On Partial Sums in Cyclic Groups

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This talk is based on joint work with Dan Archdeacon and Jeff Dinitz.

Partial Sums in Cyclic Groups

- Let $(G, +)$ be an additive abelian group with identity element 0.
- Suppose that $A \subseteq G \setminus \{0\}$, $|A| = k$.
- Let $(a_1, a_2, \ldots, a_k)$ be an ordering of the elements in A.
- Define the partial sums as

$$s_j = \sum_{i=1}^{j} a_i,$$

$1 \leq j \leq k$, where the computations are done in G.

The Conjecture

Conjecture 1

There exists an ordering of the elements of any subset $A \subseteq \mathbb{Z}_n \setminus \{0\}$ such that the partial sums are all distinct, i.e., for all $1 \leq i < j \leq k$, it holds that $s_i \neq s_j$.

Example 2

Suppose we have $A = \{1, 2, 3, 4, 5, 6\} \subseteq \mathbb{Z}_8$. Consider the ordering:

$$1 \quad 6 \quad 3 \quad 4 \quad 5 \quad 2.$$

The partial sums are

$$1 \quad 7 \quad 2 \quad 6 \quad 3 \quad 4.$$

Conjecture 1 is due to Archdeacon [1], who was motivated by a construction for embedding complete graphs so the faces are 2-colourable and each colour class is a cycle system.

Sequenceable Groups

- Conjecture 1 is also a natural generalization of the idea of sequenceable and R-sequenceable groups.
- A group G is sequenceable if there exists an ordering of all the group elements such that all the partial sums are distinct.
- It is known that $(\mathbb{Z}_n, +)$ is sequenceable if and only if n is even (Lucas-Walecki, 1892).
- More generally, it is known (Gordon, 1961) that an abelian group is sequenceable if and only if it has a unique element of order 2.
- A sequencing of $(\mathbb{Z}_{2t}, +)$ is given by

$$0, 1, 2t-2, 3, 2t-4, 5, \ldots, 4, 2t-3, 2, 2t-1.$$

Sequenceable Groups (cont.)

- When n is odd, $(\mathbb{Z}_n, +)$ cannot be sequenced because the sum of all the group elements is zero (the first element in the sequencing must be 0, so the first and last sums both equal zero).
- However, it has been shown that $(\mathbb{Z}_n, +)$ is R-sequenceable when n is odd (this generalization allows the first and last sums to both equal zero).
- Conjecture 1 can be considered as a sequencing of an arbitrary subset of the non-zero elements of the cyclic group $(\mathbb{Z}_n, +)$, which in theory should be easier (?) than sequencing the whole group.

Computational Results

- Conjecture 1 is true for $n \leq 25$. Here is the algorithm we used:
  1. For each $A \subseteq \mathbb{Z}_n \setminus \{0\}$ choose a random permutation of the elements of A.
  2. Repeat step 1 until a valid ordering of the elements in A is found.
- When $|A|$ is small compared to n, we usually only need to try very few random permutations before a valid ordering is found.
- However as $|A|$ increases, many more random permutations might be required before we find an ordering that works.
- The algorithm was programmed in Mathematica and was run on a laptop.
- It found all the orderings of the subsets of $\mathbb{Z}_{24}$ in roughly 3 days.
- The subsets of $\mathbb{Z}_{25}$ took longer.

Some Data for $n = 25$

- When $n = 25$ we needed fewer than 6 tries for nearly all subsets with $|A| \leq 7$.
- We used fewer than 100 tries when $|A| \leq 13$ and fewer than 10,000 tries when $|A| \leq 18$.
- However, when $|A| \geq 22$, there were cases where over 300,000 permutations were tried before a valid ordering was found.
- In general, between 10,000 and 75,000 permutations were checked before finding a valid ordering for larger subsets A.

Some Data for $n = 25$ (cont.)

set={2,3,4,5,7,8,9,10,11,12,14,16,17,18,19,20,21,22,23,24}
531326020174185660th permutation
ordering=(17,19,14,22,7,2,3,24,11,8,21,23,20,10,4,5,18,9,12,16)
sums = (17,11,0,22,4,6,9,8,19,2,23,21,16,1,5,10,3,12,24,15)
tries=4248

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set={1,2,3,4,5,6,7,8,9,10,11,12,13,14,16,18,19,21,22,23,24}
3836500304569l958047th permutation
ordering=(8,18,14,24,16,12,7,21,5,13,9,10,2,3,6,23,11,4,22,1,19)
sums = (8,1,15,14,5,17,24,20,0,13,22,7,9,12,18,16,2,6,3,4,23)
tries=15631

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set={1,2,3,4,6,8,9,10,11,12,13,14,15,16,17,18,19,21,22,23,24}
2767180362164384l656th permutation
ordering=(22,17,12,15,24,6,11,4,19,23,1,2,18,10,3,13,8,9,21,14,16)
sums = (22,14,1,16,15,21,7,11,5,3,4,6,24,9,12,0,8,17,13,2,18)
tries=304138

The Conjecture is True for $k \leq 5$

For $k = 1, 2, 3$, the result is easy. We give a proof for $k = 4$.

(1) Let p be the number of pairs $\{x, -x\} \subseteq A$. So $p = 0, 1$ or 2.

(2) If $p = 2$, then $A = \{x, -x, y, -y\}$ and the ordering

$$(x, y, -x, -y)$$

works.

(3) If $p = 1$, then $A = \{x, -x, y, z\}$ and the ordering

$$(z, x, y, -x)$$

works.

The Conjecture is True for $k \leq 5$ (cont.)

(4) So we can now assume $p = 0$. First choose three elements from A and order them as (a_1, a_2, a_3) in such a way that s_1, s_2 and s_3 are distinct. It is clear that $s_4 \neq s_3, s_2$. If $s_4 \neq s_1 (= a_1)$ we are done, so assume $s_4 = s_1$. Then $a_2 + a_3 + a_4 = 0$.

Now consider the sequence

$$(a'_1, a'_2, a'_3, a'_4) = (a_2, a_1, a_3, a_4).$$

Let s'_j be the sum of the first j terms in this new sequence. We only need to check that $s'_1 \neq s'_4$. This fails only if $a_1 + a_3 + a_4 = 0$, but from above we have that $a_2 + a_3 + a_4 = 0$, so $a_1 = a_2$ which is a contradiction.

The proof for $k = 5$ is messier; we did not attempt a proof for $k = 6$.

A Result on Ordering Subsets of A

Theorem 3

For any $A \subseteq \mathbb{Z}_n \setminus \{0\}$ with $|A| = k$, there exists $B \subseteq A$ such that

1. $|B| \geq \lfloor (k+1)/2 \rfloor$ *and*
2. B *can be ordered so its partial sums are distinct.*

Proof:

- Assume that the sequence $(a_1, a_2, \ldots, a_r)$ of elements from A has the property that $s_i \neq s_j$ for $1 \leq i < j \leq r$.
- If there are at least $r+1$ elements from A not already used in the sequence, then we can choose one, say $x \in A$, such that $s_r + x \neq s_i$ for all $i \leq r$.
- This is possible if $k \geq 2r+1$, i.e., if $r \leq \lfloor (k-1)/2 \rfloor$.
- Given such an x, we can extend the sequence by defining $a_{r+1} = x$.

Many Subsets of A Can Be Ordered

Theorem 4

For any $A \subseteq \mathbb{Z}_n \setminus \{0\}$ with $|A| = 2t$, there exist at least 2^t t-subsets $B \subseteq A$ that can be ordered so their partial sums are distinct.

Proof:

- Given a sequence of length r having distinct partial sums, there are at least $2t - 2r$ ways to extend it to a sequence of length $r + 1$
- We get at least $2t \times (2t - 2) \times \cdots \times 2 = 2^t \, t!$ permissible orderings of t-subsets $B \subseteq A$
- Any given t-subset B occurs at most $t!$ times.
- Therefore there are at least 2^t different t-subsets $B \subseteq A$ that can be ordered.

A similar (but slightly messier) result can be proven when $|A|$ is odd.

Ordering Random Subsets A

Lemma 5

Let $1 \leq k \leq n-1$ and let $T \in \mathbb{Z}_n$. For any set $A \in \mathbb{Z}_n$, let s_A be the sum of the elements of A. Then for a randomly chosen k-subset $A \subseteq \mathbb{Z}_n \setminus \{0\}$, the probability that $s_A = T$ is at most $2/n$.

Theorem 6

Let A be a randomly chosen k-subset of $\mathbb{Z}_n \setminus \{0\}$. Then the probability that A cannot be ordered so its partial sums are distinct is at most $k(k-1)/n$.

If we take $k \approx \sqrt{n/2}$, then we see that a randomly chosen $\sqrt{n/2}$-subset of $\mathbb{Z}_n \setminus \{0\}$ can be ordered with probability at least $1/2$.

Ordering Random Subsets A (cont.)

Proof idea (informal, non-rigourous):

- For $i < j$, observe that $s_i = s_j$ if and only if the run

$$r_{ij} := \sum_{h=i+1}^{j} a_h = 0.$$

- An ordering is "good" if all $\binom{k}{2}$ runs are non-zero.
- From the previous lemma, the probability that a particular run equals zero is at most $2/n$.
- The probability that at least one run equals zero is at most

$$\binom{k}{2} \times \frac{2}{n} = \frac{k(k-1)}{n}.$$

References

[1] Dan Archdeacon. Heffter arrays and biembedding graphs on surfaces. Preprint, 2014.

[2] M. A. Ollis. Sequenceable groups and related topics. *Electronic Journal of Combinatorics* **20** (2013), #DS10v2.

Thank You For Your Attention!