A Combinatorial Design Method for Repairing Shares in Threshold Schemes

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(t, n)-Threshold Schemes

- We informally define a **(t, n)-threshold scheme**.
- Let t and n be positive integers, $t \leq n$.
- A **secret K** is "split" into n **shares**, denoted $s_1, \ldots, s_n$.
- The following two properties should hold:
  1. a **reconstruction algorithm** can be used to reconstruct the secret, given any t of the n shares,
  2. no $t - 1$ shares reveal any information as to the value of the secret.
- Threshold schemes were invented independently by **Blakley** and **Shamir** in 1979.
- Shamir's threshold scheme is based on **polynomial interpolation** over $\mathbb{Z}_p$, where $p \geq n + 1$ is prime.
- It is really a **Reed-Solomon code** in disguise.

Shamir Threshold Scheme

- In an **initialization phase**, $x_1, x_2, \ldots, x_n$ are defined to be n distinct non-zero elements of $\mathbb{Z}_p$.
- the **dealer** gives x_i to **user** U_i, for all i, $1 \leq i \leq n$.
- The x_i's are **public** information.
- For a given secret $K \in \mathbb{Z}_p$, shares are created as follows:
  1. Let $a(x) \in \mathbb{Z}_p[x]$ be a **random polynomial** of degree at most $t-1$, such that the constant term is the secret, K.
  2. For $1 \leq i \leq n$, the share $\boldsymbol{s_i = a(x_i)}$ (so the shares are evaluations of the polynomial $a(x)$ at n non-zero points).
- Suppose we have t shares $s_{i_j} = a(x_{i_j})$, $1 \leq j \leq t$.
- Since $a(x)$ is a polynomial of degree at most $t-1$, we can determine $a(x)$ by **Lagrange interpolation**.
- Then K is computed by substituting $x = 0$ into $a(x)$.

Repairability

- Suppose that an arbitrary user U_ℓ (in a **(t,n)-threshold scheme**, say) loses their share.
- The goal is to find a **secure protocol**, involving U_ℓ and a subset of the other users, that allows the missing share s_ℓ to be reconstructed.
- We are considering a setting where the dealer is **no longer present** in the scheme after the initial setup.
- We will assume secure pairwise channels linking pairs of users.
- Three techniques for repairing shares:
  1. the **enrollment scheme** (Nojoumian, 2012)
  2. **secure regenerating codes** (Shah, Rashmi and Kumar, 2011)
  3. **combinatorial schemes** (Stinson and Wei, 2018)
- For a survey of these techniques, see (Laing and Stinson, 2018).
- In this talk, we discuss the third technique.

Repairable Threshold Schemes

- A **(t, n, d)-repairable threshold scheme**, which we abbreviate to **(t, n, d)-RTS**, is a (t, n)-threshold scheme which permits the share of an arbitrary user P_ℓ to be repaired as follows.
- Certain subsets of d users (not including P_ℓ) send a message to P_ℓ.
- The messages received by P_ℓ allow P_ℓ's share to be reconstructed.
- We note that $\boldsymbol{d \geq t}$ is an obvious **necessary condition** for the existence of such a scheme.
- If $t - 1$ users could repair a share, then they would have t shares and they could reconstruct the secret, which is not allowed.

Goals of Combinatorial Repairability

- Our method employs a **base scheme** in which users receive multiple shares.
- A certain **distribution design** specifies which shares are given to which users.
- We study three problems:
  1. **Thresholds:** What properties of the distribution design ensure that the resulting scheme is in fact a (t, n, d)-RTS?
  2. **Scalability**: How can we accommodate various numbers of users from one specific distribution design?
  3. **Reliability**: What can we say about the probability of successful repair, in a scenario where users are available with some probability p?

A Combinatorial RTS based on a $(9, 3, 1)$-BIBD

- As an example, we construct a $(2, 12, 3)$-**RTS**.
- Start with a $(9, 3, 1)$-**BIBD** (an affine plane of order 3), which has **12 blocks**.
- This is the **distribution design**, which is **public**.
- We associate a block of the design with each user:

$$
\begin{array}{rrr}
U_1 \leftrightarrow \{1,2,3\} & U_2 \leftrightarrow \{4,5,6\} & U_3 \leftrightarrow \{7,8,9\} \\
U_4 \leftrightarrow \{1,4,7\} & U_5 \leftrightarrow \{2,5,8\} & U_6 \leftrightarrow \{3,6,9\} \\
U_7 \leftrightarrow \{1,5,9\} & U_8 \leftrightarrow \{2,6,7\} & U_9 \leftrightarrow \{3,4,8\} \\
U_{10} \leftrightarrow \{1,6,8\} & U_{11} \leftrightarrow \{2,4,9\} & U_{12} \leftrightarrow \{3,5,7\}
\end{array}
$$

- Each user gets three shares from a $(5, 9)$-**threshold scheme** (the **base scheme**), as specified by the associated block.
- This threshold scheme has nine shares, denoted $s_1, \ldots, s_9$.
- Each block lists the **indices of shares** held by a given user.
- Thus U_1 has the shares s_1, s_2 and s_3, etc.

Thresholds

- The base scheme has threshold equal to $\mathbf{5}$.
- The resulting RTS has threshold equal to $\mathbf{2}$.
- This is explained as follows:

Any **two blocks** of the distribution design contain **at least five points**, whereas **one block** contains only **three points**.

- Therefore, in the resulting RTS, any **two users** can reconstruct the secret, since they (collectively) have at least **five** distinct shares from the base scheme.
- However, **one user** cannot reconstruct the secret, because it only has **three** shares from the base scheme (which is less than the required threshold of five shares).

Repairability

- When a user wants to repair their share, they contact $\boldsymbol{d = 3}$ **other users** who have the relevant subshares.
- For example, U_1 could contact U_4 to obtain subshare #1, U_8 to obtain subshare # 2 and U_{12} to obtain subshare #3:

$$
\begin{array}{ccc}
U_1 \leftarrow \{\mathbf{1}, \mathbf{2}, \mathbf{3}\} & U_2 \leftarrow \{4, 5, 6\} & U_3 \leftarrow \{7, 8, 9\} \\
U_4 \leftarrow \{\mathbf{1}, 4, 7\} & U_5 \leftarrow \{2, 5, 8\} & U_6 \leftarrow \{3, 6, 9\} \\
U_7 \leftarrow \{1, 5, 9\} & U_8 \leftarrow \{\mathbf{2}, 6, 7\} & U_9 \leftarrow \{3, 4, 8\} \\
U_{10} \leftarrow \{1, 6, 8\} & U_{11} \leftarrow \{2, 4, 9\} & U_{12} \leftarrow \{\mathbf{3}, 5, 7\}
\end{array}
$$

Scalability

- The described RTS is a scheme for 12 users, where 12 is the number of blocks in the distribution design.
- We can use a **subset** of the 12 blocks, and thereby reduce the number of users, provided that we can still repair shares.
- It suffices to choose a subset of blocks such that each point is a **contained in at least two blocks**.
- Here, we can take the **first six blocks**, along with **any subset of the last six blocks**.
- This allows us to construct a $\mathbf{(2, m, 3)}$**-RTS** for any $m \in \{6, \ldots, 12\}$.
- Note that the first six blocks comprise two parallel classes of the design, so every point occurs exactly twice in these blocks.

Required Properties of a Distribution Design

In order to be able to construct an RTS with threshold t, the distribution design must satisfy the property that

> the number of points in the **union of any t blocks** is greater than the number of points in the **union of any $t-1$ blocks**.

In order to provide **scalability**, (i.e., construct an RTS for a variable number of users), we identify a "small" **basic repairing set**, i.e.,

> a set of blocks in the distribution design such that every point is contained in **at least two** of these blocks.

Remark: As in the previous example, taking two **parallel classes** from a **resolvable design** will yield a basic repairing set of minimum possible size.

Projective Planes as Distribution Designs

Lemma 1

The union of any $t-1$ blocks (lines) in a projective plane of order q contain **at most** $\boldsymbol{q(t-1)+1}$ *points.*

Proof.

Denote the $t-1$ lines by $A_0, \ldots, A_{t-2}$. Each A_i $(i \geq 1)$ contains a point in A_0, so

$$\left|\bigcup_{i=0}^{t-2} A_i\right| \leq q + 1 + (t-2)q = q(t-1) + 1.$$

$\square$

Remark: Equality occurs if and only if the $t-1$ lines all contain a common point.

Projective Planes as Distribution Designs (cont.)

Lemma 2

For $t \leq q+1$, the union of any t lines in a projective plane of order q contain **at least** $\boldsymbol{t(q+1-(t-1)/2)}$ *points.*

Proof.

Denote the t lines by $A_0, \ldots, A_{t-1}$. Each A_i contains $q+1-i$ points that are not in $\bigcup_{h=0}^{i-1} A_h$. It follows that

$$\left|\bigcup_{i=0}^{t-1} A_i\right| \geq \sum_{i=0}^{t-1}(q+1-i) = t(q+1) - \frac{t(t-1)}{2}.$$

□

Remark: Equality occurs if and only if no three of the t lines are collinear, so they form the dual of a ***t*-arc**.

Example

- Consider a projective plane of order 5.
- One block contains 6 points.
- Two blocks contain 11 points.
- Three blocks contain **at least** 15 and **at most** 16 points.
- Four blocks contain **at least** 18 and **at most** 21 points.
- Five blocks contain **at least** 20 points.
- We can construct RTS with the following thresholds:
  - $t = 2$ (since $\mathbf{6 < 11}$),
  - $t = 3$ (since $\mathbf{11 < 15}$), and
  - $t = 4$ (since $\mathbf{16 < 18}$)
- However $t = 5$ does not work (because $\mathbf{21 \geq 20}$).

Basic Repairing Sets in Projective Planes

- Recall that a basic repairing set is a subset of blocks (lines) that contains **every point at least twice**.
- In the context of a projective plane, this is precisely the dual of a **2-blocking set** (see, e.g., Ball and Blokhuis, 1996).
- A simple construction: Choose any three noncollinear points x, y and z of the projective plane, and take all the lines that contain at least one of these points. This yields a basic repairing set of size $\boldsymbol{3q}$.
- Another construction: Suppose that q is a square of a prime power. Start with two disjoint **Baer subplanes** in $\mathbf{PG(2,q)}$ and take all the lines that contain a line from either of these two subplanes. This yields a basic repairing set of size $\boldsymbol{2(q+\sqrt{q}+1)}$, which is an improvement asymptotically over the previous construction.

More Efficient Repairing

- Suppose we use a t-design with $\boldsymbol{t > 2}$ as a distribution design.
- This could permit a more efficient repairing process.
- For example, suppose we use a **3-$(v, 4, 1)$-design** (a **Steiner quadruple system**).
- Since some pairs of blocks intersect in two points, a user's share (which consists of **four subshares**) could be repaired using information supplied by **two other users**, each of which contributes **two subshares**.
- This reduces the **number** of messages sent (but not the total **size** of the messages).
- It could also have a positive effect on the **reliability** of the scheme (to be discussed a bit later).
- Two blocks contain at least six points, and one block contains four points.
- Therefore, because $6 > 4$, we obtain a $(2, n, 2)$-RTS, where n is the number of blocks in the design.

The $\mathbf{3}$-$(\mathbf{8}, \mathbf{4}, \mathbf{1})$-design

$$
\begin{array}{rcl}
A_1 \leftrightarrow \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\} & \qquad & A_2 \leftrightarrow \{5, 6, 7, 8\} \\
B_1 \leftrightarrow \{\mathbf{1}, \mathbf{2}, 5, 6\} & & B_2 \leftrightarrow \{1, 2, 7, 8\} \\
B_3 \leftrightarrow \{1, 3, 5, 7\} & & B_4 \leftrightarrow \{1, 3, 6, 8\} \\
B_5 \leftrightarrow \{1, 4, 5, 8\} & & B_6 \leftrightarrow \{1, 4, 6, 7\} \\
B_7 \leftrightarrow \{\mathbf{3}, \mathbf{4}, 7, 8\} & & B_8 \leftrightarrow \{3, 4, 5, 6\} \\
B_9 \leftrightarrow \{2, 4, 6, 8\} & & B_{10} \leftrightarrow \{2, 4, 5, 7\} \\
B_{11} \leftrightarrow \{2, 3, 6, 7\} & & B_{12} \leftrightarrow \{2, 3, 5, 8\}
\end{array}
$$

- Suppose A_1 wants to repair their share.
- They could contact B_1 to get subshares $\#$ 1 and $\#$ 2, and B_7 to get subshares $\#$ 3 and $\#$ 4.

A Network Reliability Model

- When a user contacts other users in an attempt to repair their share, the other users may not be available.
- Suppose that each user is **available** with probability $\boldsymbol{p}$ and **unavailable** with probability $\boldsymbol{q = 1 - p}$, independent of any other users.
- We consider two interesting and natural questions that arise when a user wants to repair their share:
  1. What is the probability $\boldsymbol{\mathcal{R}(p)}$ that there is **at least one** set of available users that can repair the given share?
  2. What is the **expected number** $\boldsymbol{\mathcal{E}(p)}$ of (minimal) sets of available users that can repair the given share?
- $\mathcal{R}(p)$ and $\mathcal{E}(p)$ are **reliability polynomials** in the variable p (or q).

Repairing RTS Based on BIBDs with $\boldsymbol{\lambda = 1}$

- Suppose a fixed user U_ℓ wants to repair a share corresponding to the block $\{x_1, \ldots, x_k\}$.
- Each point of the BIBD is in $\boldsymbol{r = (v-1)/(k-1)}$ blocks.
- For $1 \leq i \leq k$, there are $r-1$ users **other than** U_ℓ who has subshare # x_i; call this subset $\boldsymbol{\mathcal{U}(x_i)}$.
- The subsets $\mathcal{U}(x_1)$, ... , $\mathcal{U}(x_k)$ are **disjoint**.
- Consider the $(9,3,1)$-BIBD:

$$
\begin{array}{rrr}
U_1 \leftrightarrow \{\mathbf{1},\mathbf{2},\mathbf{3}\} & U_2 \leftrightarrow \{4,5,6\} & U_3 \leftrightarrow \{7,8,9\} \\
U_4 \leftrightarrow \{\mathbf{1},4,7\} & U_5 \leftrightarrow \{\mathbf{2},5,8\} & U_6 \leftrightarrow \{\mathbf{3},6,9\} \\
U_7 \leftrightarrow \{\mathbf{1},5,9\} & U_8 \leftrightarrow \{\mathbf{2},6,7\} & U_9 \leftrightarrow \{\mathbf{3},4,8\} \\
U_{10} \leftrightarrow \{\mathbf{1},6,8\} & U_{11} \leftrightarrow \{\mathbf{2},4,9\} & U_{12} \leftrightarrow \{\mathbf{3},5,7\}
\end{array}
$$

- Suppose U_1 wants to repair their share.
- Then $\boldsymbol{\mathcal{U}(1) = \{U_4, U_7, U_{10}\}}$, $\boldsymbol{\mathcal{U}(2) = \{U_5, U_8, U_{11}\}}$ and $\boldsymbol{\mathcal{U}(3) = \{U_6, U_9, U_{12}\}}$

Repairing RTS Based on BIBDs with $\boldsymbol{\lambda = 1}$ (cont.)

- The probability that at least one user in a **specific** $\mathcal{U}(x_i)$ is available is $\boldsymbol{1 - q^3}$.
- $\mathcal{R}(p)$ is the probability that at least one user in **every** $\mathcal{U}(x_i)$ is available, so we have

$$\boldsymbol{\mathcal{R}(p) = (1 - q^3)^3}.$$

- Computing the expected number of minimal repairing sets is even easier; it follows from linearity of expectation that

$$\boldsymbol{\mathcal{E}(p) = (3p)^3}.$$

- In general, for a $(v, k, 1)$-BIBD, we have

$$\boldsymbol{\mathcal{R}(p) = (1 - q^{r-1})^k}$$

and

$$\boldsymbol{\mathcal{E}(p) = ((r - 1)p)^k}.$$

Repairing RTS Based on Steiner Quadruple Systems

- For SQS, the situation is more complicated, as there are various types of repairing sets to consider.
- A **minimal** repairing set could have size $\mathbf{2}$, $\mathbf{3}$ or $\mathbf{4}$.
- A standard technique from network reliability proves useful in computing the reliability polynomials.
- A **cut** is a minimal set of users with the property that repairing is impossible if **all the users in the cut are unavailable**; in this case we say that the cut **fails**.
- Suppose a user U_ℓ wants to repair a share corresponding to the block $\{x_1, \ldots, x_k\}$; then the cuts are $\mathcal{U}(x_i)$, $1 \leq i \leq k$.
- The cuts for an SQS are **not** disjoint.

Cuts in the $\mathbf{3}$-$(\mathbf{8}, \mathbf{4}, \mathbf{1})$-design

$$
\begin{array}{rlrl}
A_1 \leftrightarrow \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\} & & A_2 \leftrightarrow \{5, 6, 7, 8\} \\
B_1 \leftrightarrow \{\mathbf{1}, \mathbf{2}, 5, 6\} & & B_2 \leftrightarrow \{\mathbf{1}, \mathbf{2}, 7, 8\} \\
B_3 \leftrightarrow \{\mathbf{1}, \mathbf{3}, 5, 7\} & & B_4 \leftrightarrow \{\mathbf{1}, \mathbf{3}, 6, 8\} \\
B_5 \leftrightarrow \{\mathbf{1}, \mathbf{4}, 5, 8\} & & B_6 \leftrightarrow \{\mathbf{1}, \mathbf{4}, 6, 7\} \\
B_7 \leftrightarrow \{\mathbf{3}, \mathbf{4}, 7, 8\} & & B_8 \leftrightarrow \{\mathbf{3}, \mathbf{4}, 5, 6\} \\
B_9 \leftrightarrow \{\mathbf{2}, \mathbf{4}, 6, 8\} & & B_{10} \leftrightarrow \{\mathbf{2}, \mathbf{4}, 5, 7\} \\
B_{11} \leftrightarrow \{\mathbf{2}, \mathbf{3}, 6, 7\} & & B_{12} \leftrightarrow \{\mathbf{2}, \mathbf{3}, 5, 8\}
\end{array}
$$

Suppose A_1 wants to repair their share; then

$$
\begin{aligned}
&\boldsymbol{\mathcal{U}(1) = \{B_1, B_2, B_3, B_4, B_5, B_6\}} \\
&\boldsymbol{\mathcal{U}(2) = \{B_1, B_2, B_9, B_{10}, B_{11}, B_{12}\}} \\
&\boldsymbol{\mathcal{U}(3) = \{B_3, B_4, B_7, B_8, B_{11}, B_{12}\}} \\
&\boldsymbol{\mathcal{U}(4) = \{B_5, B_6, B_7, B_8, B_9, B_{10}\}}.
\end{aligned}
$$

Computing $\mathcal{R}(p)$ for the $\mathbf{3}$-$(\mathbf{8}, \mathbf{4}, \mathbf{1})$-design

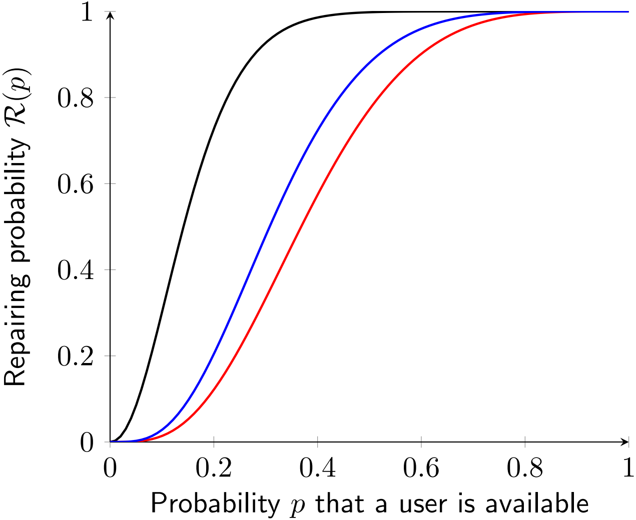
- Repairing fails if and only if **at least one of the four cuts fails**, so we can compute $\mathcal{R}(p)$ using the **principle of inclusion-exclusion** (**PIE**).
- $|\boldsymbol{\mathcal{U}(i)}| = \mathbf{6}$ for all i, so the probability that a cut $\mathcal{U}(i)$ fails is $\boldsymbol{q^6}$ (recall $q = 1 - p$).
- The probability that two given cuts both fail is $\boldsymbol{q^{10}}$, because $|\boldsymbol{\mathcal{U}(i) \cup \mathcal{U}(j)}| = \mathbf{10}$ for all $1 \leq i < j \leq 4$
- The probability that three or four given cuts all fail is $\boldsymbol{q^{12}}$.
- Applying **PIE**, we obtain

$$1 - \mathcal{R}(p) = 4q^6 - \binom{4}{2} q^{10} + \binom{4}{3} q^{12} - \binom{4}{4} q^{12}$$

- Hence,

$$\boldsymbol{\mathcal{R}(p) = 1 - 4q^6 + 6q^{10} - 3q^{12}}.$$

Graphing $\mathcal{R}(p)$ for 2-$(13, 4, 1)$, 2-$(16, 4, 1)$, and 3-$(10, 4, 1)$-designs



Computing $\mathcal{E}(p)$ for the 3-$(10,4,1)$-design

$A_0 \leftrightarrow \{1,2,4,5\}$	$B_0 \leftrightarrow \{1,2,3,7\}$	$C_0 \leftrightarrow \{1,3,5,8\}$
$A_1 \leftrightarrow \{2,3,5,6\}$	$B_1 \leftrightarrow \{2,3,4,8\}$	$C_1 \leftrightarrow \{2,4,6,9\}$
$A_2 \leftrightarrow \{3,4,6,7\}$	$B_2 \leftrightarrow \{3,4,5,9\}$	$C_2 \leftrightarrow \{3,5,7,0\}$
$A_3 \leftrightarrow \{4,5,7,8\}$	$B_3 \leftrightarrow \{4,5,6,0\}$	$C_3 \leftrightarrow \{4,6,8,1\}$
$A_4 \leftrightarrow \{5,6,8,9\}$	$B_4 \leftrightarrow \{5,6,7,1\}$	$C_4 \leftrightarrow \{5,7,9,2\}$
$A_5 \leftrightarrow \{6,7,9,0\}$	$B_5 \leftrightarrow \{6,7,8,2\}$	$C_5 \leftrightarrow \{6,8,0,3\}$
$A_6 \leftrightarrow \{7,8,0,1\}$	$B_6 \leftrightarrow \{7,8,9,3\}$	$C_6 \leftrightarrow \{7,9,1,4\}$
$A_7 \leftrightarrow \{8,9,1,2\}$	$B_7 \leftrightarrow \{8,9,0,4\}$	$C_7 \leftrightarrow \{8,0,2,5\}$
$A_8 \leftrightarrow \{9,0,2,3\}$	$B_8 \leftrightarrow \{9,0,1,5\}$	$C_8 \leftrightarrow \{9,1,3,6\}$
$A_9 \leftrightarrow \{0,1,3,4\}$	$B_9 \leftrightarrow \{0,1,2,6\}$	$C_9 \leftrightarrow \{0,2,4,7\}$

This design has a cyclic automorphism generated by
$\boldsymbol{x \mapsto x + 1 \bmod 10}$.

Computing $\mathcal{E}(p)$ for the $\mathbf{3}$-$(\mathbf{10}, \mathbf{4}, \mathbf{1})$-design: minimal repairing sets of size two

- Suppose we want to repair the block $A_0 = \{1, 2, 4, 5\}$.
- We consider minimal repairing sets of sizes **two, three** and **four**.
- A repairing set of size two consists of
  - a block containing $\mathbf{1, 2}$ and a block containing $\mathbf{4, 5}$; or
  - a block containing $\mathbf{1, 4}$ and a block containing $\mathbf{2, 5}$; or
  - a block containing $\mathbf{1, 5}$ and a block containing $\mathbf{2, 4}$.
- The total number of choices is $3 \times 3 \times 3 = 27$.
- Therefore, the expected number of repairing sets of size two is $\mathbf{27p^2}$.

Minimal repairing sets of size two

$A_0 \leftrightarrow \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}\}$	$B_0 \leftrightarrow \{\mathbf{1}, \mathbf{2}, 3, 7\}$	$C_0 \leftrightarrow \{1, 3, 5, 8\}$
$A_1 \leftrightarrow \{2, 3, 5, 6\}$	$B_1 \leftrightarrow \{2, 3, 4, 8\}$	$C_1 \leftrightarrow \{2, 4, 6, 9\}$
$A_2 \leftrightarrow \{3, 4, 6, 7\}$	$B_2 \leftrightarrow \{3, \mathbf{4}, \mathbf{5}, 9\}$	$C_2 \leftrightarrow \{3, 5, 7, 0\}$
$A_3 \leftrightarrow \{\mathbf{4}, \mathbf{5}, 7, 8\}$	$B_3 \leftrightarrow \{\mathbf{4}, \mathbf{5}, 6, 0\}$	$C_3 \leftrightarrow \{4, 6, 8, 1\}$
$A_4 \leftrightarrow \{5, 6, 8, 9\}$	$B_4 \leftrightarrow \{5, 6, 7, 1\}$	$C_4 \leftrightarrow \{5, 7, 9, 2\}$
$A_5 \leftrightarrow \{6, 7, 9, 0\}$	$B_5 \leftrightarrow \{6, 7, 8, 2\}$	$C_5 \leftrightarrow \{6, 8, 0, 3\}$
$A_6 \leftrightarrow \{7, 8, 0, 1\}$	$B_6 \leftrightarrow \{7, 8, 9, 3\}$	$C_6 \leftrightarrow \{7, 9, 1, 4\}$
$A_7 \leftrightarrow \{8, 9, \mathbf{1}, \mathbf{2}\}$	$B_7 \leftrightarrow \{8, 9, 0, 4\}$	$C_7 \leftrightarrow \{8, 0, 2, 5\}$
$A_8 \leftrightarrow \{9, 0, 2, 3\}$	$B_8 \leftrightarrow \{9, 0, 1, 5\}$	$C_8 \leftrightarrow \{9, 1, 3, 6\}$
$A_9 \leftrightarrow \{0, 1, 3, 4\}$	$B_9 \leftrightarrow \{0, \mathbf{1}, \mathbf{2}, 6\}$	$C_9 \leftrightarrow \{0, 2, 4, 7\}$

Minimal repairing sets of size four

- A minimal repairing set of size four consists of four blocks having the following form:
  - a block containing $\mathbf{1}$, but none of $\mathbf{2, 4, 5}$
  - a block containing $\mathbf{2}$, but none of $\mathbf{1, 4, 5}$
  - a block containing $\mathbf{4}$, but none of $\mathbf{1, 2, 5}$
  - a block containing $\mathbf{5}$, but none of $\mathbf{1, 2, 4}$
- There are **two choices** for each of these four blocks.
- The total number of choices is $2^4 = 16$.
- Therefore, the expected number of minimal repairing sets of size four is $\boldsymbol{16p^4}$.

Minimal repairing sets of size four (cont.)

$A_0 \leftrightarrow \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}\}$	$B_0 \leftrightarrow \{1, 2, 3, 7\}$	$C_0 \leftrightarrow \{1, 3, 5, 8\}$
$A_1 \leftrightarrow \{2, 3, 5, 6\}$	$B_1 \leftrightarrow \{2, 3, 4, 8\}$	$C_1 \leftrightarrow \{2, 4, 6, 9\}$
$A_2 \leftrightarrow \{3, \mathbf{4}, 6, 7\}$	$B_2 \leftrightarrow \{3, 4, 5, 9\}$	$C_2 \leftrightarrow \{3, \mathbf{5}, 7, 0\}$
$A_3 \leftrightarrow \{4, 5, 7, 8\}$	$B_3 \leftrightarrow \{4, 5, 6, 0\}$	$C_3 \leftrightarrow \{4, 6, 8, 1\}$
$A_4 \leftrightarrow \{\mathbf{5}, 6, 8, 9\}$	$B_4 \leftrightarrow \{5, 6, 7, 1\}$	$C_4 \leftrightarrow \{5, 7, 9, 2\}$
$A_5 \leftrightarrow \{6, 7, 9, 0\}$	$B_5 \leftrightarrow \{6, 7, 8, \mathbf{2}\}$	$C_5 \leftrightarrow \{6, 8, 0, 3\}$
$A_6 \leftrightarrow \{7, 8, 0, \mathbf{1}\}$	$B_6 \leftrightarrow \{7, 8, 9, 3\}$	$C_6 \leftrightarrow \{7, 9, 1, 4\}$
$A_7 \leftrightarrow \{8, 9, 1, 2\}$	$B_7 \leftrightarrow \{8, 9, 0, \mathbf{4}\}$	$C_7 \leftrightarrow \{8, 0, 2, 5\}$
$A_8 \leftrightarrow \{9, 0, \mathbf{2}, 3\}$	$B_8 \leftrightarrow \{9, 0, 1, 5\}$	$C_8 \leftrightarrow \{9, \mathbf{1}, 3, 6\}$
$A_9 \leftrightarrow \{0, 1, 3, 4\}$	$B_9 \leftrightarrow \{0, 1, 2, 6\}$	$C_9 \leftrightarrow \{0, 2, 4, 7\}$

Minimal repairing sets of size three

- A **minimal** repairing set of size three can have three possible forms:
  - type **pair - pair - pair**: three pairs intersecting in a point, e.g., $\mathbf{12, 14, 15}$. There are **four** configurations of this type.
  - type **pair - pair - point**: two pairs intersecting in a point, and a disjoint point e.g., $\mathbf{12, 14, 5}$. There are **twelve** configurations of this type.
  - type **pair - point - point**: one pair, and two disjoint points, e.g., $\mathbf{12, 4, 5}$. There are **six** configurations of this type.
- After some counting, the expected number of minimal repairing sets of size three is seen to be

$$(\mathbf{4} \times \mathbf{3^3} + \mathbf{12} \times \mathbf{3^2} \times \mathbf{2} + \mathbf{6} \times \mathbf{3} \times \mathbf{2^2})\boldsymbol{p^3} = \mathbf{396}\boldsymbol{p^3}.$$

- Therefore,

$$\boldsymbol{\mathcal{E}(p) = 27p^2 + 396p^3 + 16p^4}.$$

Summary and Open Problems

1. We have a formula to compute $\mathcal{R}(p)$ for any t-$(v,k,1)$-design. However, we only have formulas for $\mathcal{E}(p)$ for 2-$(v,k,1)$-designs and for 3-$(v,4,1)$-designs.
2. Are there **probabilistic** existence results for "good" distribution designs?
3. What other types of combinatorial structures yield "good" distribution designs?
4. By using **ramp schemes** for the base scheme, it is possible to get more efficient RTS for certain distribution designs; see Stinson and Wei (2018).
5. We have also been investigating how to design **efficient algorithms** to find a repairing set (Kacsmar and Stinson).

Happy Birthday Charlie!