Optimal orthogonal arrays with repeated rows

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Definitions and a Bound

- Let $k \geq 2$, $n \geq 2$ and $\lambda \geq 1$ be integers.
- An **orthogonal array** $\mathbf{OA}_\lambda(\mathbf{k}, \mathbf{n})$ is a λn^2 by k array, A, with entries from a set X of cardinality n such that, within any two columns of A, every ordered pair of symbols from X occurs in exactly λ rows of A.
- In this talk, we are interested in $\mathbf{OA}_\lambda(\mathbf{k}, \mathbf{n})$ that contain a row that is repeated m times, where m is as large as possible.

Theorem 1

Let $k \geq 2$, $n \geq 2$ and $\lambda \geq 1$ be integers. If there is an $\mathbf{OA}_\lambda(\mathbf{k}, \mathbf{n})$ containing a row that is repeated m times, then

$$m \leq \frac{\lambda n^2}{k(n-1)+1}.$$

Comments

- Theorem 1 is a straightforward extension of the classical *Plackett-Burman bound* for OAs of strength two. It improves the *Plackett-Burman bound* by a factor of m.
- Note the resemblance to *Mann's inequality*, which states that $b \geq mv$ for a BIBD that contains a block of multiplicity m. This improves *Fisher's inequality* by a factor of m.
- A short proof of Theorem 1, based on the original proof of the *Plackett-Burman bound*, is given in Stinson [4].
- The proof uses a standard "variance" technique.
- However this theorem is also an immediate corollary of a much more general theorem of Mukerjee, Qian and Wu [3].

Definitions and Background

- An $\mathbf{OA}_\lambda(\mathbf{k}, \mathbf{n})$ containing a row that is repeated

$$m = \frac{\lambda n^2}{k(n-1)+1} \tag{1}$$

  times will be termed **optimal**.
- Another way to view the optimality property is to observe that the ratio m/λ is as large as possible in an optimal OA.
- We note that, in a recent paper, Culus and Toulouse [2] discuss an application where it is beneficial to construct optimal orthogonal arrays.
- They also construct several small examples of optimal OAs using linear programs.

Structural Properties

- Without loss of generality we can assume that the m-times repeated row has the form $x\ x \cdots x$ for any specified symbol x.

Theorem 2

Let $k \geq 2$, $n \geq 2$ and $\lambda \geq 1$ be integers. Suppose there is an optimal $\mathbf{OA}_\lambda(\mathbf{k}, \mathbf{n})$, say A, containing a row $x\ x \cdots x$ that is repeated m times. Then every other row of A contains exactly $(k-1)/n$ occurrences of the symbol x and thus $k \equiv 1 \bmod n$.

- Therefore, the following are necessary conditions for the existence of an optimal $\mathbf{OA}_\lambda(\mathbf{k}, \mathbf{n})$ with an m-times repeated row:
  - $k \geq 2$ and $n \geq 2$,
  - $m = \frac{\lambda n^2}{k(n-1)+1}$, and
  - $\overline{a} = \frac{k-1}{n}$ is a positive integer.

Basic OAs

- An optimal $\mathbf{OA_\lambda(k, n)}$ is **basic** if $\gcd(m, \lambda) = 1$.
- A basic OA cannot consist of multiple copies of OAs with smaller λ.
- As an example, we construct a basic $\mathbf{OA_3(5, 2)}$ from the following two starting rows:

$$\begin{array}{ccccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}$$

- We cyclically rotate these starting rows five times, and then adjoin $m = 2$ rows of 0's.

$$\begin{array}{ccccc} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{ccccc} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}$$

Another Example of a Basic OA

- We construct a basic $\mathbf{OA_5(7,3)}$ with $m = 3$.
- We have three starting rows, consisting of symbols from the set $\{\infty\} \cup \mathbb{Z}_2$:

$$\begin{array}{ccccccc} \infty & \infty & 0 & 0 & 0 & 0 & 1 \\ \infty & 1 & \infty & 0 & 1 & 1 & 0 \\ \infty & 1 & 1 & \infty & 1 & 0 & 1 \end{array}$$

- First, cyclically rotate each starting row seven times.
- Then develop each row modulo 2 (the point ∞ is fixed).
- Finally, adjoin three rows of ∞'s.
- The resulting $3 \times 7 \times 2 + 3 = 45$ rows form a basic $\mathbf{OA_5(7,3)}$.

Basic OAs with $n = 2$

Lemma 3

A basic $\mathbf{OA}_\lambda(\mathbf{k}, \mathbf{2})$ *with* $m > 1$ *has* $m = 2$, $\lambda = 2t + 1$ *and* $k = 4t + 1$ *for some positive integer* t.

Proof.

- $(k-1)/n = (k-1)/2$ is an integer, so $k = 2s + 1$.
- We have

$$m = \frac{\lambda n^2}{k(n-1)+1} = \frac{4\lambda}{k+1} = \frac{2\lambda}{s+1},$$

  so $2\lambda = m(s+1)$.
- $gcd(m, \lambda) = 1$ so $m = 1$ or 2. But $m > 1$, so $m = 2$.
- Then $\lambda = s + 1$.
- Since $gcd(m, \lambda) = 1$, s is even, so $s = 2t$.
- Then $\lambda = 2t + 1$ and $k = 4t + 1$.

Basic OAs with $n = 2$ (cont.)

Theorem 4

There exists a basic $\mathbf{OA_{2t+1}(4t+1, 2)}$ *if and only if there is a* $(4t+1, 2t+1, 2t+1)$*-BIBD.*

Proof.

($\Leftarrow$) Let M be the b by v incidence matrix of the given BIBD. Construct the matrix

$$A = \begin{pmatrix} 0\ 0 \cdots 0 \\ 0\ 0 \cdots 0 \\ M \end{pmatrix}.$$

Then A is a basic $\mathbf{OA_{2t+1}(4t+1, 2)}$.

The other direction of the proof is similar. □

Constructing the Basic OAs with $n = 2$

Theorem 5

If there exists a Hadamard matrix of order $8t + 4$, then there exists a $(4t + 1, 2t + 1, 2t + 1)$-$BIBD$ (and hence there exists a basic $\mathbf{OA_{2t+1}(4t + 1, 2)}$).

Proof.

It is well-known that a Hadamard matrix of order $8t + 4$ is equivalent to a symmetric $(8t + 3, 4t + 1, 2t)$-$BIBD$. The derived BIBD is a $(4t + 1, 2t, 2t - 1)$-$BIBD$. If we then complement every block in this BIBD, we obtain a $(4t + 1, 2t + 1, 2t + 1)$-$BIBD$. □

A General Construction for Optimal OAs

- Suppose we are given values for k and n, where $k \geq n+1$.
- Take all possible k-tuples that contain precisely $\overline{a} = (k-1)/n$ occurrences of 0, where 0 is one of the symbols.
- Then adjoin

$$m = \frac{\lambda n^2}{k(n-1)+1}$$

rows of 0s, where

$$\lambda = \binom{k-2}{\overline{a}-1}(n-1)^{k-\overline{a}-1}.$$

- We obtain an optimal $\mathbf{OA_{\lambda}(k,n)}$.
- For example, suppose $k=7$ and $n=3$. Then $\overline{a}=2$, $\lambda=80$ and $m=48$.
- The above-described process yields an optimal $\mathbf{OA_{80}(7,3)}$.

An Improvement

- It turns out that the optimal $\mathbf{OA}_{\lambda}(\mathbf{k}, \mathbf{n})$ described above can be partitioned into $n-1$ optimal $\mathbf{OA}_{\lambda/(\mathbf{n}-\mathbf{1})}(\mathbf{k}, \mathbf{n})$.
- Relabel the points so that the m repeated rows each consist of $\infty\ \infty \cdots \infty$.
- Take the $n-1$ remaining symbols to be the elements of $\mathbb{Z}_{n-1}$.
- For any row $\mathbf{r}$ in this OA, say A, let $s(\mathbf{r})$ denote the sum modulo $n-1$ of the non-infinite elements in this row.
- For any $i \in \mathbb{Z}_{n-1}$, let A_i consist of all the rows $\mathbf{r}$ of A such that $s(\mathbf{r}) = i$.
- Then each A_i is an optimal $\mathbf{OA}_{\lambda/(\mathbf{n}-\mathbf{1})}(\mathbf{k}, \mathbf{n})$.

A Further Improvement

Theorem 6

Suppose $k \geq n+1$ *and suppose* $\overline{a} = (k-1)/n$ *is an integer. Suppose*

$$\gamma = \left\lfloor \frac{k}{\overline{a}+3} \right\rfloor \geq 1.$$

Then there is an optimal $\mathbf{OA}_{\lambda}(\mathbf{k}, \mathbf{n})$, *where*

$$\lambda = \binom{k-2}{\overline{a}-1}(n-1)^{k-\overline{a}-1},$$

that can be partitioned into $(n-1)^{\gamma}$ *optimal* $\mathbf{OA}_{\lambda/(\mathbf{n}-\mathbf{1})^{\gamma}}(\mathbf{k}, \mathbf{n})$.

- Suppose we take $k = 16$ and $n = 3$.
- Then $\overline{a} = 5$ and $\gamma = 2$.
- We obtain an optimal $\mathbf{OA}_{\lambda}(\mathbf{16}, \mathbf{3})$, where

$$\lambda = \binom{14}{4}2^{8}.$$

Further Results

- Theorem 1 states that

$$m \leq \frac{\lambda n^2}{k(n-1)+1}.$$

- Optimal OAs are OAs where we have equality, which can only occur if the expression the right side is an integer.
- In general, we have

$$m \leq \left\lfloor \frac{\lambda n^2}{k(n-1)+1} \right\rfloor.$$

- An OA in which we have equality is termed m-**optimal**.
- We show that, if a "small" number of columns is deleted from an optimal OA, the result is an m-optimal OA.

Open Problems

- Find infinite classes of basic OAs with $n \geq 3$.
- Find improved "general" constructions for optimal OAs (i.e., find constructions with smaller λ values than the known constructions).
- Are there recursive constructions for optimal or basic OAs?

References

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Thank You For Your Attention!