Some new results and conjectures on Costas and honeycomb arrays

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Costas array definition

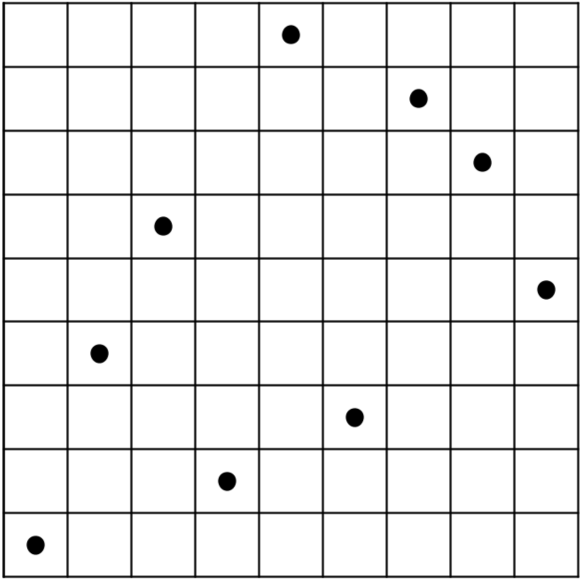
An $n \times n$ Costas array consists of n dots in an $n \times n$ array such that:

- The dots form a permutation (there is exactly one dot in each row and each column).
- The $n(n-1)$ difference vectors are distinct.

Costas arrays were introduced by J.P. Costas in 1975. They have applications in the design of sonar systems, and to radar, synchronisation and alignment systems.

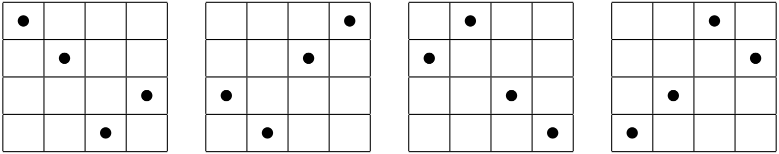
More recently, there have been some interesting cryptographic applications of Costas arrays in the design of key predistribution schemes for wireless sensor networks in grids.

A Costas array of order 9



Costas latin squares

- Two Costas arrays of order n are disjoint if there is no cell in which both arrays have a dot.
- Here are four disjoint Costas arrays of order 4:



- n disjoint Costas arrays of side n yield a Costas latin square of order n, denoted $CLS(n)$.
- Here is a $CLS(4)$ constructed from the four disjoint Costas arrays of order 4:

1	3	4	2
3	1	2	4
2	4	3	1
4	2	1	3

An infinite class of Costas latin squares

- A singly periodic Costas array of order n is a Costas array of order n in which every cyclic rotation modulo n is also a Costas array of order n.
- If there exists a singly periodic Costas array of side n, then there exists a $CLS(n)$.
- The following is the well known Welch construction for Costas arrays. Let p be prime and let α be a primitive element in the field $\mathbb{F}_p$. Let $n = p - 1$. A Costas array of order n is obtained by placing a dot in cell (i, j) if and only if $i = \alpha^j$, for $1 \leq j \leq n$ and $1 \leq j \leq n$.
- It is known that Costas arrays constructed via the Welch construction are singly periodic. So we have the following

Theorem

There exists a $CLS(p-1)$ for any prime p.

A Costas latin square of order 6

Take $p = 7$, $n = 6$, and $\alpha = 3$ is a primitive element of $\mathbb{F}_7$. The Costas array of order 6 produced from the Welch construction is the following:

●					
		●			
	●				
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The resulting $CLS(6)$ is as follows:

1	2	3	4	5	6
5	6	1	2	3	4
6	1	2	3	4	5
2	3	4	5	6	1
4	5	6	1	2	3
3	4	5	6	1	2

$4\frac{3}{5}$ disjoint Costas arrays of order 5

A Costas latin square of order 5 does not exist. However, there exist four disjoint Costas arrays of order 5, and a fifth Costas array containing three dots disjoint from the other four arrays:

1	2	3	4	5
3	5	4	1	2
4	3	2	5	1
	4	1, 5	2	3
2, 5	1		3	4

Numerical results and a conjecture

Let $D(n)$ denote the maximum number of disjoint Costas arrays of order n.

n	1	2	3	4	5	6	7	8	9	10
$D(n)$	1	2	2	4	4	6	6	8	8	10

n	11	12	13	14	15	16	17	18	19	20
$D(n)$	10	12	≤ 12	≤ 13		16		18		

n	21	22	23	24	25	26	27	28	29	30
$D(n)$	11	22	9	8	5	6	8			

The data above suggests the following

Conjecture

There does not exist a $CLS(n)$ for any odd $n \geq 3$.

Near Costas arrays

An $n \times n$ near Costas array consists of $n-1$ dots in an $n \times n$ array such that:

- There is at most one dot in each row and each column.
- The $(n-1)(n-2)$ difference vectors are distinct.

Let $D_{near}(n)$ denote the maximum number of near Costas arrays of order n. Observe that $D_{near}(n) \leq n+1$.

The following example shows that $D_{near}(6) = 7$:

6	1	7	4	2	3
5		6	3	1	7
7	2	1	5	3	4
3	5	4	1	6	2
4	6	5	2	7	1
2	4	3	7	5	6

The Golomb construction

The Golomb construction for Costas arrays is as follows:

- Let α and β be primitive elements in $\mathbb{F}_q$.
- Let $n = q - 2$. For $1 \leq i \leq n$, $1 \leq j \leq n$, place a dot in cell (i, j) of an n by n array if and only if $\alpha^i + \beta^j = 1$.
- The resulting array, which we denote by $G(\alpha, \beta)$, is a Costas array.

Theorem

Suppose that $q \equiv 3 \bmod 4$ is a prime power and suppose that $p = (q-1)/2$ is an odd prime. Then $D_{near}(2p-1) \geq p - 1$.

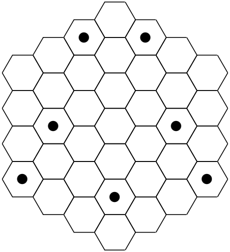
Proof sketch

$\mathbb{F}_q$ has $\phi(2p) = p - 1$ primitive roots, say $\alpha_1, \ldots, \alpha_{p-1}$. It can be shown that the $p - 1$ arrays $G(\alpha_1, \alpha_i)$ $(1 \leq i \leq p - 1)$ all contain a common dot in position (i, p), where ${\alpha_1}^i = 2$. On removing this dot, we obtain $p - 1$ disjoint near-Costas arrays of order $2p - 1$.

As an example, let $q = 11$, so $n = 9$ and $p = 5$. The four primitive elements modulo 11 are 2, 8, 7 and 6. We present the superposition of $G(2, 2)$, $G(2, 8)$, $G(2, 7)$, $G(2, 6)$. The common dot is in position $(1, 5)$.

				•				
8		2				6		7
	2		8		7		6	
7		6				2		8
2		7				8		6
	6		7		8		2	
	7		2		6		8	
	8		6		2		7	
6		8				7		2

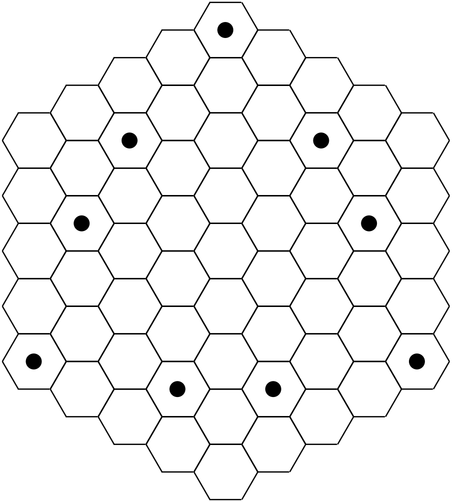
Definition of a honeycomb array



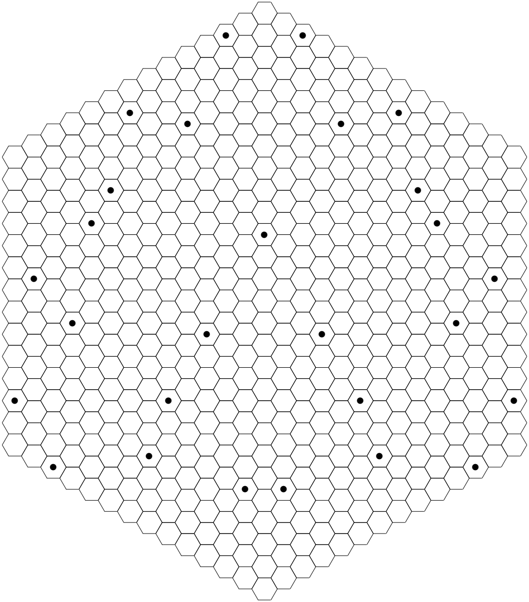
Honeycomb arrays were introduced by Golomb and Taylor in 1984:
A honeycomb array with n dots is a set of n dots in the hexagonal grid such that

- (The hexagonal permutation property) In each of the three natural directions, the dots occupy exactly n consecutive 'rows' of the grid. (One dot per row.)
- (The distinct differences property) The $n(n-1)$ difference vectors are distinct.

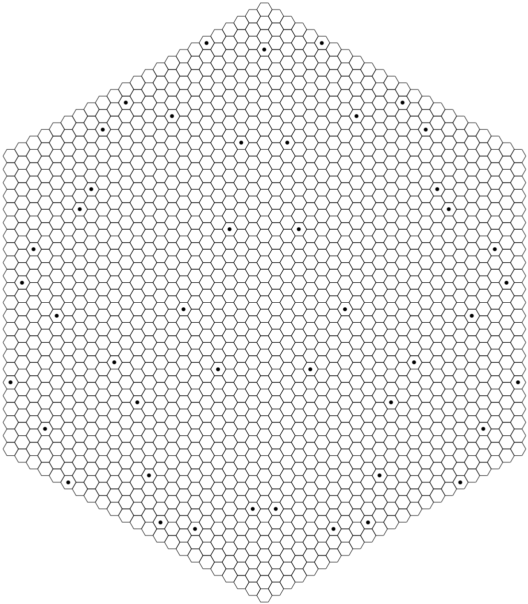
A honeycomb array with 9 dots



A honeycomb array with 27 dots

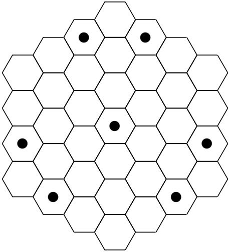


A honeycomb array with 45 dots



The shape of a honeycomb array

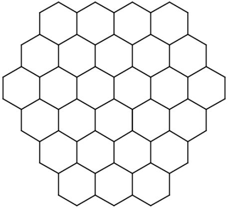
For each direction i, let R_i be the set of rows occupied by dots. Let $R = R_1 \cap R_2 \cap R_3$. What does R look like? Golomb and Taylor (1984) observed that all examples they knew had R equal to a Lee sphere of radius r:

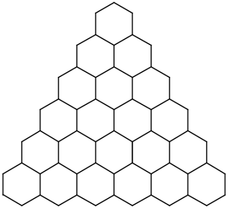


Such an array has $2r + 1$ dots. We call it a honeycomb array of radius r.

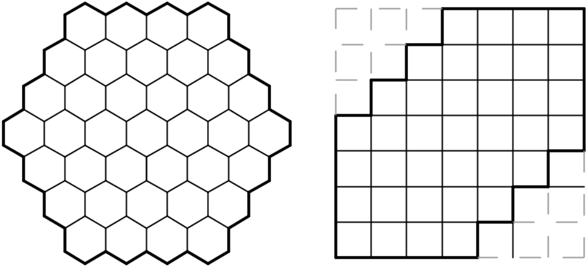
Other shapes that could be possible

For example, the tricentred Lee sphere and triangle are not ruled out by the definition:





Honeycomb arrays are Costas arrays



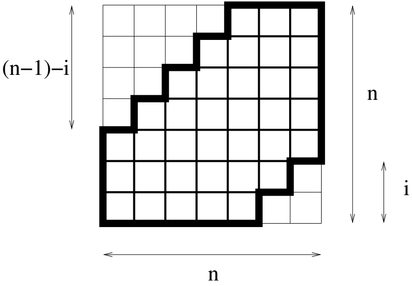
In the Costas array resulting from a honeycomb array, there are n consecutive SW–NE diagonals, each of which contains exactly one dot.

All honeycomb arrays are of radius r

Theorem (Blackburn, Paterson, Panoui, Stinson)

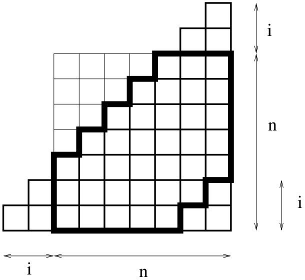
Any honeycomb array is a honeycomb array of radius r for some r. Consequently, any honeycomb array has an odd number of dots.

Proof. A honeycomb array on n dots must look like this:



where (WLOG) $i \leq \frac{n-1}{2}$. We need to show $i = \frac{n-1}{2}$.

All honeycomb arrays are of radius r



A honeycomb array on n dots implies the existence of n non-attacking brooks on a triangular board of width $w = n + i$, where a brook is a chess piece that can move up–down, left–right or SW–NE.

All honeycomb arrays are of radius r

Theorem (Nivasch, Lev ('05); Vanderlind, Guy, Larson ('02))

The maximum number of non-attacking brooks on a triangular board of width w is $\left\lfloor \frac{2w+1}{3} \right\rfloor$.

Remark: Blackburn, Paterson and Stinson have a new proof of this bound on non-attacking brooks, which uses linear programming techniques.

To complete the proof, observe that the inequalities

$$n \leq \left\lfloor \frac{2(n+i)+1}{3} \right\rfloor \text{ and } i < \frac{n-1}{2}$$

lead to a contradiction. Therefore, $i = \frac{n-1}{2}$, as required. This also implies that n is odd.

Remark: We haven't used the distinct differences property of honeycomb arrays.

The size of honeycomb arrays

Golomb and Taylor (1984) asked whether there exist honeycomb arrays of arbitrary size. They conjectured yes; however:

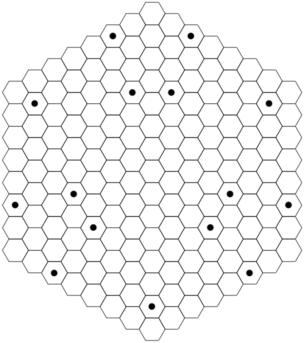
Theorem (Blackburn, Etzion, Martin, Paterson)

There are no honeycomb arrays with n dots when $n \geq 1289$.

Remark: The proof only uses the distinct differences property of the honeycomb array.

The number of honeycomb arrays

There are 12 honeycomb arrays known; the largest has 45 dots.



All but one of these examples has a 6-fold symmetry! (The sole exception is an array on 7 dots.)

Conjecture (S.R. Blackburn)

There are exactly 12 honeycomb arrays.

References

S.R. Blackburn, T. Etzion, K.M. Martin and M.B. Paterson, Two-Dimensional Patterns with Distinct Differences – Constructions, Bounds, and Maximal Anticodes. IEEE Trans. Inform. Theory, Vol. 56 (2010), pp. 1216–1229.

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