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A Brief Introduction to Design Theory

Jeffrey H. Dinitz and Douglas R. Stinson

1	BALANCED INCOMPLETE BLOCK DESIGNS	1
2	t -DESIGNS	4
3	AUTOMORPHISMS AND ISOMORPHISMS OF DESIGNS	6
4	PAIRWISE BALANCED DESIGNS	6
5	LATIN SQUARES	9
6	GRAPHS	10
7	REMARKS	12
	REFERENCES	12

In this introduction, we give some basic definitions, establish standard notation, and present some fundamental “classical” results in combinatorial design theory.

1 BALANCED INCOMPLETE BLOCK DESIGNS

A *balanced incomplete block design* (BIBD) with parameters (v, b, r, k, λ) is a pair (X, \mathcal{A}) that satisfies the following properties:

1. X is a set of v elements (called *points*).
2. \mathcal{A} is a family of b subsets of X , each of cardinality k (called *blocks*).
3. Every point occurs in exactly r blocks.
4. Every pair of distinct points occurs in exactly λ blocks.

It is generally required that $k < v$; that is why BIBDs are called *incomplete block designs*. We will use the abbreviation (v, b, r, k, λ) -BIBD to denote a balanced incomplete block design with parameters (v, b, r, k, λ) . It is easy to

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see that the five parameters are not independent; simple counting yields the following two relations:

$$vr = bk \quad \text{and} \quad \lambda(v - 1) = r(k - 1) \quad (1)$$

Hence, since the other two parameters can then be deduced, it is not uncommon to write a BIBD using the three parameters v , k , and λ , that is, as a (v, k, λ) -BIBD.

Suppose $(\mathbf{X}, \mathcal{A})$ is a (v, b, r, k, λ) -BIBD, where $\mathbf{X} = \{x_i : 1 \leq i \leq v\}$ and $\mathcal{A} = \{A_j : 1 \leq j \leq b\}$. The *incidence matrix* of the BIBD is the v by b matrix $M = (m_{ij})$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } x_i \in A_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then the incidence matrix satisfies the equation $MM^T = (r - \lambda)I + \lambda J$, where I is a $v \times v$ identity matrix and J is a $v \times v$ matrix of 1's.

Determining necessary and sufficient conditions for the existence of BIBDs is one of the central questions in design theory. We will now briefly survey some necessary and sufficient conditions for existence.

The most basic necessary condition is known as *Fisher's inequality*. It states that a (v, b, r, k, λ) -BIBD exists only if $b \geq v$ (or equivalently, if $r \geq k$). A BIBD with $b = v$ (or equivalently, $r = k$) is called a *symmetric* BIBD. Some further necessary conditions are known for the existence of symmetric BIBDs. The following result is known as the Bruck-Ryser-Chowla theorem:

Theorem 1.1 (Bruck-Ryser-Chowla Theorem). *If a symmetric (v, k, λ) -BIBD exists, then:*

1. *if v is even, $k - \lambda$ is a perfect square,*
2. *if v is odd, the Diophantine equation $x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2} \lambda z^2$ has a solution in integers, not all of which are zero.*

Let us now turn to sufficient conditions for existence of BIBDs. BIBDs with $k = 2$ or $k = v$ exist trivially, so we will usually restrict our attention to the cases where $2 < k < v$. A $(v, 3, 1)$ -BIBD is known as a *Steiner triple system* and denoted $\text{STS}(v)$. An $\text{STS}(v)$ was shown to exist for all $v \equiv 1$ or 3 modulo 6 in the nineteenth century. In fact, in the cases $k = 3, 4$, or 5 , all designs satisfying condition 1 exist, with the single exception of the parameter list $(15, 5, 2)$, for which a BIBD does not exist. The existence of BIBDs with $k = 6$ is not yet resolved. The smallest design in this class whose existence is undetermined has parameters $(46, 6, 1)$.

Several specific classes of BIBDs should be mentioned. A symmetric $(n^2 + n + 1, n + 1, 1)$ -BIBD is called a *projective plane* of order n . An $(n^2, n, 1)$ -BIBD is called an *affine plane* of order n . It is not difficult to show that an affine plane of order n exists if and only if a projective plane of order n exists.

A projective plane of order n exists if n is a prime power. No examples of projective planes of nonprime power order are known to exist. A projective plane of order 6 is ruled out by the Bruck-Ryser-Chowla theorem. A recent computer-aided search by C. Lam showed that a plane of order 10 does not exist. Thus, the smallest unresolved order is now order 12.

Let q be a prime power, and let $d \geq 2$. If we take the lines of $\text{PG}(d, q)$, the d -dimensional projective geometry over $\text{GF}(q)$, as blocks, we get a $((q^{d+1} - 1)/(q - 1), q + 1, 1)$ -BIBD. If we take the lines of $\text{AG}(d, q)$, the d -dimensional affine geometry over $\text{GF}(q)$, as blocks, we get a $(q^d, q, 1)$ -BIBD. One further class of BIBDs that comes from geometries are the *unitals*, having parameters $(q^3 + 1, q + 1, 1)$.

A symmetric $(4n + 3, 2n + 1, n)$ -BIBD is called a *Hadamard design* and is equivalent to a Hadamard matrix of order $4n + 4$. Hadamard designs are conjectured to exist for all integers $n > 0$. Despite much study this conjecture is still open.

A set of blocks in a BIBD that partitions the point set is called a *parallel class*. A *resolution* of a BIBD is a partition of the family of blocks into parallel classes. Note that a resolution contains exactly r parallel classes. A BIBD is said to be *resolvable* if it has at least one resolution. Clearly, a resolvable BIBD can exist only if k divides v , in addition to the necessary conditions 1. We note that any affine plane is resolvable.

Existence of resolvable BIBDs for $k = 3$ and 4 has been completely determined; the necessary conditions are sufficient. (A resolvable STS(v) is called a *Kirkman triple system* of order v and denoted KTS(v).) The spectrum of resolvable $(v, 5, 1)$ -BIBDs has almost been completed. The necessary condition $v \equiv 5$ modulo 20 is sufficient with a few possible exceptions, the smallest of which is $v = 45$.

Fisher's inequality can be strengthened in the case of resolvable designs: if a resolvable (v, k, λ) -BIBD exists, then $b \geq v + r - 1$. In the case of equality, the BIBD is said to be *affine resolvable*. Notice that an affine plane is affine resolvable.

Suppose that a BIBD has two resolutions, with the property that any parallel class from the first resolution and any parallel class from the second resolution have at most one block in common. Then the two resolutions are said to be *orthogonal*. A $(v, 2, 1)$ -BIBD having two orthogonal resolutions is equivalent to a *Room square* of side $v - 1$.

There are several methods by which new BIBDs can be obtained from old ones. The *complement* of a BIBD is obtained by replacing every block by its complement with respect to the point set. The complement of a (v, b, r, k, λ) -BIBD is a $(v, b, b - r, v - k, b - 2r + \lambda)$ -BIBD.

A symmetric BIBD has the property that any two blocks intersect in exactly λ points. Hence, the dual incidence structure obtained by interchanging the roles of points and blocks is also a BIBD with the same parameters (whence the term symmetric). Given a symmetric (v, k, λ) -BIBD, and given a block A of this design, delete the block A , and delete all points in A from all other

blocks. The resulting design is called the *residual design* and has parameters $(v - k, v - 1, k, k - \lambda, \lambda)$. If instead we delete the block A and delete all points not in A from all other blocks, we get a design called the *derived design*. Its parameters are $(k, v - 1, k - 1, \lambda, \lambda - 1)$. Any $(v - k, v - 1, k, k - \lambda, \lambda)$ -BIBD is called *quasi-residual*. It is an interesting question to ask which quasi-residual BIBDs are in fact residual. It is known that any quasi-residual BIBD with $\lambda = 1$ or 2 is residual; but for $\lambda = 3$, there are examples of quasi-residual BIBDs that are not residual.

Next, we discuss the idea of subdesigns. Suppose we have a (v, k, λ) -BIBD, and we choose a subset Y of w points. If it happens that every block of the BIBD contains either exactly k or at most one of the points in Y , then we obtain a (w, k, λ) -BIBD by taking those blocks that contain k points from Y . This BIBD on w points is called a *subdesign* and is denoted $\text{sub-}(w, k, \lambda)$ -BIBD. It is easy to see that if there is a $(v, k, 1)$ -BIBD containing a $\text{sub-}(w, k, 1)$ -BIBD, then $v \geq (k - 1)w + 1$. (Of course, the parameter lists $(v, k, 1)$ and $(w, k, 1)$ must satisfy the necessary conditions 1.) In the cases $k = 3$ and $k = 4$, these necessary conditions for existence are sufficient.

We mention one further way in which a BIBD can occur “inside” another BIBD. Suppose we can delete one point from every block of a (v, k, λ) -BIBD in such a way that we are left with a $(v, k - 1, \mu)$ -BIBD. Then the smaller BIBD is said to be *nested* in the larger one. Clearly, $\mu = \lambda(k - 2)/k$. In the case $k = 4$, $\lambda = 2$, one can construct a $(v, 4, 2)$ -BIBD containing a nested $(v, 3, 1)$ -BIBD if and only if $v \equiv 1$ modulo 6.

In situations where BIBDs do not exist, it is interesting to determine how “close” to a BIBD one can get. This motivates the ideas of covering and packing designs. A *packing design* with parameters (v, k, λ) is a pair (X, \mathcal{A}) that satisfies the following properties:

1. X is a set of v elements (called *points*).
2. \mathcal{A} is a family of subsets of X , each of cardinality k (called *blocks*).
3. Every pair of distinct points occurs in at most λ blocks.

A $(v, 3, 1)$ packing design is also known as a *partial triple system* of order v and is denoted $\text{PTS}(v)$. A *covering design* with parameters (v, k, λ) will satisfy the same conditions, except that every pair of distinct points should occur in at least λ blocks. Usually, it is desired to find a packing designs with the maximum number of blocks and covering designs with the minimum number of blocks.

2 t -DESIGNS

In this section, we discuss a generalization of BIBDs. A t - (v, k, λ) design is a pair (X, \mathcal{A}) that satisfies the following properties:

1. X is a set of v elements (called *points*).

2. \mathcal{A} is a family of subsets of X , each of cardinality k (called *blocks*).
3. Every t -subset of distinct points occurs in exactly λ blocks.

A t -(v, k, λ) design is also denoted $S_\lambda(t, k, v)$. Observe that a (v, k, λ) -BIBD is equivalent to a 2-(v, k, λ) design. A t -design is called *simple* if it contains no repeated blocks.

By elementary counting, it can be shown that if $s < t$, a t -(v, k, λ) design is also an s -(v, k, μ) design, where

$$\mu = \frac{\lambda \binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

Since μ must be an integer, this equation yields a necessary condition for existence of the t -design, for any $s < t$. We also observe that if we take all the blocks of a t -(v, k, λ) design through a point x , and delete x , we get a $(t-1)$ -($v-1, k-1, \lambda$) design.

Much less is known about the existence of t -(v, k, λ) designs with $t \geq 3$ as compared to BIBDs. For $t = 3$, there are several infinite families known. For any prime power q , and for any $d \geq 2$, there exists a 3-($q^d + 1, q + 1, 1$) design, known as an *inversive geometry*. When $d = 2$, these designs are known as *inversive planes*. A 3-($v, 4, 1$) design is called a *Steiner quadruple system* and is denoted SQS(v). These designs exist for all $v \equiv 2$ or 4 modulo 6.

The major existence result for t -designs is Teirlinck's theorem, proved in 1987, which guarantees the existence of simple t -designs for arbitrarily large t . It should be noted that the λ -values of these designs are extremely large.

Theorem 2.1 (Teirlinck's Theorem). *A t -($v, t + 1, ((t + 1)!)^{2t+1}$) design exists if $v \geq t + 1$ and $v \equiv t$ modulo $((t + 1)!)^{2t+1}$.*

In contrast, for designs with $\lambda = 1$, examples are known only for $t \leq 5$. Construction of a 6-($v, k, 1$) design remains one of the outstanding open problem in the study of t -designs.

Even for $t = 4$ and 5, only a few examples of t -($v, k, 1$) designs are known. The known examples with $v \leq 30$ are the following: 4-(11, 5, 1), 5-(12, 6, 1), 4-(23, 7, 1), 5-(24, 8, 1), 4-(27, 6, 1), and 5-(28, 7, 1). For $t = 6$ and $v \leq 30$, the only t -(v, k, λ) designs known to exist are those having parameters 6-(14, 7, 4), 6-(20, 9, 112), 6-(22, 7, 8), and 6-(30, 7, 12).

There is a generalization of Fisher's inequality to t -designs, which is due to Ray-Chaudhuri and Wilson. If a t -(v, k, λ) design exists, where $t = 2s$ is even, then the number of blocks $b \geq \binom{v}{s}$.

3 AUTOMORPHISMS AND ISOMORPHISMS OF DESIGNS

Two (v, k, λ) -BIBDs or t -designs are said to be *isomorphic* if there exists a bijection of the respective point sets that preserves blocks. An *automorphism* of a design is an isomorphism with itself. The set of all automorphisms of a design forms a group under functional composition, called the *automorphism group*.

Some specific types of automorphisms have been studied extensively. An automorphism of a design on v points consisting of a single cycle of length v is called a *cyclic* automorphism. A symmetric BIBD having a cyclic automorphism is equivalent to a *difference set* in the group \mathbf{Z}_p .

A major result concerning difference sets was proved by Hall and Ryser in 1951.

Theorem 3.1 (Multiplier Theorem). *Suppose that there exists a symmetric (v, k, λ) -BIBD having a cyclic automorphism. Suppose that p is a prime such that $p > \lambda$, p is relatively prime to v , and p divides $k - \lambda$. Then the function $x \rightarrow px$ is also an automorphism of the BIBD.*

It is a long-standing open question as to whether the condition $p > \lambda$ in Theorem 3.1 is really necessary. As an infinite class of difference sets, we mention the famous result of Singer which proves that the projective plane $\text{PG}(2, q)$ admits a cyclic automorphism.

Nonsymmetric BIBDs with cyclic automorphisms have also been studied extensively. For example, there is an $\text{STS}(v)$ having a cyclic automorphism for all $v \equiv 1$ or 3 modulo 6 , $v \neq 9$.

In the cases where BIBDs or t -designs are known to exist, it is often useful to enumerate the nonisomorphic designs with a given parameter set. For example, the number of nonisomorphic Steiner triple systems of order v have been enumerated for $v \leq 15$. Up to isomorphism, there is a unique design of order 3, 7, and 9; there are precisely two nonisomorphic designs of order 13, and 80 of order 15. At that point, an explosion occurs: The number of nonisomorphic $\text{STS}(19)$ exceeds 2,000,000. There is also the asymptotic result of Wilson that the number of nonisomorphic $\text{STS}(v)$ is at least $(e^{-5}v)^{v^2/6}$.

A very useful table of parameters of BIBDs with $r \leq 41$ has been published by Mathon and Rosa [9]. This table lists all admissible parameter sets in the given range, together with existence, enumeration and resolvability results. A list of parameter sets of t -designs with $v \leq 30$, together with existence results, has been tabulated by Chee, Colbourn and Kreher [6].

4 PAIRWISE BALANCED DESIGNS

A *pairwise balanced design* (v, K, λ) -PBD is an ordered pair $(\mathbf{X}, \mathcal{A})$ where \mathbf{X} is a set (of *points*) of size v and \mathcal{A} is a collection of subsets of \mathbf{X} (called *blocks*)

with the property that every pair of elements of \mathbf{X} occur together in exactly λ blocks and that for every block $A \in \mathcal{A}$, $|A| \in K$. Note that a (v, k, λ) -BIBD is a special case of a PBD in which the blocks are only permitted to be of one size k . When $\lambda = 1$, it is sometimes omitted from the notation, and the design is called a (v, K) -PBD. A pairwise balanced design with $\lambda = 1$ is also sometimes called a *finite linear space*.

One can observe that the existence of a (v, K) -PBD (with $v > 0$) implies that

$$v \equiv 1 \pmod{\alpha(K)} \quad \text{and} \quad v(v-1) \equiv 0 \pmod{\beta(K)}, \quad (2)$$

where $\alpha(K)$ is the greatest common divisor of the integers $\{k-1 : k \in K\}$ and $\beta(K)$ is the greatest common divisor of the integers $\{k(k-1) : k \in K\}$. The following theorem of Wilson shows that the above conditions are “asymptotically sufficient” for the existence of a (v, K) -PBD:

Theorem 4.1 (Wilson’s Theorem). *Given K , there exists a constant c_k such that a (v, K) -PBD exists for all $v \geq c_k$ that satisfy the congruences $v \equiv 1 \pmod{\alpha(K)}$ and $v(v-1) \equiv 0 \pmod{\beta(K)}$.*

As an example suppose that $K = \{4, 7\}$. Then, $\alpha(K) = 3$ and $\beta(K) = 6$. Wilson’s Theorem guarantees the existence of a $(v, \{4, 7\})$ -PBD for all sufficiently large $v \equiv 1 \pmod{3}$. It should be noted here that the constant c_k in Wilson’s theorem is, in general unspecified. In practice, considerable further work is usually required to obtain a concrete upper bound on c_k (e.g., when $K = \{4, 7\}$, it can be shown that $c_k = 22$).

A concept that plays an important role in the construction of BIBDs is that of a *group divisible design* (or GDD). A (K, λ) -GDD denotes a triple $(\mathbf{X}, \mathcal{G}, \mathcal{A})$, where

1. \mathbf{X} is a set (of points),
2. \mathcal{G} is a partition of \mathbf{X} into subsets (called *groups*),
3. \mathcal{A} is a family of subsets of \mathbf{X} (called *blocks*) such that a group and a block contain at most one common point,
4. every pair of points from distinct groups occurs in exactly λ blocks.

If $\lambda = 1$, a (K, λ) -GDD is often denoted by K -GDD. The *group-type* (or *type*) of a GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. Usually, an “exponential notation” is used to describe the type of a GDD: A GDD of type $t_1^{u_1} t_2^{u_2} \dots t_k^{u_k}$ is a GDD where there are u_i groups of size t_i for $1 \leq i \leq k$. A *transversal design* $\text{TD}(k, n)$ is a k -GDD of type n^k (i.e., one having k groups of size n and uniform block size k). One other common notation is $\text{GD}(K, \lambda, M, v)$, which denotes a (K, λ) -GDD in which $|\mathbf{X}| = v$, and $|G| \in M$ for every group $G \in \mathcal{G}$.

If all groups $G \in \mathcal{G}$ have size $|G| \geq 2$, then $(\mathbf{X}, \mathcal{G} \cup \mathcal{A})$ is a PBD. Also, note that if \mathcal{G} consists of all singleton subsets of \mathbf{X} , then $(\mathbf{X}, \mathcal{A})$ is a PBD if and only if $(\mathbf{X}, \mathcal{G}, \mathcal{A})$ is a GDD.

There are numerous recursive constructions for GDDs and PBDs. The following theorem of Wilson gives the flavor of many of these constructions:

Theorem 4.2 (The Fundamental Construction). *Let $(X, \mathcal{G}, \mathcal{A})$ be a GDD, and let s_x be a positive integral weight assigned to each point $x \in X$. Let $(S_x : x \in X)$ be pairwise disjoint sets with $|S_x| = s_x$. With the notation*

$$S_Y = \bigcup_{x \in Y} S_x$$

for $Y \subseteq X$, put

$$X^* = S_X \quad \text{and} \quad \mathcal{G}^* = \{S_G : G \in \mathcal{G}\}.$$

Suppose that for each block $A \in \mathcal{A}$, a GDD $(S_A, \{S_x : x \in A\}, B_A)$ exists and denote $A^* = \bigcup_{A \in \mathcal{A}} B_A$. Then $(X^*, \mathcal{G}^*, A^*)$ is a GDD.

An extremely important idea in this area is that of closure. Given a set K of positive integers, let $\mathbf{B}(K)$ denote the set of positive integers v for which there exists a (v, K) -PBD. The mapping $K \rightarrow \mathbf{B}(K)$ is a *closure operation* on the set of subsets of the positive integers, as it satisfies the properties

1. $K \subseteq \mathbf{B}(K)$,
2. If $K_1 \subseteq K_2$, then $\mathbf{B}(K_1) \subseteq \mathbf{B}(K_2)$,
3. $\mathbf{B}(\mathbf{B}(K)) = \mathbf{B}(K)$.

The set $\mathbf{B}(K)$ is called the *closure* of the set K . If K is any set of positive integers, then K is *PBD-closed* (or *closed*) if $\mathbf{B}(K) = K$. A consequence of Theorem 4.1 is that if K is a closed set, then there exists a finite subset $J \subseteq K$ such that $K = \mathbf{B}(J)$. This set J is called a *generating set* for the PBD-closed set K . If J is a generating set for K and if $s \in J$ is such that $J \setminus \{s\}$ is also a generating set for K , then s is said to be *inessential* in K ; otherwise, s is said to be *essential*. A generating set consisting of essential elements is called a *basis*.

Another concept that plays an important role in many recursive constructions is the idea of designs with a “hole,” which are known as *incomplete designs*. For PBDs with $\lambda = 1$, we have the following definition: An *incomplete* (v, w, K) -PBD is an ordered triple (X, Y, \mathcal{A}) where the following properties are satisfied:

1. X is a set (of *points*).
2. Y is a subset of X (called the *hole*).
3. \mathcal{A} is a set of subsets of X (called *blocks*).
4. $|A| \in K$ for every block $A \in \mathcal{A}$.
5. No block contains two points from the hole.

6. Every pair of points $\{x, y\}$, not both in the hole, occurs in a unique block.

Of course, holes may be filled in: If an incomplete (v, w, K) -PBD exists and a (w, K) -PBD exists, then so does a (v, K) -PBD exist.

Just as BIBDs can be generalized to t -designs, so can PBDs be generalized, to designs called *t -wise balanced designs* or t BDs. Here, we require that every t -subset of points occurs in exactly λ blocks. We will use the notation (v, K, λ) - t BD or $S_\lambda(t, K, v)$.

5 LATIN SQUARES

A *Latin square* of side n is an $n \times n$ array based on some set S of n symbols with the property that every row and every column contains every symbol exactly once. Obviously, any permutation of the rows, columns or symbols of a Latin square A results in a Latin square. Two Latin squares A and B are *equivalent* if it is possible to obtain B from A by some sequence of permutations on the rows, columns or symbols of A . Otherwise, they are *inequivalent*. All Latin squares of order 3 are equivalent, but for every order $n > 3$, there exist inequivalent Latin squares of order n . In fact, the maximum number of inequivalent Latin squares of order n approaches ∞ as $n \rightarrow \infty$.

A Latin square A is *standardized* if the first row of A is $1, 2, \dots, n$. It is clear that every Latin square is equivalent to a standardized Latin square. A *transversal* in a Latin square is a set of n cells, one from each row and each column, that contain each symbol exactly once. If the main diagonal of a Latin square of side n is $(1, 2, \dots, n)$, then the Latin square is said to be *idempotent*; if it is (a, a, \dots, a) (for some $a \in \{1, 2, \dots, n\}$), then it is termed *unipotent*. Obviously, an idempotent Latin square has a transversal (the main diagonal). For every even order n , there exists a Latin square of order n having no transversals. However, it is an open question as to whether every Latin square of odd order has a transversal.

A *subsquare* of order r in a Latin square of order n is a set of r^2 cells that comprise the intersection of r rows and r columns and that is itself a Latin square of order r (on some subset of r symbols). Note that every Latin square has subsquares of order 0 and order 1.

Two Latin squares $A = (a_{ij})$ and $B = (b_{ij})$ of order n are said to be *orthogonal* if the n^2 ordered pairs (a_{ij}, b_{ij}) for $1 \leq i, j \leq n$ are all distinct. Clearly, the relation of orthogonality is symmetric. A set of Latin squares $\{A_1, A_2, \dots, A_k\}$ is called a set of k *mutually orthogonal Latin squares* (MOLS, for short) if A_i and A_j are orthogonal for all $i, j \in \{1, \dots, k\}$ and $i \neq j$. It is customary to denote by $N(n)$ the maximum number of MOLS of order n . For $n \geq 2$, $N(n) \leq n - 1$; we usually define $N(0) = N(1) = \infty$.

It is known that the existence of $n - 1$ MOLS of order n is equivalent to the existence of a projective plane of order n (an $(n^2 + n + 1, n + 1, 1)$ -BIBD).

It is also easy to show that the existence of a set of $k - 2$ mutually orthogonal Latin squares of order n is equivalent to the existence of a transversal design $\text{TD}(k, n)$. Another equivalent formulation of a set of $k - 2$ MOLS of order n is an *orthogonal array* $\text{OA}(n, k)$. This is a $k \times n^2$ array of n symbols such that any two rows contain all n^2 -ordered pairs of symbols once each.

It is straightforward to show that taking the direct product of orthogonal Latin squares preserves orthogonality. By this we mean that if A_1 is orthogonal to A_2 and B_1 is orthogonal to B_2 , then the direct product $A_1 \times B_1$ is orthogonal to $A_2 \times B_2$. It follows that if $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, where the p_i are distinct primes and each $e_i \geq 1$, then $N(n) \geq \min\{p_i^{e_i} - 1 : 1 \leq i \leq r\}$. Many other lower bounds on $N(n)$ are known; a table of lower bounds on $N(n)$ for $n \leq 10,000$ has been compiled by Brouwer [5].

The famous result of Chowla, Erdős, and Straus states that $N(n) \rightarrow \infty$ as $n \rightarrow \infty$. Hence, we can define $n_k = \max\{s : N(s) < k\}$, and n_k is finite for any k . It is known that $n_2 = 6$, $n_3 \leq 10$, $n_4 \leq 42$, $n_5 \leq 62$, and $n_6 \leq 76$. Upper bounds on n_k have been proven for other values of k , as well.

The *transpose* of a Latin square A , denoted A^T , is the Latin square that results from A when the role of rows and columns are exchanged (i.e., $A^T(i, j) = A(j, i)$). A Latin square is called *self-orthogonal* if it is orthogonal to its transpose. Self-orthogonal Latin squares exist for all orders $n \neq 2, 3, 6$. More generally, one can obtain $3! = 6$ *conjugates* of a Latin square by interchanging the roles of rows, columns and symbols.

A Latin square A of order n is *symmetric* if $A = A^T$. Two symmetric Latin squares A and B of order n can obviously never be orthogonal. However, A and B are defined to be *orthogonal symmetric* Latin squares if they are idempotent and if for any two elements x and y there exists at most one ordered pair (i, j) with $i < j$ such that $A(i, j) = x$ and $B(i, j) = y$. It is easy to see that symmetric idempotent Latin squares exist only for odd orders. Analogous to the nonsymmetric case one can define $\nu(n)$ to be the maximum number of symmetric orthogonal Latin squares of order n . It is known that $\nu(7) = 3$, $\nu(9) = 4$, and that $\nu(n) \geq 5$ for all odd $n \geq 11$, $n \neq 15$.

6 GRAPHS

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set of elements called *vertices* and $E(G)$ is a finite set of distinct unordered pairs of distinct elements of $V(G)$ called *edges*. The number of vertices of G is called the *order* of G , and will usually be denoted by ν , while the number of edges (sometimes called the *size* of the graph) will be denoted by ϵ . Often, the edge $\{u, v\}$ will be denoted as uv or (u, v) .

If $e = uv$ is an edge of G , then u and v are called the two *ends* of e . The ends of an edge are said to be *incident* with the edge, and vice versa. Two vertices that are incident with a common edge are *adjacent* as are two edges that are incident with a common vertex. A set of vertices of which no two

are adjacent is called an *independent set* of vertices, while a set of vertices of which every pair is adjacent is called a *clique*. A *k-clique* is a clique with k vertices.

The *degree* or *valence* of a vertex v in G is the number of edges incident with v and is denoted by $d(v)$. The maximum and minimum degrees of the vertices in the graph G are denoted $\Delta(G)$ and $\delta(G)$, respectively. If all the vertices of a graph have the same degree, then the graph is called *regular*. If each degree is k , then G is called *k-regular*.

A graph $H = (V(H), E(H))$ is a *subgraph* of the graph $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is called a *spanning* subgraph of G . A *k-factor* of G is a spanning k -regular subgraph of G . A one-factor is also called a *perfect matching*. A partition of the edges of a k -regular graph into one-factors is called a *one-factorization*. A one-regular subgraph is termed a *matching*. A partition of the edges of a graph into matchings is called an *edge-coloring* of the graph and the set of edges in one of the matchings is referred to as a *color class*.

A *path* of length $t - 1$ in a graph G is a sequence of t distinct vertices $v_1 v_2 \dots v_t$ such that $v_i v_{i+1}$ is an edge of G for $1 \leq i \leq t - 1$. A graph G is *connected* if any two vertices are joined by a path. A *cycle* or *circuit* of length t in a graph G is a sequence of t distinct vertices $v_1 v_2 \dots v_t$ such that $v_i v_{i+1}$ is an edge of G for $1 \leq i \leq t - 1$ and $v_t v_1$ is also an edge of G . A circuit is called a *hamiltonian* circuit if its length is equal to v . A *tree* is a connected graph that contains no cycle as a subgraph.

There are two graphs that are of particular importance in design theory. The graph K_n is the *complete graph* on n vertices and is defined by the fact that every pair of vertices are adjacent. A *multipartite graph* is one in which the vertex set can be partitioned into sets (called *parts*) in such a way that each edge joins a vertex in two different parts. A *bipartite graph* is a multipartite graph with two parts. The *complete multipartite graph* K_{n_1, \dots, n_s} is a multipartite graph where the vertex set is partitioned into s parts having sizes n_1, \dots, n_s , where every vertex is adjacent to every vertex in a different part.

As one example of the connection between graphs and designs, we note that a $(v, k, 1)$ -BIBD is equivalent to an edge-decomposition of K_v into k -cliques and, in particular, that a Steiner triple system of order v is equivalent to an edge-decomposition of K_v into triangles. More generally, a k -GDD of type $\{n_1, \dots, n_s\}$ is an edge-decomposition of K_{n_1, \dots, n_s} into k -cliques. It is also quite easy to show that a Latin square of order n is equivalent to a one-factorization of the graph $K_{n,n}$ and that an idempotent symmetric Latin square of side $2n - 1$ is equivalent to a one-factorization of the graph K_{2n} .

There is also considerable interest in finding edge-decompositions of complete graphs into isomorphic subgraphs, such as cycles of a specified length or paths of a specified length. An edge-decomposition of K_v into cycles of length d is called a *d-cycle system* of order v .

The generalization of graphs where each edge has a direction assigned to it is called a *directed graph* or *digraph*. If one starts with the undirected graph

K_v and then assigns a direction to each edge, the resulting digraph is called a *tournament* on v vertices.

This is only a small sampling of the terms and ideas of graph theory. For further information on graph theory, the reader is referred to the textbook by Bondy and Murty [4], for example.

7 REMARKS

Combinatorial design theory is an extremely diverse and active area of discrete mathematics, and this introductory chapter presents only a few of the basic definitions and concepts. In the last few years, several books devoted to the subject have been published. Most of the results in this introduction are proved in one or more of these books. They are Anderson [1], Batten [2], Beth, Jungnickel and Lenz [3], Hall [7], Hughes and Piper [8], Street and Street [10], and Wallis [11].

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