Nonincident Points and Blocks in Designs

Douglas R. Stinson

David R. Cheriton School of Computer Science
University of Waterloo

SIAM Conference on Discrete Mathematics
June 18-21, 2012
Dalhousie University
Halifax, Nova Scotia

The problem (projective plane version)

- Suppose $\Pi = (X, \mathcal{L})$ is a projective plane of order q, where X is the set of points and $\mathcal{L}$ is the set of lines in Π.
- For $Y \subseteq X$ and $\mathcal{M} \subseteq \mathcal{L}$, we say that $(Y, \mathcal{M})$ is a nonincident set of points and lines if $y \notin M$ for every $y \in Y$ and every $M \in \mathcal{M}$.
- Define $f(\Pi)$ to be the maximum integer s such that there exists a nonincident set of s points and s lines in Π.
- Equivalently, $f(\Pi)$ is the size of the largest square submatrix of zeroes in the incidence matrix of Π.

An example

$f(\mathsf{PG}(2,4)) \geq 6$:

3	**6**	7	12	14	10	13	14	19	0	**17**	20	**0**	5	7
4	**7**	**8**	**13**	**15**	**11**	**14**	**15**	**20**	**1**	18	**0**	1	**6**	8
5	**8**	**9**	**14**	**16**	12	15	16	**0**	**2**	**19**	1	**2**	7	9
6	9	10	15	**17**	13	16	**17**	1	**3**	20	**2**	**3**	8	10
7	10	11	16	18	14	**17**	18	**2**	4	**0**	**3**	4	9	11
8	11	12	**17**	**19**	15	18	**19**	**3**	5	**1**	**4**	**5**	**10**	**12**
9	**12**	**13**	**18**	**20**	**16**	**19**	**20**	**4**	**6**	**2**	5	**6**	11	13

$Y = \{\mathbf{0}, \mathbf{2}, \mathbf{3}, \mathbf{6}, \mathbf{17}, \mathbf{19}\}$.
The six lines containing green points are disjoint from Y.

A bound

Theorem

For any set Y *of* s *points in a projective plane* $\Pi = (X, \mathcal{L})$ *of order* q*, the number of lines disjoint from* Y *is at most*

$$\frac{q^3 + q^2 + q - qs}{q + s}.$$

Proof

- For a subset $Y \subseteq X$ of s points, let $\mathcal{B}$ consist of the nonempty intersections of the lines in $\mathcal{L}$ with the set Y.
- Denote $b = |\mathcal{B}|$.
- We have the following equations:

$$\begin{aligned}\sum_{B \in \mathcal{B}} 1 &= b \\ \sum_{B \in \mathcal{B}} |B| &= (q+1)s \\ \sum_{B \in \mathcal{B}} \binom{|B|}{2} &= \binom{s}{2}.\end{aligned}$$

- From the above equations, it follows that

$$\sum_{B \in \mathcal{B}} |B|^2 = s(q+s).$$

Proof (cont.)

- Define $\beta = (q+s)/(q+1)$ and compute as follows:

$$\begin{aligned} 0 &\leq \sum_{B \in \mathcal{B}} (|B| - \beta)^2 \\ &= s(q+s) - 2\beta(q+1)s + \beta^2 b, \end{aligned}$$

- It follows that

$$\begin{aligned} b &\geq \frac{s(2\beta(q+1) - (q+s))}{\beta^2} \\ &= \frac{(q+1)^2 s}{q+s}. \end{aligned}$$

- Therefore, the number of lines disjoint from Y is at most

$$q^2 + q + 1 - \frac{(q+1)^2 s}{q+s} = \frac{q^3 + q^2 + q - qs}{q+s}.$$

Characterization of equality

- A maximal (s, β)-arc in a projective plane of order q is a set Y of s points such that every line meets Y in 0 or β points.
- It is well-known that a maximal (s, β)-arc has $s = 1 + (q+1)(\beta - 1)$ and the number of lines that intersect the maximal arc is precisely

$$\frac{s(q+1)}{\beta} = \frac{s(q+1)^2}{q+s}.$$

Corollary

Let Y be a set of s points in a projective plane of order q. Then the number of lines disjoint from Y is equal to $(q^3 + q^2 + q - qs)/(q+s)$ if and only if Y is a maximal (s, β)-arc, where $s = (q+1)(\beta - 1) + 1$.

Solution to the projective plane problem

Setting $(q^3 + q^2 + q - qs)/(q + s) = s$ and solving for s, we obtain the following result.

Theorem

For any projective plane Π of order q, it holds that $f(\Pi) \leq 1 + (q + 1)(\sqrt{q} - 1)$.

- Denniston proved that there is a maximal $(s, 2^u)$-arc in PG$(2, 2^v)$ whenever $0 < u < v$.
- Suppose $q = 2^v$ where v is even, and take $u = v/2$. There is a Denniston maximal (s, β)-arc, where $q = 2^v$, $\beta = \sqrt{q}$ and $s = 1 + (q + 1)(\sqrt{q} - 1)$.

Theorem

If q is an even power of 2, then $f(PG(2, q)) = 1 + (q + 1)(\sqrt{q} - 1)$.

Related work

- De Winter, Schillewaert and Verstraete (preprint, 2012) independently obtained the same results using similar methods.
- Our main theorem can also be derived from powerful results to Haemers (1995) based on eigenvalue interlacing techniques.
- Haemers proved that if there exists a nonincident set of s points and t blocks in a (v, b, r, k, λ)-BIBD, then

$$st \leq \frac{(r-\lambda)(v-s)(b-t)}{kr}. \quad (1)$$

- The obvious generalization of our bound to BIBDs yields

$$t \leq b - \frac{r^2 s}{r + \lambda(s-1)}. \quad (2)$$

- The two bounds (1) and (2) are equivalent for BIBDs.

The analogous problem for Steiner triple systems

- A Steiner triple system of order v (or STS(v)), is a pair $(X, \mathcal{B})$, where X is the set of v points and $\mathcal{B}$ is a set of $b = v(v-1)/6$ blocks, such that each block contains three points and every pair of points occurs in a unique block.
- It is well-known that $v \equiv 1, 3 \bmod 6$ is a necessary and sufficient condition for the existence of an STS(v).
- Define $f(v)$ to be the maximum integer s such that there exists a nonincident set of s points and s blocks in some STS(v).
- As an example, consider an STS(9) that is a subdesign of an STS(21). There are 12 blocks in the subdesign and 12 points not in the subdesign.
- This implies that $f(21) \geq 12$.

A bound

We summarise the main results:

Theorem

For any set Y of s points in an STS(v), the number of blocks disjoint from Y is at most

$$\frac{1}{3}\binom{v-s}{2} = \frac{(v-s)(v-s-1)}{6}.$$

Corollary

Suppose that $(X, \mathcal{B})$ is an STS(v), $Y \subset X$ and $|Y| = s$. Then the number of blocks in $\mathcal{B}$ disjoint from Y is equal to $(v-s)(v-s-1)/6$ if and only if $(X \setminus Y, \mathcal{B}_Y)$ is a sub-STS$(v-s)$ of $(X, \mathcal{B})$, where $\mathcal{B}_Y$ denotes the blocks in $\mathcal{B}$ that are disjoint from Y.

Upper bound on $f(v)$

Setting $s = (v - s)(v - s - 1)/6$ and solving for s, we obtain the following upper bound:

Theorem

For any positive integer $v \equiv 1, 3 \bmod 6$*, it holds that*

$$f(v) \leq \frac{2v + 5 - \sqrt{24v + 25}}{2}.$$

Meeting the bound

In order for $f(v)$ to attain the upper bound, we require an STS(v) containing a sub-STS$(v-s)$, where $s = (2v+5-\sqrt{24v+25})/2$.

Theorem (Doyen-Wilson Theorem)

There exists an STS(v) containing a sub-STS(w) if and only if $v \geq 2w+1$, $v, w \equiv 1, 3 \bmod 6$.

Therefore, it suffices to determine the positive integers v such that the following conditions are satisfied:

1. $v \equiv 1, 3 \bmod 6$
2. $s = (2v+5-\sqrt{24v+25})/2$ is an integer
3. $v - s \equiv 1, 3 \bmod 6$, and
4. $v \geq 2(v-s)+1$.

The optimal solutions

Theorem

Suppose $v \equiv 1, 3 \mod 6$ *is a positive integer. Then*

$$f(v) = \frac{2v + 5 - \sqrt{24v + 25}}{2}$$

if and only if $v = 216z^2 + 42z + 1$, $216z^2 + 186z + 39$, $216z^2 + 138z + 21$ *or* $216z^2 + 282z + 91$, *where* z *is a non-negative integer.*

The three smallest cases where $f(v)$ attains its optimal value are when $v = 21$, $s = 12$; $v = 39$, $s = 26$; and $v = 91$, $s = 70$.

References

[1] S. De Winter, J. Schillewaert and J. Verstraete. Large incidence-free geometries. Preprint (2012).

[2] W.H. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra and Its Applications* **227–228** (1995), 593–616.

[3] D.R. Stinson. A problem concerning nonincident points and lines in projective planes. ArXiv report 1109/1573, `http://arxiv.org/abs/1109.1573` (2011).

[4] D.R. Stinson. A problem concerning nonincident points and blocks in Steiner triple systems. ArXiv report 1109/3847, `http://arxiv.org/abs/1109.3847` (2011).

Thank you for your attention!