \in \mathbf{Z}_m$, define $C(\{x,y\},z;i)$ to be the cycle obtained by adding i to the second coordinate of each point in the cycle $C(\{x,y\},z;0)$, and reducing modulo m.

For any unordered pair $\{x,y\}$ from different holes of the frame F, define $\mathrm{Row}(x,y)$ to be the row of F containing $\{x,y\}$ in some cell, and define $\mathrm{Col}(x,y)$ to be the column of F containing $\{x,y\}$ in some cell.

We construct our cycle decomposition as follows. For every unordered pair $\{x,y\}$ from different holes of F, and for every $i \in \mathbf{Z}_m$, take the cycle $C(\{x,y\}, \mathrm{Row}(x,y);i)$, and nest it with the point $\mathrm{Col}(x,y)$. It is not too difficult to verify that this produces a nested cycle decomposition of the complete multipartite graph; the details of the verification are contained in [3]. □

A *group-divisible design,* (or GDD), is a triple $(X,\mathcal{G},\mathcal{A})$ which satisfies the following properties:

1) $\mathcal{G}$ is a partition of X into subsets called *groups*,
2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and
3) every pair of points from distinct groups occurs in a unique block.

The *group-type* (or *type*) of a GDD $(X,\mathcal{G},\mathcal{A})$ is the multiset $\{|G|; G \in \mathcal{G}\}$. As before, we use an "exponential" notation to describe group-types. We will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathcal{A}$.

Our next construction uses group-divisible designs in a recursive construction.

GDD construction. Let $(X,\mathcal{G},\mathcal{A})$ be a GDD having type T, and let $w : X \to \mathbf{Z}^+ \cup 0$ (we say that w is a *weighting*). For every $A \in \mathcal{A}$, suppose there is a nested m-cycle decomposition for the complete multipartite graph having type $\{w(x) : x \in A\}$. Then there is a nested m-cycle decomposition for a complete multipartite graph having type $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$.

Proof. For every $x \in X$, let $s(x)$ be $w(x)$ "copies" of x. For any subset $Y \subset X$, define $s(Y) = \bigcup_{x\in Y} s(x)$. For every $A \in \mathcal{A}$, suppose that $(s(A), \mathcal{C}(A))$ is a nested cycle decomposition of the complete multipartite graph of type $\{w(x) : x \in A\}$ having holes $s(x), x \in A$. Let f_A be a nesting of $(s(A), \mathcal{C}(A))$. Then $(S(X), \bigcup_{A\in\mathcal{A}} \mathcal{C}(A))$ is a nested cycle decomposition of the complete multipartite graph of type $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$ having holes $s(G), G \in \mathcal{G}$. We define a nesting of this cycle decomposition by $f(C) = f_A(C)$ if and only if $C \in \mathcal{C}(A)$. □

Once we have constructed a nested cycle decomposition of a complete multipartite graph, we can produce a nested cycle system by the usual technique of filling in holes.

Filling in holes construction. Suppose there is a nested m-cycle decomposition for the complete multipartite graph $K(T)$, where T is the type $t_1^{u_1}t_2^{u_2}\ldots t_k^{u_k}$. For $1 \le i \le k$, suppose there is a nested m-cycle system of order t_i+1. Then there is a nested m-cycle system of order $\sum_{i=1}^{k}(t_iu_i+1)$.

We also use the following class of nested cycle systems which are constructed by difference methods.

LEMMA 2.1. *For all integers $r \ge 3$, there is a nested r-cycle system of order $2r+1$.*

Proof. Define $k = \lfloor \frac{r-1}{2} \rfloor$, and define $\mathbf{a} = (a_1, \ldots, a_r)$ by

$$a_i = (-1)^i i, \text{ if } 1 \le i \le k-1$$
$$a_i = (-1)^{i+1} i, \text{ if } k \le i \le r,$$

where each a_i is reduced modulo $2r+1$. Let $\mathcal{C} = \{\mathbf{a}+j : j \in \mathbf{Z}_{2r+1}\}$, where $\mathbf{a}$ represents the cycle $a_1a_2\ldots a_ra_1$. Then, it is easy to see that $\mathcal{C}$ is a cycle system of order $2n+1$. We define a nesting f of $\mathcal{C}$ by $f(\mathbf{a}+j) = j$, for every cycle $\mathbf{a}+j \in \mathcal{C}$. □

LEMMA 2.2. [6, *Lemma* 1] *Suppose $k \equiv 1$ modulo 4 is a prime power. Then there is a nested 4-cycle decomposition of the complete multipartite graph $K(2^k)$.*

Proof. As the vertex set for $K(2^k)$ we take $GF(k) \times \mathbf{Z}_2$, and we let the holes be $\{y\} \times \mathbf{Z}_2, y \in GF(k)$. Let α be a primitive element in $GF(k)$. Write $k = 4t+1$, and define $\beta = \alpha^t$. For $0 \le i \le t-1$, and for any element $a \in GF(k) \times \mathbf{Z}_2$, define a cycle

$$C(i,a) = (a + (\alpha^i, 1); a + (\alpha^i\beta, 0); a + (\alpha^i\beta^2, 0); a + (\alpha^i\beta, 1)).$$

For each cycle $C(i,a)$, define the nested point to be $f(C(i,a)) = a$. Then, it is not difficult to verify that $\mathcal{C} = \{C(a,i)\}$ is a 4-cycle decomposition of $K(2^k)$ and f is a nesting of $\mathcal{C}$. □

3. The existence results. First, we consider nested m-cycle systems for odd values of m. We shall employ the following known class of skew Room frames.

THEOREM 3.1. [3, *Theorem* 2.2] *For all $n \ge 5, n \notin \{6,22,23,24,26,27,28,30,34,38\}$, there is a skew Room frame of type 2^n.*

LEMMA 3.2. *Suppose $m \ge 3$ is odd and $u \notin \{1,2,3,4,6,22,23,24,26,27,28,30,34,38\}$. Then there is a nested m-cycle decomposition of $K((2m)^u)$.*

Proof. This follows from applying the skew Room frame construction to a skew Room frame of type 2^u (which exists by Theorem 3.1). □

We now have the following immediate consequence.

THEOREM 3.3. *Suppose $m \geq 3$ is odd, $n = 2um+1$, and $u \notin \{2,3,4,6,22,23,24,26,27,28,30,34,38\}$. Then there is a nested m-cycle system of order n.*

Proof. If $u \neq 1$, fill in the holes with nested m-cycle systems of order $2m+1$ (Lemma 2.1). For $u = 1$, Lemma 2.1 gives the result immediately. □

More generally, we have the following result for even cycle lengths that are not a power of two.

THEOREM 3.4. *Suppose $m \geq 3$ is odd, $n = 2um+1$, $u \notin \{2,3,4,6,22,23,24,26,27,28,30,34,38\}$, and $i \geq 0$. Then there exists a nested $(2^i m)$-cycle system of order $2^{i+1}um+1$.*

Proof. For $u = 1$, the result is given in Lemma 2.1. For $u > 1$, proceed as follows. Apply the multiplication construction to the m-cycle decompositions obtained in Lemma 3.2 using $k = 2^i$. We obtain a nested $(2^i m)$-cycle decomposition of $K((2^{i+1}m)^u)$. Now, fill in the holes with nested $(2^i m)$-cycle systems of order $2^{i+1}m+1$ which exist by Lemma 2.1. □

Finally, we address the question of constructing nested 2^i-cycle systems. Our construction for nested 2^i-cycle systems ($i \geq 3$) depends on the existence of the following group-divisible designs.

THEOREM 3.5. [4, *Theorem* 4.14] *Suppose $u \geq 5, u \notin \{7,8,12,14,18,19,23,24,33,34\}$. Then there is a $\{5,9,13,17,29,49\}$-GDD having group-type 4^u.*

The existence of the following nested 4-cycle decompositions will prove useful.

LEMMA 3.6. *Suppose $u \geq 5, u \notin \{7,8,12,14,18,19,23,24,33,34\}$. Then there is a nested 4-cycle decomposition of $K(8^u)$.*

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a $\{5,9,13,17,29,49\}$-GDD having group-type 4^u. Apply the GDD construction, giving every point weight 2. For every block A, $|A| \in \{5,9,13,17,29,49\}$, so there is a nested 4-cycle decomposition of $K(2^{|A|})$ by Lemma 2.2. We get a nested 4-cycle decomposition of $K(8^{|X|/4})$. □

LEMMA 3.7. *Suppose $u \geq 5, u \notin \{7,8,12,14,18,19,23,24,33,34\}$, and $i \geq 2$. Then there is a nested (2^i)-cycle decomposition of $K((2^{i+1})^u)$.*

Proof. Apply the multiplication construction to the m-cycle decompositions obtained in Lemma 3.6 using $k = 2^{i-2}$. We obtain a nested 2^{i+1}-cycle decomposition of $K((2^{i+1})^u)$. □

THEOREM 3.8. *Suppose $u \geq 1, u \neq 2,3,4,7,8,12,14,18,19,23,24,33$, or 34, and $i \geq 2$. Then there is a nested (2^i)-cycle system of order $2^{i+1}u+1$.*

Proof. For $u = 1$, apply Lemma 2.1. For $u > 1$, we proceed as follows. Construct a nested (2^i)-cycle decomposition of $K((2^{i+1})^u)$, using Lemma 3.7, and then fill in the holes with nested (2^i)-cycle systems of order $2^{i+1}+1$ which exist by Lemma 2.1. □

Summarizing the results proved above, we have the following.

COROLLARY 3.9. *Suppose $m \geq 3$ is any integer, $n \equiv 1$ modulo $2m$, and $n \geq 70m+1$. Then there is a nested m-cycle system of order n.*

4. Further results for small cycle lengths. For some small odd cycle lengths, it is possible to remove most or all of the 13 possible exceptions given in Theorem 3.3. For odd $m \leq 15$, this was done in [3]. We summarize the results from [3] below.

m	spectrum of nested m-cycle systems
3	$n \equiv 1$ modulo 6
5	$n \equiv 1$ modulo 10
7	$n \equiv 1$ modulo 14, except possibly 57 and 85
9	$n \equiv 1$ modulo 18, except possibly 55
11	$n \equiv 1$ modulo 22, except possibly 133
13	$n \equiv 1$ modulo 26, except possibly 105
15	$n \equiv 1$ modulo 30, except possibly 91

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