15 Boxes Puzzle

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Abstract

In this expository note, I discuss a "15 boxes puzzle" from Gil Kalai's blog.

1 Introduction

On September 3, 2024, Gil Kalai posted an interesting probability puzzle on his blog "Combinatorics and More" [1]. Here is the description of the puzzle:

> Andrew and Barbara are playing a game. Fifteen boxes are arranged in a 3-by-5 grid, labeled with the letters A through O, as shown below.
>
> A B C D E
> F G H I J
> K L M N O
>
> The organizers of the game have chosen two boxes randomly and placed a prize in each of the two chosen boxes. (It is the same prize in both boxes.) Andrew and Barbara each secretly submit to the organizers a search order, meaning the order in which they wish to examine the boxes. Andrew decides to search the boxes row by row, from left to right, top to bottom. His search order is therefore ABCDEFGHIJKLMNO. Barbara, however, decides to search the boxes column by column, from top to bottom, left to right. So her search order is AFKBGLCHMDINEJO.
>
> After receiving the two search orders, the organizers now step through the two search orders in parallel. The first player to hit one of the prizes is declared the winner (ties are possible).

We want to compute the probabilities that Andrew and Barbara win.

It is interesting to first consider the simpler puzzle where there is only one prize placed in the boxes. Let's write the the two search orders, one above the other:

Andrew	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Barbara	A	F	K	B	G	L	C	H	M	D	I	N	E	J	O

We can see that there are three letters that appear in the same positions in the two search orders: A, H and O. There are six letters that appear earlier in Andrew's search order: B, C, D, E, I and J. The remaining six letters, namely F, G, K, L, M, N, appear earlier in Barbara's search order. It follows that Andrew and Barbara each have a winning probability of $6/15 = 2/5$, and the probability of a tie is $3/15 = 1/5$.

The situation is more complicated when there are two prizes. But it is possible to compute the relevant probabilities by a fairly straightforward case analysis. There are $\binom{15}{2} = 105$ distributions of the two prizes to consider. We will list these in lexicographic order as

$$\text{AB, AC, } \ldots, \text{ AO, BC, BD, } \ldots, \text{ BO, CD, } \ldots, \text{ CO, } \ldots, \text{ NO.}$$

When we do our case analysis, we will break down the 105 cases as follows:

case Ax: 14 pairs of the form Ax, where x > A;

case Bx: 13 pairs of the form Bx, where x > B;

cases Cx to Lx: the number of pairs in these cases (in order) is 12, 11, . . . , 3;

case Mx: 2 pairs of the form Mx, where x > M;

case Nx: 1 pair, NO.

We analyze all the cases in turn. First, case Ax results in a tie for all 14 possible pairs, as A is in the first position in both search orders.

BF is a tie, because B is second in Andrew's search order and F is second in Barbara's search order. The other Bx possibilities are wins for Andrew.

C is in the third position in Andrew's search order, and in a later position in Barbara's search order. So we see that CK is a tie; CF is a win for Barbara; and the remaining Cx possibilities are wins for Andrew.

D is in the fourth position in Andrew's search order, and in a later position in Barbara's search order. DF and DK are wins for Barbara, while the other nine Dx possibilities are wins for Andrew.

E is in the fifth position in Andrew's search order, and in a later position in Barbara's search order. EG is a tie; EF and EK are wins for Barbara; and and the remaining Ex possibilities are wins for Andrew.

F is in the sixth position in Andrew's search order and it is in an earlier position in Barbara's search order. So all Fx possibilities are wins for Barbara. Similar reasoning shows that all the Gx possibilities are also wins for Barbara.

Table 1: Case analysis of winning probabilities

case	wins for Andrew	wins for Barbara	ties	total
Ax	0	0	14	14
Bx	12	0	1	13
Cx	10	1	1	12
Dx	9	2	0	11
Ex	7	2	1	10
Fx	0	9	0	9
Gx	0	8	0	8
Hx	0	2	5	7
Ix	3	2	1	6
Jx	2	3	0	5
Kx	0	4	0	4
Lx	0	3	0	3
Mx	0	2	0	2
Nx	0	1	0	1
total	43	39	23	105

H is in the eighth position in both search orders. HK and HL are wins for Barbara, while the remaining Hx possibilities are ties.

I is in the ninth position in Andrew's search order, and in a later position in Barbara's search order. IM is a tie; and IK and IL are wins for Barbara. IJ, IN and IO are wins for Andrew.

J is in the tenth position in Andrew's search order, and in a later position in Barbara's search order. JK, JL and JM are wins for Barbara, while JN and JO are wins for Andrew.

The remaining cases, namely Kx, Lx, Mx and Nx, constituting ten pairs, are all wins for Barbara.

A summary of the analysis can be found in Table 1. We conclude that Andrew's probability of winning is 43/105, Barbara's winning probability is 39/105, and the probability of a tie is 23/105.

References

[1] G. Kalai. Test Your Intuition 56: Fifteen Boxes Puzzle. `https://gilkalai.wordpress.com/2024/09/03/test-your-intuition-56-fifteen-boxes-puzzle/`