1 Balanced Incomplete Block Designs (BIBDs)

Suppose $1 < k < v$ are integers and $\lambda \geq 1$ is an integer.

A $(v, k, \lambda)$-BIBD is a collection of $k$-subsets (called blocks) of a $v$-set (whose elements are called points), such that every pair of points is in exactly $\lambda$ blocks.

**Question:** for what choices of parameters $(v, k, \lambda)$ can we construct a $(v, k, \lambda)$-BIBD?

The case $k = 2$ is trivial— take every pair $\lambda$ times. For example, a $(3, 2, 1)$-BIBD has blocks $\{1, 2\}, \{1, 3\}, \{2, 3\}$.

The blocks $\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 6\}, \{3, 4, 5\}, \{2, 5, 7\}, \{3, 6, 7\}, \{2, 4, 6\}$ form a $(7, 3, 1)$-BIBD.

An alternative construction for a $(7, 3, 1)$-BIBD: The points are the elements of $\mathbb{Z}_7$. Start with the base block $\{0, 1, 3\}$. Then develop the base block modulo 7, obtaining the blocks $\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}$. We add 1 (mod 7) to every point in a block to get the next block.

This works because the base block contains every difference modulo 7 exactly once: $0 - 1 = 6, 1 - 0 = 1, 0 - 3 = 4, 3 - 0 = 3, 1 - 3 = 5, 3 - 1 = 2$. 

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Combinatorial Structures
Part 1: Block Designs
CS 858 Notes

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Two other parameters in a \((v, k, \lambda)\)-BIBD are \(r\) and \(b\). Every point occurs in \(r\) blocks, where \(r = \lambda(v-1)/(k-1)\). The total number of blocks is \(b = vr/k\). Note that \(r\) and \(b\) must be integers.

**Example:** In a \((7, 3, 1)\)-BIBD, \(r = 1 \times 6/2 = 3\) and \(b = 7 \times 3/3 = 7\).

**Example:** If a \((6, 3, 2)\)-BIBD exists, then \(r = 2 \times 5/2 = 5\) and \(b = 6 \times 5/3 = 10\).

Sometimes we write the parameters of a BIBD as \((v, b, r, k, \lambda)\).

**Example:** If an \((11, 3, 1)\)-BIBD exists, then \(r = 5\) and \(b = 11 \times 5/3 = 55/3\). The value \(b\) is not an integer, so the BIBD does not exist.

**Fisher’s Inequality:** If a \((v, b, r, k, \lambda)\)-BIBD exists, then \(b \geq v\). (Equivalently, \(r \geq k\).)

**Example:** If a \((16, 6, 1)\)-BIBD exists, then \(r = 3\) and \(b = 8\). Therefore, this BIBD does not exist, because Fisher’s Inequality is violated.

If a \((v, k, \lambda)\)-BIBD has \(b = v\) (equivalently, \(r = k\)), then the BIBD is called a symmetric BIBD and it is denoted an SBIBD.

**Theorem:** Any two blocks in a \((v, k, \lambda)\)-SBIBD contain exactly \(\lambda\) common points.

**Example:** A \((7, 3, 1)\)-BIBD is symmetric. Therefore, any two blocks intersect in exactly one point.

**Example:** An \((11, 5, 2)\)-BIBD is symmetric. It can be constructed by developing the base block \(\{1, 3, 4, 5, 9\}\) modulo 11. Any two blocks of this BIBD intersect in exactly two points. The base block consists of the quadratic residues (i.e., perfect squares) modulo 11: \(1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 5\) and \(5^2 = 3\), where all arithmetic is modulo 11.

An \((n^2 + n + 1, n + 1, 1)\)-BIBD is called a projective plane of order \(n\). It is a symmetric BIBD, so every pair of blocks intersect in exactly one point.

A projective plane of order \(n\) exists if \(n\) is a prime power. Therefore projective planes of orders 2, 3, 4, 5, 7, 8 and 9 all exist. There is no projective plane of order 6 or 10.
Here is a construction for a projective plane of order \( q \), where \( q \) is a prime power. Let \( \mathbb{F}_q \) denote the finite field of order \( q \) (side comment: \( \mathbb{F}_q \) is the same thing as \( \mathbb{Z}_q \) if \( q \) is prime). The points of the design are the 1-dimensional subspaces of \( (\mathbb{F}_q)^3 \) and the blocks are the 2-dimensional subspaces of \( (\mathbb{F}_q)^3 \).

A projective plane of order \( q \), where \( q \) is a prime power, can also be constructed from a base block in \( \mathbb{Z}_{q^2+q+1} \).

**Example:** \{7, 14, 3, 6, 12\} is a base block (modulo 21) for a projective plane of order 4.

**Bruck-Ryser-Chowla Theorem:** Suppose that a \((v, k, \lambda)\)-SBIBD exists. Then (1) if \( v \) is even, then \( k - \lambda \) is a perfect square, and (2) if \( v \) is odd, then the equation \( x^2 = (k - \lambda)y^2 + (\lambda - 1)^{v-1}/2\lambda z^2 \) has a nontrivial integral solution (i.e., a solution \((x, y, z)\) where \( x, y \) and \( z \) are integers that are not all equal to 0).

**Example:** A \((22, 7, 2)\)-SBIBD does not exist, because 22 is even and \( 7 - 2 = 5 \) is not a perfect square.

**Example:** We can use the Bruck-Ryser-Chowla Theorem to show that a projective plane of order 6 does not exist. Such a BIBD would be a \((43, 7, 1)\)-SBIBD. If it existed, then the equation \( x^2 = 6y^2 + z^2 \) would have a nontrivial integral solution. It can be shown that the equation has no nontrivial integral solution, which means that the BIBD does not exist.

An \((n^2, n, 1)\)-BIBD is called an **affine plane of order** \( n \). It has \( r = n + 1 \) and \( b = n^2 + n \).

A projective plane of order \( n \) is equivalent to an affine plane of order \( n \).

**Example:** A projective plane of order 3 can be constructed by developing the base block \{0, 1, 3, 9\} modulo 13. We obtain the following blocks:

\[
\{0, 1, 3, 9\}, \{1, 2, 4, 10\}, \{2, 3, 5, 11\}, \{3, 4, 6, 12\}, \{4, 5, 7, 0\}, \{5, 6, 8, 1\}, \{6, 7, 9, 2\}, \{7, 8, 10, 3\}, \{8, 9, 11, 4\}, \{9, 10, 12, 5\}, \{10, 11, 0, 6\}, \{11, 12, 1, 7\}, \{12, 0, 2, 8\}. \]

To construct an affine plane of order 3, pick a block in the projective plane, say \{0, 1, 3, 9\} and delete the points in this block from all other blocks. Since \{0, 1, 3, 9\} intersects every other block in exactly one point, we are deleting one point from every other block. We obtain the following 12 blocks:

\[
\{2, 4, 10\}, \{2, 5, 11\}, \{4, 6, 12\}, \{4, 5, 7\}, \{5, 6, 8\}, \{6, 7, 2\}, \{7, 8, 10\}, \{8, 11, 4\}, \{10, 12, 5\}, \{10, 11, 6\}, \{11, 12, 7\}, \{12, 2, 8\}. \]

These are the blocks of an affine plane of order 3 on the nine points 2, 4, 5, 6, 7, 8, 10, 11, 12. Note that this is a \((9, 3, 1)\)-BIBD.
The above-described process can be reversed. The 12 blocks of the affine plane can be partitioned into four parallel classes, each of which consists of three disjoint blocks. Add a new point \( x_i \) to each block in the \( i \)th parallel class, for \( 1 \leq i \leq 4 \). Finally, add a new block \( \{x_1, x_2, x_3, x_4\} \).

A Steiner triple system is a \((v,3,1)\)-BIBD. It is also denoted as STS(v). It has \( r = (v-1)/2 \), so \( r \) is odd. Then \( b = (2r+1)r/3 \), so \( 3 \mid r \) or \( 3 \mid 2r + 1 \). Hence, \( r \equiv 0, 1 \pmod{3} \) and \( v \equiv 1, 3 \pmod{6} \) is a necessary condition for existence of an STS(v). We can also write \( b = v(v-1)/6 \).

**Example:** We have already constructed STS(7) and STS(9). An STS(13) has \( b = 26 \) blocks. It can be constructed by developing the two base blocks \( \{0,1,4\} \) and \( \{0,2,8\} \) modulo 13.

**Theorem:** An STS(v) exists for all \( v \equiv 1, 3 \pmod{6} \), \( v \geq 7 \).

A Hadamard design is a \((4n-1,2n-1,n-1)\)-BIBD. The Hadamard designs are is a symmetric BIBDs.

**Example:** We have already constructed a \((7,3,1)\)-BIBD and a \((11,5,2)\)-BIBD. These are Hadamard designs corresponding to \( n = 2 \) and \( n = 3 \), respectively.

Hadamard designs are known to exist for \( 2 \leq n \leq 166 \). The smallest unknown case is a \((667,333,166)\)-BIBD.

A Hadamard matrix of order \( 4n \) is a \( 4n \) by \( 4n \) matrix \( H \), whose entries are all \( \pm 1 \), which satisfies the property \( HH^T = 4nI_{4n} \) (where \( I_{4n} \) is the identity matrix of order \( 4n \)).

A Hadamard matrix of order \( 4n \) is equivalent to a \((4n-1,2n-1,n-1)\)-BIBD (i.e., a Hadamard design).

**Example:** We construct a Hadamard matrix of order 8 from a \((7,3,1)\)-BIBD. Recall that the BIBD has blocks \( \{0,1,3\} \), \( \{1,2,4\} \), \( \{2,3,5\} \), \( \{3,4,6\} \), \( \{4,5,0\} \), \( \{5,6,1\} \), \( \{6,0,2\} \). We first construct the incidence matrix of the BIBD. The rows are indexed by the points, the columns are indexed by the blocks, and an entry is 1 if the given point is a member of the given block, and 0, otherwise. The incidence matrix is as follows:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
\end{pmatrix}
\]
Now replace all 0’s by −1’s and adjoin a row and column of 1’s:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}
\]

The result is a Hadamard matrix of order 8.

It is a bit more complicated to construct a \((4n - 1, 2n - 1, n - 1)\)-BIBD from a Hadamard matrix of order \(4n\). First, the Hadamard matrix must be modified in a suitable manner so it contains a border of 1’s. Then the border can be stripped off and all −1’s are changed to 0’s.

2 \(t\)-designs

A \(t\)-(\(v\), \(k\), \(\lambda\))-design is a collection of \(k\)-subsets (called blocks) of a \(v\)-set (whose elements are called points), such that every \(t\)-subset of points is in exactly \(\lambda\) blocks. If \(t = 2\), we have a BIBD.

A Steiner quadruple system is a 3-(\(v\), 4, 1)-design. It is also denoted as SQS(\(v\)). An SQS(\(v\)) exists if and only if \(v \equiv 2, 4 \pmod{6}\).

Example: We construct an SQS(8). We start with two blocks: \{1, 2, 3, 4\} and \{5, 6, 7, 8\}. Next we divide these blocks into pairs as follows:

<table>
<thead>
<tr>
<th>{1, 2}</th>
<th>{1, 3}</th>
<th>{1, 4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{3, 4}</td>
<td>{2, 4}</td>
<td>{2, 3}</td>
</tr>
<tr>
<td>{5, 6}</td>
<td>{5, 7}</td>
<td>{5, 8}</td>
</tr>
<tr>
<td>{7, 8}</td>
<td>{6, 8}</td>
<td>{6, 7}</td>
</tr>
</tbody>
</table>

Now we form 12 blocks as follows:

<table>
<thead>
<tr>
<th>{1, 2, 5, 6}</th>
<th>{1, 3, 5, 7}</th>
<th>{1, 4, 5, 8}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 7, 8}</td>
<td>{1, 3, 6, 8}</td>
<td>{1, 4, 6, 7}</td>
</tr>
<tr>
<td>{3, 4, 5, 6}</td>
<td>{2, 4, 5, 7}</td>
<td>{2, 3, 5, 8}</td>
</tr>
<tr>
<td>{3, 4, 7, 8}</td>
<td>{2, 4, 6, 8}</td>
<td>{2, 3, 6, 7}</td>
</tr>
</tbody>
</table>

These 12 blocks, along with the original two blocks, form the desired SQS(8).
The preceding construction can be generalized to show that an SQS(2v) can be obtained from an SQS(v).

Another infinite class of 3-designs are the *inversive planes*, which are 3-(n^2 + 1, n + 1, 1)-designs. These designs are known to exist if n is a prime power.

If we fix a point x in an inversive plane, delete all blocks that do not contain x, and then delete x from all the remaining blocks, we get an affine plane.

Very few explicit examples of t-designs with t ≥ 4 are known. However, a result of Keevash from 2014 shows that t-(v, k, 1)-design exist for all t, albeit with enormously large values of v.