Multicollision attacks on iterated hash functions

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references and summary

This talk is based on joint work with Mridul Nandi and Jalaj Upadhyay:

- M. Nandi and D.R. Stinson. Multicollision attacks on some generalized sequential hash functions. *IEEE Transactions on Information Theory* **53** (2007), 759–767.
- D.R. Stinson and J. Upadhyay. On the complexity of the herding attack and some related attacks on hash functions. *IACR ePrint 2010/30*.

I will talk about two recent results on multicollision attacks for hash functions:

1. a generalization of Joux's multicollision attack to a wide variety of hash functions, and
2. a second look at constructing diamond structures, which were invented by Kelsey and Kohno to use in their herding attacks on iterated hash functions.

hash functions

- Typically, a hash function takes a "long" input string and produces a random-looking "short" output string called a message digest.
- Hash functions have been used for many years in computer science to create hash tables for efficient methods for information retrieval.
- In this context, it is important that collisions occur as infrequently as possible, where a collision for a hash function $hash$ is a pair of distinct inputs x, x' such that $hash(x') = hash(x)$.
- Hash functions are also used frequently in cryptography, where additional properties are required. Such hash functions are termed cryptographic hash functions.
- A cryptographic hash function maps an arbitrary-length input string to a fixed-length output string: $hash : \{0,1\}^* \rightarrow \{0,1\}^n$.

three security properties of hash functions

Collision resistance

It should be difficult to find $x, x' \in \{0,1\}^*$ such that $x' \neq x$ and $hash(x') = hash(x)$.
(Here, x and x' collide.)

Preimage resistance

Given $z \in \{0,1\}^n$, it should be difficult to find $x \in \{0,1\}^*$ such that $hash(x) = z$.
(Here, x is a preimage of z.)

Second preimage resistance

Given $x \in \{0,1\}^*$, it should be difficult to find $x' \in \{0,1\}^*$ such that $x' \neq x$ and $hash(x') = hash(x)$.
(Here, x' is a second preimage of $h(x)$.)

difficulty of the three problems

- Suppose we postulate the existence of an "ideal" hash function that outputs a random value $hash(x)$ for every input x.
- Such a hash function is called a random oracle.
- It is easy to analyse the difficulty of the three problems in the random oracle model.
- Preimages and Second preimages can be found by exhaustive search in expected time $\Theta(2^n)$.
- Collisions can be found using the birthday paradox in expected time $\Theta(2^{n/2})$.
- When we construct a "real" hash function, our goal is that the three problems cannot be solved more quickly than in the ideal case (but proving things like this are extremely difficult!).

multicollisions

- There has been recent interest in studying the difficulty of finding multicollisions in hash functions.
- A γ-multicollision is a γ-subset $\{x_1, \ldots, x_\gamma\} \subseteq \{0,1\}^*$ such that $hash(x_1) = hash(x_2) = \cdots = hash(x_\gamma)$.
- It is commonly asserted that the complexity of finding a γ-multicollision in the random oracle model is $\Theta(2^{n(\gamma-1)/\gamma})$.
- Using estimates due to Diaconis and Mosteller (1989), Nandi and Stinson observed that the true complexity is $\Theta(\gamma\, 2^{n(\gamma-1)/\gamma})$.
- For additional, more detailed analysis along these lines, see Suzuki, Tonien, Kurosawa, and Toyota (2008).

iterated hash functions

- The most common design strategy for hash functions is the iterated hash function.
- MD4, MD5, and SHA-1 are all iterated hash functions.
- We need a padding function, which takes an input string x, where $|x| \geq n + t + 1$, and constructs a "padded" string y, such that $|y| \equiv 0 \bmod t$.
- We also need a compression function, $compress : \{0,1\}^{n+t} \rightarrow \{0,1\}^n$.
- IV is a public initial value which is a bitstring of length n.

constructing an iterated hash function

preprocessing step

Given x, construct the padded string y, where $|y| \equiv 0 \bmod t$. Denote

$$y = y_1 \parallel y_2 \parallel \cdots \parallel y_r,$$

where $|y_i| = t$ for $1 \leq i \leq r$. The y_i's are called message blocks.

processing step

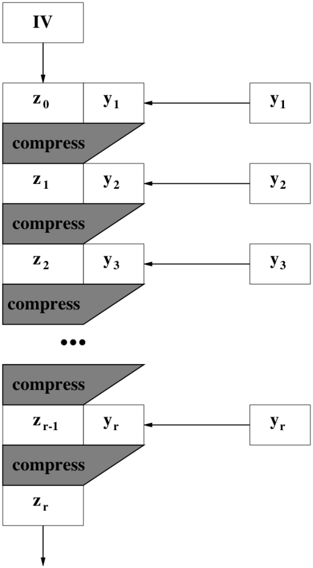
Compute the following chaining values:

$$\begin{aligned} z_0 &\leftarrow IV \\ z_1 &\leftarrow \mathit{compress}(z_0 \parallel y_1) \\ \vdots &\quad \vdots \quad \vdots \\ z_r &\leftarrow \mathit{compress}(z_{r-1} \parallel y_r). \end{aligned}$$

output

Define $h(x) = z_r$.

constructing an iterated hash function



Joux's multicollision attack

- Joux (2004) discovered a simple multicollision attack on iterated hash functions.
- The expected complexity to find a 2^r-multicollision is $\Theta(r\,2^{n/2})$, which is much smaller than the birthday attack having complexity $\Theta(2^r \times 2^{n(2^r-1)/2^r})$.
- The idea is to find r successive collisions in the compression function, each of which requires time $\Theta(2^{n/2})$ to find.
- For $z, z' \in \{0,1\}^n$ and $y \in \{0,1\}^t$, we use the notation $z \xrightarrow{y} z'$ (a labelled arc) to mean $compress(z, y) = z'$, where $|z| = |z'| = n$ and $|y| = t$.
- We can extend this notation in a natural way to incorporate multiple message blocks, e.g., $z \xrightarrow{y_1, y_2, y_3} z'$.

Joux's multicollision attack (cont.)

$$\begin{array}{llll} z_0 \xrightarrow{y_1^1} z_1 & \text{and} & z_0 \xrightarrow{y_1^2} z_1 & \text{for some } z_1\text{, where } y_1^1 \neq y_1^2 \\ z_1 \xrightarrow{y_2^1} z_2 & \text{and} & z_1 \xrightarrow{y_2^2} z_2 & \text{for some } z_2\text{, where } y_2^1 \neq y_2^2 \\ \vdots & & & \\ z_{r-1} \xrightarrow{y_r^1} z_r & \text{and} & z_{r-1} \xrightarrow{y_r^2} z_r & \text{for some } z_r\text{, where } y_r^1 \neq y_r^2. \end{array}$$

Then the set

$$\{y_1^1, y_1^2\} \times \{y_2^1, y_2^2\} \times \cdots \times \{y_r^1, y_r^2\}$$

is a 2^r-multicollision:

Question: Can Joux's attack be generalised to other types of hash functions?

generalised iterated hash functions

- hash twice uses every message block twice: $hashtwice(y) = hash(hash(IV, y), y)$ where y is the padded message.
- That is, we process the message blocks in the order $y_1, \ldots, y_r, y_1, \ldots, y_r$.
- zipper hash processes the message blocks in the order $y_1, \ldots, y_r, y_r, \ldots, y_1$.
- Let $\mathcal{S} = \{1, 2, \ldots, r\}$ denote the set of indices of the r message blocks.
- A generalised sequential hash function (GSHF) is based on a sequence $\alpha = \langle \alpha_1, \cdots, \alpha_s \rangle$ where $\alpha_i \in \mathcal{S}$ for all i.
- The GSHF based on α is defined as follows:

$$\begin{aligned} z_0 &= IV \\ z_i &= compress(z_{i-1}, y_{\alpha_i}), 1 \leq i \leq s. \end{aligned}$$

a partial order relation

- We define a relation on the symbol set $\mathcal{S}$.
- For $x, x' \in \mathcal{S}$, $x \neq x'$, define $x \prec x'$ if every occurrence of x in α precedes every occurrence of x' in α.
- The relation "$\prec$" is antisymmetric and transitive; hence "$\prec$" is a partial order.
- Two symbols $x \neq x'$ are incomparable if it is not the case that $x \prec x'$ or $x' \prec x$.
- A list of symbols $x_1, \ldots, x_d$ is a chain if $x_1 \prec x_2 \prec \cdots \prec x_d$.
- A set of chains is a chain decomposition if the chains are disjoint and their union is $\mathcal{S}$.

an attack based on a chain

- We present an attack on the hash function based on the sequence

$$\alpha = \langle 1, 2, 1, 3, 2, 4, 3, 5, 4, 5 \rangle$$

- Note that $1 \prec 3 \prec 5$ is a chain.
- We decompose α into three subsequences:

$$\langle 1, 2, 1 \rangle, \langle 3, 2, 4, 3 \rangle, \langle 5, 4, 5 \rangle$$

- Define $y_2 = y_4 = y^*$ for some arbitrary t-bit string y^*.
- The attack consists of three successive birthday attacks:

$$\begin{array}{lcl} z_0 \xrightarrow{y_1^1, y^*, y_1^1} z_1 & \text{and} & z_0 \xrightarrow{y_1^2, y^*, y_1^2} z_1 \\ z_1 \xrightarrow{y_3^1, y^*, y^*, y_3^1} z_2 & \text{and} & z_1 \xrightarrow{y_3^2, y^*, y^*, y_3^2} z_2 \\ z_2 \xrightarrow{y_5^1, y^*, y_5^1} z_3 & \text{and} & z_2 \xrightarrow{y_5^2, y^*, y_5^2} z_3 \end{array}$$

- We get a 2^3-multicollision with collision value z_3.

an attack based on an initial interval

- For hash twice, we have $\alpha = \langle 1, 2, \ldots, r, 1, 2, \ldots, r \rangle$, which does not have a chain of length longer than 1.
- We have another approach, based on the fact that the first r message blocks to be processed are all different.

(1) Use Joux's multicollision attack to find a 2^r-multicollision $\mathcal{C}$ for the first r message blocks.

(2) Let $r = uv$ for "appropriate" u and v. Divide the index interval $[r+1, 2r]$ into u equal intervals, each of size v. For $i = 1, \ldots, u$, (if possible) use a standard birthday attack to find two v-tuples from the appropriate part of $\mathcal{C}$ which collide.

(3) Provided that the u birthday attacks in step (2) all succeed, we get a multicollision set (of size 2^u) for hash twice.

combining the two attacks

We consider sequences in which every symbol occurs at most twice.

The next theorem follows from Dilworth's Theorem, which states that for any a partial order "$\prec$" on a finite set $\mathcal{S}$, the maximum number of mutually incomparable elements in $\mathcal{S}$ is equal to the minimum number of chains in any chain decomposition.

Theorem (Nandi and Stinson (2007))

Let α be a sequence of elements from symbol set $\mathcal{S} = \{1, \ldots, r\}$ such that $1 \leq \mathrm{freq}(x, \alpha) \leq 2$ for all $x \in \mathcal{S}$. Suppose that $r \geq r_1 r_2$. Then one of the following holds:

1. $\mathrm{maxchain}(\alpha) \geq r_1$, *or*
2. *there exists an initial interval $[1, w]$ such that $\alpha[1, w]$ contains at least r_2 symbols each having frequency 1.*

These attacks have subsequently been extended by Hoch and Shamir (2006) to sequences where each symbol occurs at most c times, for some fixed positive integer c.

proof sketch

- Let $\rho_1 = \mathrm{maxchain}(\alpha)$.
- If $\rho_1 \geq r_1$, we're done.
- Otherwise, when $\rho_1 < r_1$, let ρ_2 denote the maximum number of incomparable elements.
- By Dilworth's Theorem, there is a chain decomposition having ρ_2 chains.
- Each chain has length at most ρ_1, so

$$\rho_2 \geq n/\rho_1 > n/r_1 \geq r_2.$$

- Take an initial subsequence of α that contains the first occurrences of the ρ_2 incomparable elements.
- This works precisely because these elements are incomparable.

the herding attack

Kelsey and Kohno (2006) described the following hash function property, presented as a game between an attacker and a challenger:

Chosen-target-forced-prefix resistance

An attacker commits to a message digest, z, and is then challenged with a prefix, P. It should be infeasible for the attacker to be able to find a suffix S such that $hash(P \parallel S) = z$.

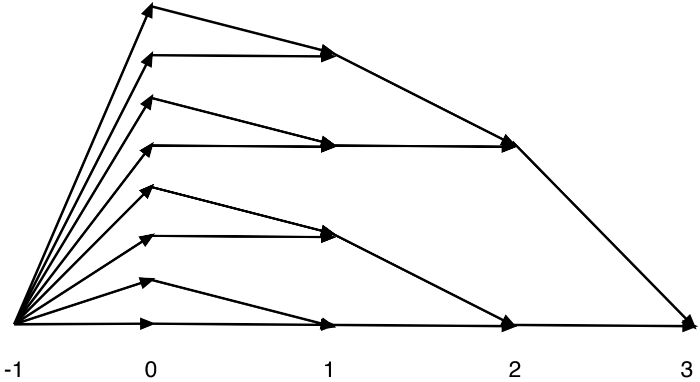
- Intuitively, it does not seem that a chosen-target-forced-prefix attack should be easier than finding a preimage, which generally takes time $\Theta(2^n)$.
- An attack that violates CTFP resistance is often called a herding attack.
- Kelsey and Kohno described a herding attack on iterated hash functions using a precomputed data structure called a diamond structure.

diamond structures

- First we'll talk about diamond structures; we'll present the herding attack a bit later.
- A 2^k-diamond structure contains a complete binary tree of depth k.
- There are $2^{k-\ell}$ nodes at level ℓ, for $k \geq \ell \geq 0$.
- There is also a single node at level -1, which we will call the source node.
- The source node is joined to every node at level 0.
- The nodes at level 0 are called the leaves of the diamond structure and the node at level k is called the root of the tree.

diamond structures (cont.)

Here is a diagram of a 2^3 diamond structure:



diamond structures (cont.)

- Every edge e in the diamond structure is labeled by a string $\sigma(e)$ which consists of one or more message blocks.
- We also assign a label $h(N)$ to every node N in the structure at level at least 0, as follows:
- Consider the unique directed path P from the source node to the node N in the diamond structure.
- P will consist of some edges $e_0 e_1 \cdots e_\ell$, where N is at level ℓ in the tree. Then we define

$$h(N) = hash(\sigma(e_0) \parallel \sigma(e_1) \parallel \cdots \parallel \sigma(e_\ell)).$$

- At any level ℓ of the structure there are $2^{k-\ell}$ hash values.
- These must be paired up in such a way that, when the next message blocks are appended, $2^{k-\ell-1}$ collisions occur.
- Thus there are $2^{k-\ell-1}$ hash values at the next level.
- The entire structure yields a 2^k-multicollision.

building a diamond structure

- A diamond structure is constructed one level at a time.
- We describe how to construct the nodes at level 1.
- For each of the 2^k nodes at level 0, construct a list of L random message blocks and compute the relevant hashes.
- Look for collisions in different lists and try to find 2^{k-1} disjoint pairs of collisions.
- For example, suppose $k = 2$, $L = 4$ and $n = 4$, and we get the following lists of hash values:

List 1:	0011	1011	0101	1100
List 2:	0010	1000	1010	0001
List 3:	0101	0001	1111	0000
List 4:	1110	1101	1011	1001

- Then we can pair up lists 1 and 4 (having collision 1011) and lists 2 and 3 (having collision 0001).

Kelsey and Kohno's analysis

Kelsey and Kohno argued as follows:

> *The work done to build the diamond structure is based on how many messages must be tried from each of 2^k starting values, before each has collided with at least one other value. Intuitively, we can make the following argument, which matches experimental data for small parameters: When we try $2^{n/2+k/2+1/2}$ messages spread out from 2^k starting hash values (lines), we get $2^{n/2+k/2+1/2-k}$ messages per line, and thus between any pair of these starting hash values, we expect about $(2^{n/2+k/2+1/2-k})^2 \times 2^{-n} = 2^{n+k+1-2k-n} = 2^{-k+1}$ collisions. We thus expect about $2^{-k+k+1} = 2$ other hash values to collide with any given starting hash value.*

the flaw in the analysis

Unfortunately, this line of reasoning does not imply that the 2^k nodes can be paired up in such a way that we get 2^{k-1} collisions:

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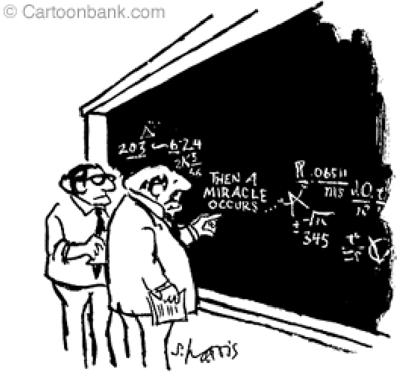


"I think you should be more explicit here in step two."

random graph formulation

- It is useful to think of this problem in a graph-theoretic setting.
- Suppose we label the nodes as $1, 2, \ldots, 2^k$.
- Then we construct a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the vertex set is $\mathcal{V} = \{v_1, \ldots, v_{2^k}\}$ and $(v_i, v_j) \in \mathcal{E}$ if the nodes v_i and v_j collide at the next level of the diamond structure.
- Let $\mathcal{G}(\nu, p)$ denote a random graph on ν labelled vertices, obtained by selecting each pair of vertices to be an edge randomly and independently with a fixed probability p.
- Based on the analysis given above, we see that the graph $\mathcal{G}$ is precisely a random graph in $\mathcal{G}(2^k, 2^{-k+1})$.
- Now, the question is if this random graph contains a perfect matching, as this is precisely what is required in order to be able to find the desired 2^{k-1} pairs of collisions.

threshold functions for random graphs

- As p increases from 0 to 1, a random graph in $\mathcal{G}(\nu, p)$ becomes more and more dense.
- Many natural monotone graph-theoretic properties become true within a very small range of values of p.
- Given a monotone graph-theoretic property, there is typically a value of p (which will be a function $t(\nu)$ depending on ν, the number of vertices) called the called threshold function.
- The given property holds in the model $\mathcal{G}(\nu, p)$ with probability close to 0 for $p < t(\nu)$, and the property holds with probability close to 1 for $p > t(\nu)$.
- A threshold function for having a perfect matching is any function having the form

$$t(\nu) = \frac{\ln \nu + f(\nu)}{\nu}$$

for any $f(\nu)$ such that $\lim_{\nu \to \infty} f(\nu) = \infty$.

fixing the analysis

- $\mathcal{G}(2^k, 2^{-k+1})$ has $p = 2/\nu$, which is much lower than required threshold, so the Kelsey-Kohno analysis is not valid.
- We assume a random graph in $\mathcal{G}(\nu, \ln \nu/\nu)$ has a perfect matching.
- We construct $\nu = 2^k$ lists, each containing L messages.
- The probability that any two given messages collide is 2^{-n}. The probability that there is at least one collision between two given lists is $p \approx L^2/2^n$.
- We want $p \approx \ln \nu/\nu$, so we take

$$L \approx \sqrt{k \ln 2} \times 2^{(n-k)/2} \approx 0.83 \times \sqrt{k} \times 2^{(n-k)/2}.$$

- The message complexity (i.e., the number of hash computations) at level 0 is therefore

$$2^k L \approx 0.83 \times \sqrt{k} \times 2^{(n+k)/2}.$$

fixing the analysis (cont.)

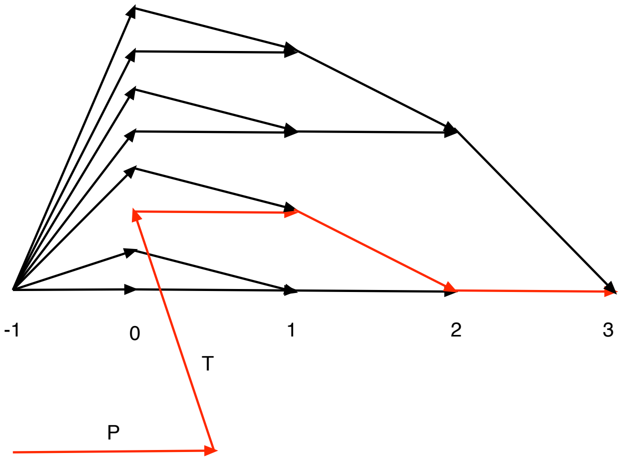
- Ignoring constant factors, this is a factor of about $\sqrt{k}$ bigger than the estimate in Kelsey-Kohno.
- The lower levels of the diamond structure are analysed in a similar way, replacing k by $k-1, k-2$, etc.
- The total message complexity is also $\Theta(\sqrt{k} \times 2^{(n+k)/2})$.
- Thus we obtain a rigourous analysis (in the random oracle model) with a precise estimate of the message complexity.
- Overall, it turns out that Kelsey and Kohno's estimate (for the entire structure) was too small by a factor of $\sqrt{k}$.
- Note this has some effect on various other attacks in the literature that make use of diamond structures.

Kelsey-Kohno's herding attack

- First, we construct a diamond structure with k levels.
- We commit to the hash value $z = h(\mathrm{root})$ and the challenger provides a prefix P.
- We choose random strings T until we find a linking message, i.e., a string T such that $hash(P \parallel T) = h(N)$ for some node N in the diamond structure.
- This takes, on average, 2^{n-k-1} attempts.
- Once we have found the linking message T, construct S by concatenating T with the message blocks in the diamond structure on the path from N to root.
- The total complexity of the attack is $\Theta(2^{n-k} + \sqrt{k} \times 2^{(n+k)/2})$.
- The value of k can be chosen as desired. If $k \approx n/3$, then the message complexity of the attack is about $\Theta(\sqrt{n} \times 2^{2n/3})$, which is a significant improvement over $\Theta(2^n)$.

Kelsey-Kohno's herding attack (cont.)

A linking message for a 2^3 diamond structure:



thank you for your attention!