Putting Dots in Triangles

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the plan

In this talk, we give a complete answer to a simply stated combinatorial problem. The solution that we found is quite short, but perhaps surprising.

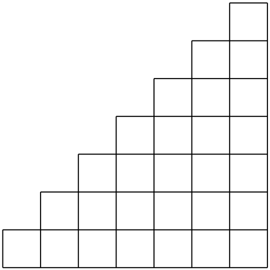
The main focus of this elementary talk is not the proof of the main result, but how we arrived at the proof, including a few wrong turns along the way.

After carrying out this research, we found that the problem had been solved previously using somewhat different techniques:

- G. Nivasch, E. Lev. Nonattacking queens on a triangle, *Mathematics Magazine*, 2005.
- P. Vaderlind, R.K. Guy, L.C. Larson. Problem 252 in *The Inquisitive Problem Solver*, 2002.

the problem

Consider a "triangle" of squares in a grid whose sides are n squares long, as illustrated by the following diagram, for which $n = 7$.



$\leftarrow n \rightarrow$

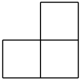
We denote by $N(n)$ the maximum number of dots that can be placed into the cells of the triangle such that each row, each column, and each diagonal parallel to the third side of the triangle contains at most one dot.

$n = 1$

$n = 1$

$$N(1) = 1$$

$n = 2$



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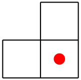
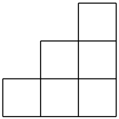
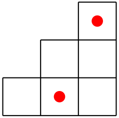


$$N(2) = 1$$

$n = 3$

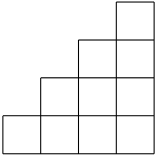


$n = 3$



$$N(3) = 2$$

$n = 4$



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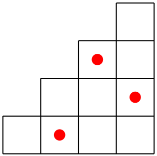
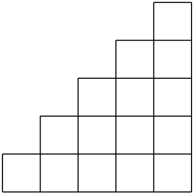


$$N(4) = 3$$

$n = 5$



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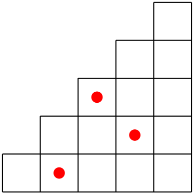
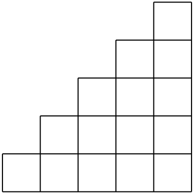


$$N(5) = 3$$

$N(5) \neq 4$



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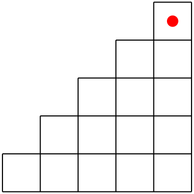

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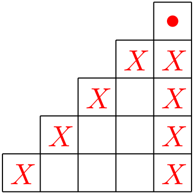

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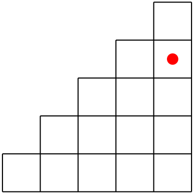

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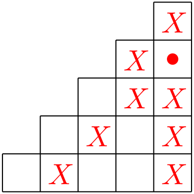

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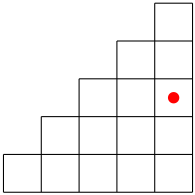

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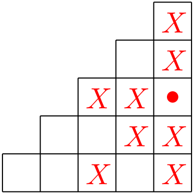

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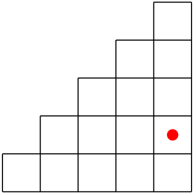

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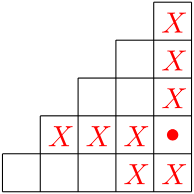

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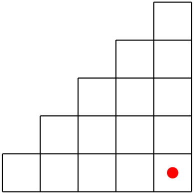

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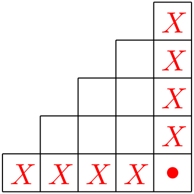
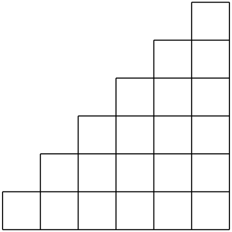


$n = 6$



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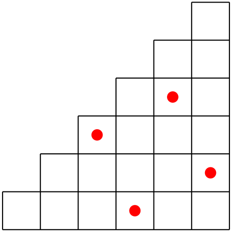
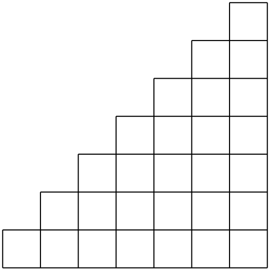
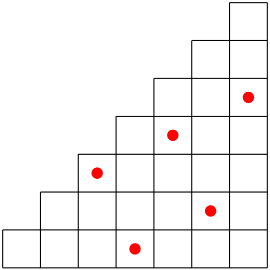


$$N(6) = 4$$

$n = 7$



$n = 7$



$$N(7) = 5$$

$N(n)$ for small values of n

n	1	2	3	4	5	6	7	8	9
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Conjecture: $N(n) = N_f(n)$, where

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Simplification:

$$N_f(n) = \left\lfloor \frac{2n+1}{3} \right\rfloor$$

a construction meeting the lower bound

- First, we show that $N(3t+1) \geq 2t+1$:
  1. Place a dot in the leftmost cell of the $(2t+1)$st row.
  2. Place t more dots, each two squares to the right and one square up from the previous dot.
  3. Place a dot in the $(t+2)$nd cell from the left in the bottom row.
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- Next, $N(3t+2) \geq N(3t+1) \geq 2t+1$ (add a row of empty cells).
- Finally, $N(3t) \geq N(3t+1) - 1 \geq 2t$ (delete the bottom row of cells, which contain at most one dot).

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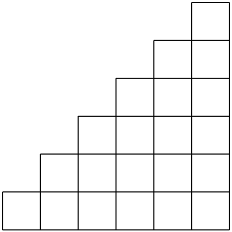
- An inductive proof seems promising, but we couldn't make the induction proof work out, despite trying various approaches.
- It is possible to prove some weak partial results such as the following: If there are two dots in the top three rows, then the total number of dots is at most $N(n-3) + 2$.

two dots in the top three rows



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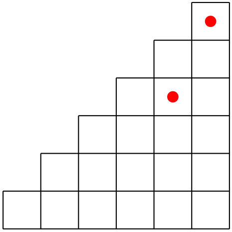

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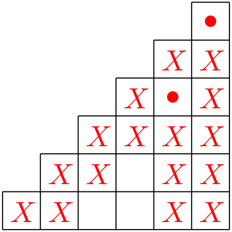

a triangle of side $n - 4$ remains

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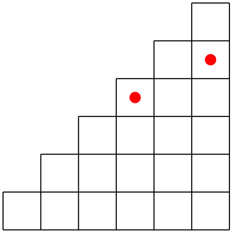

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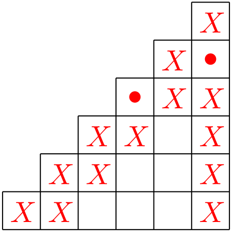
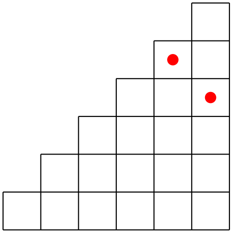
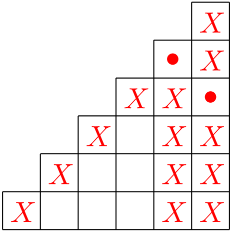


a triangle of side $n - 3$ remains

two dots in the top three rows

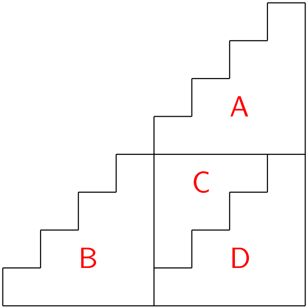


two dots in the top three rows



a triangle of side $n-3$ remains

a tiny step: a not-very-good upper bound



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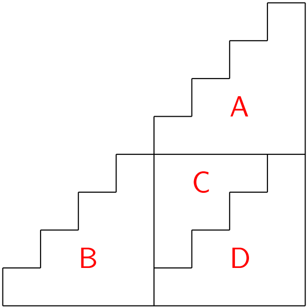
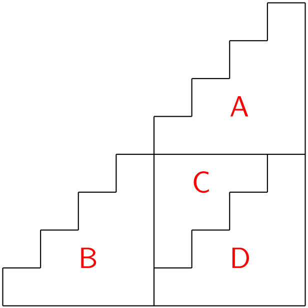


$$A + B + C \leq \frac{n}{2}$$
$$B + C + D \leq \frac{n}{2}$$
$$A + C + D \leq \frac{n}{2}$$

a tiny step: a not-very-good upper bound



$$\begin{aligned} A+B+C &\le \tfrac{n}{2} \\ B+C+D &\le \tfrac{n}{2} \\ A+C+D &\le \tfrac{n}{2} \end{aligned} \quad \Rightarrow \quad A+B+C+D \le \frac{3n}{4}$$

another dead end?

- A more refined analysis yields the result that $N(n) < 3n/4$ for all even $n > 4$ (note that $N(4) = 3 = 4 \times 3/4$).

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- Decomposing the triangle into $1 + 3 + 5 = 9$ smaller triangles also yielded the same bound of roughly $3n/4$.
- This is far from the conjectured bound of (roughly) $2n/3$.
- But, if we decompose the triangle into $n(n+1)/2$ individual cells, then we have an integer program which will yield the exact value of $N(n)$ (in principle, at least).

integer program formulation

The computation of $N(n)$ can be formulated as an integer program. Suppose we number the cells as indicated in the following diagram (where $n = 6$):

					$x_{1,1}$
				$x_{2,2}$	$x_{2,1}$
			$x_{3,3}$	$x_{3,2}$	$x_{3,1}$
		$x_{4,4}$	$x_{4,3}$	$x_{4,2}$	$x_{4,1}$
	$x_{5,5}$	$x_{5,4}$	$x_{5,3}$	$x_{5,2}$	$x_{5,1}$
$x_{6,6}$	$x_{6,5}$	$x_{6,4}$	$x_{6,3}$	$x_{6,2}$	$x_{6,1}$

Define $x_{i,j} = 1$ if the corresponding cell contains a dot; define $x_{i,j} = 0$ otherwise.

integer program formulation

The sum of the variables in each row, column, and diagonal is at most 1. This leads to constraints of the form

$$\sum_{j=1}^{i} x_{i,j} \leq 1, \qquad \text{for } i = 1, 2, \ldots, n$$

$$\sum_{i=j}^{n} x_{i,j} \leq 1, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$\sum_{i=k+1}^{n} x_{i,i-k} \leq 1, \qquad \text{for } k = 0, 1, \ldots, n-1.$$

Finally, $x_{i,j} \in \{0, 1\}$ for all i, j.

Objective function: Maximize $\sum x_{i,j}$ subject to the above constraints; this maximum is $N(n)$.

linear program formulation

The only change is that the variables can take on any real values in the closed interval $[0,1]$. So the constraints are

$$\sum_{j=1}^{i} x_{i,j} \leq 1, \qquad \text{for } i = 1,2,\ldots,n$$

$$\sum_{i=j}^{n} x_{i,j} \leq 1, \qquad \text{for } j = 1,2,\ldots,n$$

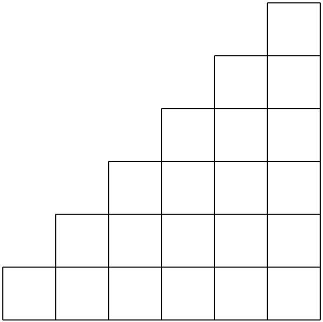
and

$$\sum_{i=k+1}^{n} x_{i,i-k} \leq 1, \qquad \text{for } k = 0,1,\ldots,n-1.$$

Finally, $\boxed{0 \leq x_{i,j} \leq 1 \text{ for all } i,j}$.

Objective function: Maximize $\sum x_{i,j}$ subject to the above constraints; call this maximum $LP(n)$.

solution of the linear program for $n = 6$



solution of the linear program for $n = 6$

					0
				0	$\frac{2}{7}$
			0	$\frac{5}{7}$	$\frac{2}{7}$
		$\frac{5}{7}$	0	0	$\frac{2}{7}$
	$\frac{2}{7}$	0	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
0	0	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	0

solution of the linear program for $n = 6$

					0
				0	$\frac{2}{7}$
			0	$\frac{5}{7}$	$\frac{2}{7}$
		$\frac{5}{7}$	0	0	$\frac{2}{7}$
	$\frac{2}{7}$	0	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
0	0	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	0

This solution is optimal, so $LP(6) = 4\frac{2}{7}$.

solutions to the LP for small values of n

n	$N(n)$	$LP(n)$	$LP(n) - N(n)$
4	3	3	0
5	3	$3\frac{3}{5}$	$\frac{3}{5}$
6	4	$4\frac{2}{7}$	$\frac{2}{7}$
7	5	5	0
8	5	$5\frac{5}{8}$	$\frac{5}{8}$
9	6	$6\frac{3}{10}$	$\frac{3}{10}$
10	7	7	0
11	7	$7\frac{7}{11}$	$\frac{7}{11}$
12	8	$8\frac{4}{13}$	$\frac{4}{13}$

another conjecture

Define

$$
\begin{aligned}
LP_f(3t) &= 2t + \frac{t}{3t+1} \\
LP_f(3t+1) &= 2t+1 \\
LP_f(3t+2) &= 2t+1+\frac{2t+1}{3t+2}
\end{aligned}
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LP Conjecture: $LP(n) = LP_f(n)$

a possible approach to a proof?

- Because $N(n)$ is an integer and $N(n) \leq LP(n)$, it is clear that

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- It is also easy to verify that

$$\lfloor LP_f(n) \rfloor = N_f(n).$$

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$$\lfloor LP_f(n) \rfloor = N_f(n).$$
- We already showed that $N(n) \geq N_f(n)$.

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- Now, suppose we could prove the LP Conjecture.

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- It is also easy to verify that
$$\lfloor LP_f(n) \rfloor = N_f(n).$$
- We already showed that $N(n) \geq N_f(n)$.
- Now, suppose we could prove the LP Conjecture.
- Then it would immediately follow that
$$N(n) = N_f(n).$$

is this going anywhere?

- We have a simple conjectured formula for $N(n)$ along with a simple construction that achieves the conjectured bound.

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- Actually, we have one very powerful weapon when dealing with LPs, namely, duality theory.

primal and dual LPs, and weak duality

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maximize	$c^T x$
subject to	$Ax \leq b$, $x \geq 0$.

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weak duality: The objective function value of the dual LP at any feasible solution is always greater than or equal to the objective function value of the primal LP at any feasible solution.

the dual LP

- Label the rows $r_1, r_2, \ldots, r_n$ such that r_i is the row containing i squares, and label the columns and diagonals similarly.

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- In fact, there is a bijection from the set of $n(n+1)/2$ cells to the set of triples

$$\mathcal{T} = \{(i,j,k) : i + j + k = 2n + 1, i, j, k \geq 1\}.$$

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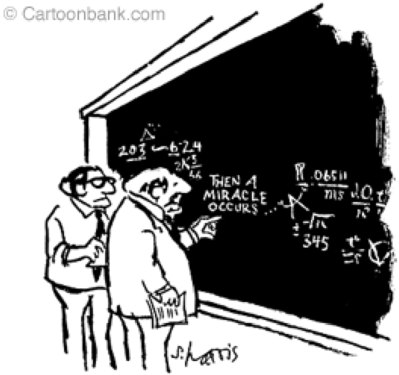
- The objective function is to minimize $\sum r_i + \sum c_j + \sum d_k$.

seeking divine intervention?



"I think you should be more explicit here in step two."

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a simpler proof strategy

- Suppose we prove that
  1. The solutions presented above are feasible for the dual LP, and
  2. The value of the objective function (for the dual LP) at these solutions is $LP_f(n)$.

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- This is sufficient to prove that $N(n) = N_f(n)$.
- Note that, using this approach, we do not have to prove the LP conjecture (namely, that $LP(n) = LP_f(n)$).

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- We have that

$$\begin{aligned} r_i + c_j + d_k &\geq \frac{i-t-1}{3t+1} + \frac{j-t-1}{3t+1} + \frac{k-t}{3t+1} \\ &= \frac{i+j+k-(3t+2)}{3t+1} \\ &= \frac{6t+3-(3t+2)}{3t+1} \\ &= 1. \end{aligned}$$

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- Therefore all constraints are satisfied.

computing the objective function

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- The value of the objective function is

$$
\begin{aligned}
&\frac{1}{3t+1}\left(\sum_{i=t+1}^{3t+1}(i-t-1)+\sum_{i=t+1}^{3t+1}(i-t-1)+\sum_{i=t}^{3t+1}(i-t)\right)\\
&=\frac{1}{3t+1}\left(\frac{2t(2t+1)}{2}+\frac{2t(2t+1)}{2}+\frac{(2t+1)(2t+2)}{2}\right)\\
&=\frac{(2t+1)(3t+1)}{3t+1}\\
&=2t+1\\
&=LP_f(3t+1).
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main result and proof summary

The proofs for $n = 3t + 2, 3t$ are very similar. So we have our main result:

Theorem $N(n) = \left\lfloor \frac{2n+1}{3} \right\rfloor$ for all integers $n \geq 1$.

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In the end, the proof is quite short and simple.

Proof summary:

1. By a suitable direct construction, prove that $N(n) \geq \left\lfloor \frac{2n+1}{3} \right\rfloor$.
2. Show that the dual LP has a feasible solution whose objective function value is less than $\left\lfloor \frac{2n+1}{3} \right\rfloor + 1$.

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  Then the solution to the LP is optimal (this is often called strong duality).
- When $n \equiv 1 \bmod 3$, our work in fact proves the LP conjecture.
- However, when $n \not\equiv 1 \bmod 3$, we do not have solutions to the primal LP whose objective function value matches the solutions to the dual LP. Although we are confident that the LP conjecture is also true for these values of n, proving it could get messy!

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"proofs from the book" are not required: It's not necessary that the solution to every problem be a "proof from the book". Good research is possible without possessing amazing levels of ingenuity.

thank you for your attention!