Block-avoiding sequencings of points in Steiner triple systems

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Definitions

- A **Steiner triple system of order** v, or **STS**(v), is a pair $(X, \mathcal{B})$, where X is a set of v **points** and $\mathcal{B}$ is a set of 3-subsets of X (called **blocks**), such that every pair of points occur in exactly one block.
- An STS(v) contains exactly $v(v-1)/6$ blocks, and an STS(v) exists if and only if $v \equiv 1, 3 \bmod 6$.
- For an STS(v), $(X, \mathcal{B})$, we ask if there is a permutation (or **sequencing**) of the points in X so that no three consecutive points in the sequencing comprise a block in $\mathcal{B}$.
- That is, can we find a sequencing $[x_1\ x_2\ \cdots\ x_v]$ of X such that $\{x_i, x_{i+1}, x_{i+2}\} \notin \mathcal{B}$ for all i, $1 \leq i \leq v-2$?
- Such a sequencing will be termed a 3-**good** sequencing for the given STS(v).

Definitions (cont.)

- More generally, we could ask if there is a sequencing of the points such that no ℓ consecutive points in the sequencing contain a block in $\mathcal{B}$.
- Such a sequencing will be termed **ℓ-good** for the given STS(v).
- As an example, consider the STS(7), $(X, \mathcal{B})$, where $X = \mathbb{Z}_7$ and

$$\mathcal{B} = \{013, 124, 235, 346, 450, 451, 562\}.$$

- The sequencing

$$[0\ 1\ 2\ 3\ 4\ 5\ 6]$$

  is easily seen to be 3-good.
- However, it is not 4-good, as the block 013 is contained in the first four points of the sequencing.
- Notation: For any two points $x, y \in X$, define *third*$(x, y) = z$ if and only if $\{x, y, z\} \in \mathcal{B}$.

Main Results

- We have proven that
  1. every STS(v) with $v > 3$ has a 3-good sequencing, and
  2. every STS(v) with $v > 72$ has a 4-good sequencing.
- In this talk, we will give two proofs of 1. We also sketch a proof of 2, which is considerably more complicated.
- We conjecture that, for any integer $\ell \geq 3$, there is an integer $n(\ell)$ such that any STS(v) with $v > n(\ell)$ has an ℓ-good sequencing.

  Update: I think I can now prove this conjecture.

A Counting Argument

- Let $(X, \mathcal{B})$ be an STS(v) on points $X = \{1, \ldots, v\}$, where $v > 3$.
- For a sequencing $\pi = [x_1\ x_2\ \cdots\ x_v]$ of X, and for any i, $1 \leq i \leq v - 2$, define π to be **i-forbidden** if

$$\{x_i, x_{i+1}, x_{i+2}\} \in \mathcal{B}.$$

- Let *forbidden*(i) denote the set of i-forbidden sequencings.
- Also, define a sequencing to be **forbidden** if it is i-forbidden for at least one value of i and let *forbidden* denote the set of forbidden sequencings.
- Clearly, a sequencing is 3-good if and only if it is not forbidden.

A Counting Argument (cont.)

- Clearly,

$$\textit{forbidden} = \bigcup_{i=1}^{v-2} \textit{forbidden}(i).$$

- For any two points, x_i and x_{i+1},

$$\{x_i, x_{i+1}, x_{i+2}\} \in \mathcal{B} \Leftrightarrow x_{i+2} = \textit{third}(x_i, x_{i+1}).$$

- Therefore, for any i, it holds that $|\textit{forbidden}(i)| = v!/(v-2)$.
- From the union bound,

$$|\textit{forbidden}| \leq \sum_{i=1}^{v-2} |\textit{forbidden}(i)| = (v-2) \times \frac{v!}{(v-2)} = v!$$

- Equality would be obtained if and only if the sets $\textit{forbidden}(i)$, $1 \leq i \leq v-2$, are pairwise disjoint.
- But this is impossible (consider any two intersecting blocks).

Greedy Algorithm

- We want to construct a 3-good sequencing $[x_1\ x_2\ \cdots\ x_v]$.
- Start by choosing any two distinct values for x_1 and x_2.
- Then consider any i such that $3 \leq i \leq v - 1$.
- Clearly we must have $x_i \notin \{x_1, \ldots, x_{i-1}\}$.
- Also, $x_i \neq$ *third*(x_{i-2}, x_{i-1}).
- So there are at most i values for x_i that are ruled out.
- Since $i \leq v - 1$, there is at least one value for x_i that does not violate the required conditions.
- So we can choose $x_1, x_2, \ldots, x_{v-1}$ so that $[x_1\ x_2\ \cdots\ x_{v-1}]$ is a partial 3-good sequencing.

Greedy Algorithm (cont.)

- After choosing $x_1, x_2, \ldots, x_{v-1}$ as described above, there is only one unused value remaining for x_v.
- But this might not result in a 3-good sequencing, if it happens that $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$.
- In this case, consider swapping x_1 and x_v.
- $x_1 \neq \textit{third}(x_{v-2}, x_{v-1})$, so the last three points are now OK.
- But if we are unlucky, $x_v = \textit{third}(x_2, x_3)$.
- Suppose we had previously chosen x_5 such that $\{x_2, x_3, x_5\} \in \mathcal{B}$, i.e., $x_5 = \textit{third}(x_2, x_3)$.
- It is easy to check that this is an allowable choice for x_5, because $x_1 \neq \textit{third}(x_2, x_3)$ and $x_4 \neq \textit{third}(x_2, x_3)$.
- If $x_5 = \textit{third}(x_2, x_3)$, then $x_v \neq \textit{third}(x_2, x_3)$ provided $v > 5$.

Greedy Algorithm (summary)

1. Choose a block $\{b, c, e\} \in \mathcal{B}$, let $a \neq b, c, e$ and let $d \neq a, b, c, e$.
2. Define $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$ and $x_5 = e$.
3. For $i = 6$ to $v - 1$ define x_i to be any element of X that is distinct from the values $x_1, \ldots, x_{i-1}$ and *third*(x_{i-2}, x_{i-1}).
4. Define x_v to be the unique value that is distinct from $x_1, \ldots, x_{v-1}$.
5. If $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$ then swap x_1 and x_v.
6. Return $([x_1\ x_2\ \cdots\ x_v])$.

Greedy Algorithm for 4-good Sequencings

- Now we consider how to construct a 4-good sequencing $[x_1\ x_2\ \cdots\ x_v]$.
- When we choose a value for x_i, it must be distinct from $x_1, \ldots, x_{i-1}$, of course.
- It is also required that

$$x_i \neq \textit{third}(x_{i-3}, x_{i-2}), \textit{third}(x_{i-3}, x_{i-1}) \text{ or } \textit{third}(x_{i-2}, x_{i-1}).$$

- Thus we can define $x_1, x_2, \ldots, x_{v-3}$ in such a way that $[x_1\ x_2\ \cdots\ x_{v-3}]$ is a partial 4-good sequencing.
- Denote the three remaining values by $\alpha_1, \alpha_2, \alpha_3$.

Greedy Algorithm for 4-good Sequencings (cont.)

- By relabelling $\alpha_1, \alpha_2, \alpha_3$ if necessary, it is possible to ensure that the only possible blocks contained in four consecutive points in

$$[x_1 \; x_2 \; \cdots \; x_{v-4} \; x_{v-3} \; \alpha_1 \; \alpha_2 \; \alpha_3]$$

  involve α_1.
- There are in fact seven possible 3-subsets contained in four consecutive points of the above sequencing that could conceivably be a block in the STS:

$$\begin{array}{ccc} \{x_{v-5}, x_{v-4}, \alpha_1\} & \{x_{v-5}, x_{v-3}, \alpha_1\} & \{x_{v-4}, x_{v-3}, \alpha_1\} \\ \{x_{v-4}, \alpha_1, \alpha_2\} & \{x_{v-3}, \alpha_1, \alpha_2\} & \{x_{v-3}, \alpha_1, \alpha_3\} \\ & \{\alpha_1, \alpha_2, \alpha_3\} & \end{array}$$

- There are therefore at most seven "bad" choices for x_{v-2}, and hence one of $x_1, \ldots, x_8$ must be a good choice.
- Suppose we swap α_1 with a good x_i, where $1 \leq i \leq 8$, which we denote by x_κ.

Greedy Algorithm for 4-good Sequencings (cont.)

- After we replace x_κ by α_1, we need to make sure that there is no block contained in any four consecutive points in $x_1, x_2, x_3, \ldots, x_{11}$.
- Thus, we require that

$$\alpha_1 \notin Z = \{\mathit{third}(x_i, x_j) : 1 \leq i < j \leq 11, |i - j| \leq 3\}.$$

- There are at most $10 + 9 + 8 = 27$ points in the set Z:

$$\begin{gathered} \mathit{third}(x_1, x_2), \ldots, \mathit{third}(x_{10}, x_{11}), \\ \mathit{third}(x_1, x_3), \ldots, \mathit{third}(x_9, x_{11}), \\ \mathit{third}(x_1, x_4), \ldots, \mathit{third}(x_8, x_{11}) \end{gathered}$$

- Suppose we ensure that all elements in Z have been used as an x_i-value with $i \leq v - 6$.
- Then $\alpha_1 \notin Z$ and we can swap it in for x_κ.

Greedy Algorithm for 4-good Sequencings (cont.)

- It turns out that we can fit all the elements of Z into $\{x_1, \ldots, x_{66}\}$ without any problems arising.
- Define

$$Y = Z \setminus \{x_1, \ldots, x_{11}\}.$$

  and denote the points in Y as y_1, ... , y_m, where $m \leq 27$.
- We can define $\{x_{12}, \ldots, x_{2m+12}\}$ so the following holds:

x_{12}	x_{13}	y_1	x_{15}	y_2	x_{17}	y_3	x_{19}	$\cdots$

x_{2m+7}	y_{m-2}	x_{2m+9}	y_{m-1}	x_{2m+11}	y_m

- These $2m + 12 \leq 66$ points should not overlap the last six points, so we require $v \geq 72$.

Motivation and Related Problems

- A **sequenceable** STS(v) is an STS(v) in which the points can be ordered (i.e., sequenced) so that no t consecutive points can be partitioned into $t/3$ blocks, for any $t \equiv 0 \bmod 3$, $t < v$.
- Brian Alspach gave a talk entitled Strongly Sequenceable Groups at the 2018 Kliakhandler Conference held at MTU.
- In this talk, among other things, the notion of sequencing diffuse posets was introduced and the following research problem was posed:

"Given a triple system of order n with $\lambda = 1$, define a poset P by letting its elements be the triples and any union of disjoint triples. This poset is not diffuse in general, but it is certainly possible that P is sequenceable."

Motivation and Related Problems (cont.)

- One possible relaxation of the definition of sequenceable STS(v) would be to require a sequencing of the points so that no t consecutive points can be partitioned into $t/3$ blocks, for any $t \equiv 0 \bmod 3$ such that $t \leq w$, where $w < v$ is some specified integer.
- Such an STS(v) could be termed **w-semi-sequenceable**.
- A 3-semi-sequenceable STS(v) has a sequencing of the points so that no three consecutive points form a block. This is identical to a 3-good sequencing.
- There is a connection between w-semi-sequenceable STS(v) and STS(v) having ℓ-good sequencings.

Theorem

Theorem 1

An STS(v) having a $(2u+1)$-good sequencing is $3u$-semi-sequenceable.

Proof.

Suppose there are $3u$ consecutive points, say $x_1, \ldots, x_{3u}$, in a sequencing π, that can be partitioned into u blocks of the STS(v), say $B_1, \ldots, B_u$. For $1 \leq j \leq u$, let

$$m_{lo}(j) = \min\{i : x_i \in B_j\} \quad \text{and} \quad m_{hi}(j) = \max\{i : x_i \in B_j\}.$$

Clearly there is a block B_j such that $m_{lo}(j) \geq u$. It also holds that $m_{hi}(j) \leq 3u$. Therefore $B_j \subseteq \{x_u, \ldots, x_{3u}\}$, which means that the sequencing π is not $(2u+1)$-good. □

Thank You For Your Attention!