Fast and Small:
Multiplying Polynomials without Extra Space

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CECM Day
SFU, Vancouver, 24 July 2009

Preliminaries

We study algorithms for univariate polynomial multiplication:

The Problem

Given: A ring R, an integer n,
and $f, g \in \mathrm{R}[x]$ with degrees less than n

Compute: Their product $f \cdot g \in \mathrm{R}[x]$

The Model

- Ring operations have unit cost
- Random reads from input, random reads/writes to output
- Space complexity determined by size of auxiliary storage

Univariate Multiplication Algorithms

	Time Complexity	**Space Complexity**
Classical Method	$O(n^2)$	$O(1)$
Divide-and-Conquer Karatsuba/Ofman ’63	$O(n^{\log_2 3})$ or $O(n^{1.59})$	$O(n)$
FFT-based Schönhage/Strassen ’71 Cantor/Kaltofen ’91	$O(n \log n \log \log n)$	$O(n)$

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Goal: Keep time complexity the same, reduce space

The Evolution of Multiplication

Small and slow

The Evolution of Multiplication

Big and fast


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The Evolution of Multiplication

Small and fast

Previous Work

- Savage & Swamy 1979 $O(n^2)$ time-space lower bound for straight line programs
- Abrahamson 1985: $O(n^2)$ time-space lower bound for branching programs

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- Monagan 1993: Importance of space efficiency for multiplication over $\mathbb{Z}_p[x]$
- Maeder 1993: Bounds extra space for Karatsuba multiplication so that storage can be preallocated — about $2n$ extra memory cells required.
- Thomé 2002: Karatsuba multiplication for polynomials using n extra memory cells.

Present Contributions

- New Karatsuba-like algorithm with $O(\log n)$ space
- New FFT-based algorithm with $O(1)$ space
  under certain conditions
- Implementations in `C` over $\mathbb{Z}/p\mathbb{Z}$

Standard Karatsuba Algorithm

Idea: Reduce one degree-$2k$ multiplication to three of degree k.

- Originally noticed by Gauss (multiplying complex numbers), rediscovered and formalized by Karatsuba & Ofman

Input: $f, g \in \mathrm{R}[x]$ each with degree less than $2k$.

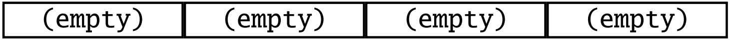
Write $f = f_0 + f_1 x^k$ and $g = g_0 + g_1 x^k$.

Low-Space Karatsuba Algorithms

Version “0”

Read-Only Input Space:

Read/Write Output Space:



To Compute: $f \cdot g$

Low-Space Karatsuba Algorithms

Version “1”

① The low-order coefficients of the output are initialized as h, and the product $f \cdot g$ is added to this.

Read-Only Input Space:

f01	f11	g0	g1

Read/Write Output Space:

h0	h1	(empty)	(empty)

To Compute: $f \cdot g + h$

Low-Space Karatsuba Algorithms

Version “2”

1. The low-order coefficients of the output are initialized as h, and the product $f \cdot g$ is added to this.
2. The first polynomial f is given as a sum $f^{(0)} + f^{(1)}$.

Read-Only Input Space:

Read/Write Output Space:

To Compute: $(f^{(0)} + f^{(1)}) \cdot g + h$

Dirty Details

Restrict modulus to 29 bits to allow for delayed reductions

In the Karatsuba step

- Only 4 values are added/subtracted in one position
- Delay reductions, perform two “corrections”

Classical algorithm

- Switch over at $n \leq 32$ (determined experimentally)
- Perform arithmetic in double-precision `long longs`;
  delay reductions (a la Monagan)

Problem: code explosion

3 “versions” of algorithms (based on extra constraints)

×

Karatsuba or classical

×

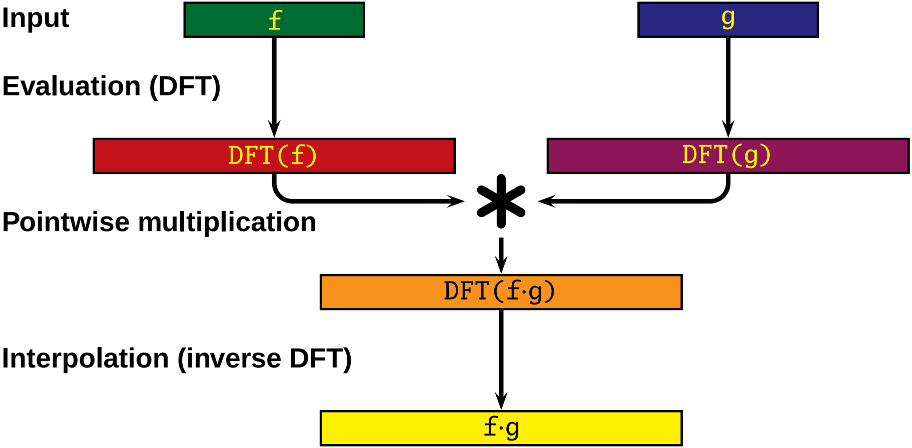
odd-sized or even-sized operands

×

equal-sized operands or “one different”

Solution: Use “supermacros” in C:
Same file is included multiple times with some parameter values changed (crude form of code generation).

DFT-Based Multiplication



Simplifying Assumptions

From now on:

- $\deg f + \deg g < n = 2^k$ for some $k \in \mathbb{N}$
- The base ring R contains a 2^k-PRU ω

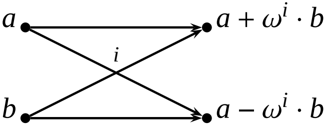
That is, assume “virtual roots of unity” have already been found, and optimize from there.

Usual Formulation of the FFT

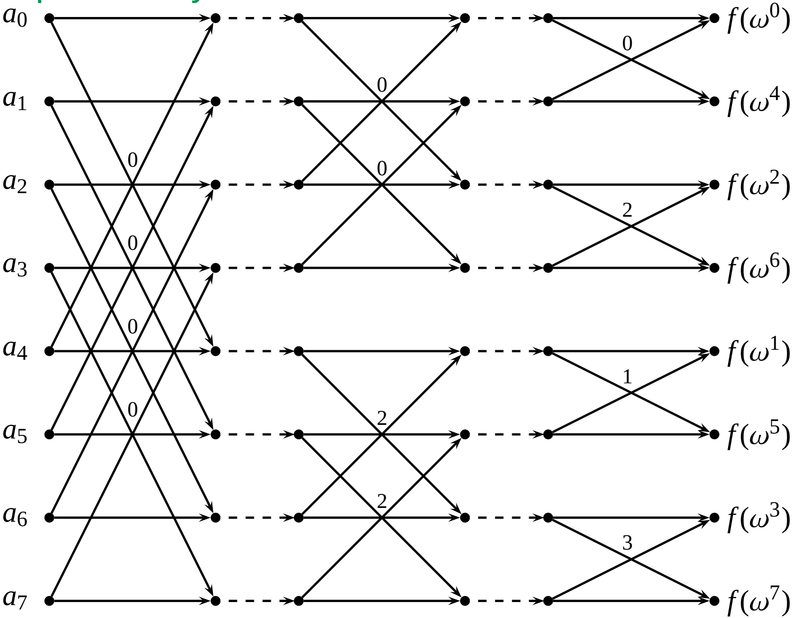
Perform two $\frac{n}{2}$-DFTs followed by $\frac{n}{2}$ 2-DFTs:

- Write $f(x) = f_{\text{even}}(x^2) + x \cdot f_{\text{odd}}(x^2)$ (i.e. $\deg f_{\text{even}}, \deg f_{\text{odd}} < n/2$)
- Compute $\mathrm{DFT}_{\omega^2}(f_{\text{even}})$ and $\mathrm{DFT}_{\omega^2}(f_{\text{odd}})$
- Compute each $f(\omega^i) = f_{\text{even}}(\omega^{2i}) + \omega \cdot f_{\text{odd}}(\omega^{2i})$

Make use of “butterfly circuit” for each size-2 DFT:



Example: 8-Way FFT



Reverted Binary Ordering

In-Place FFT permutes the ordering into reverted binary:

$0=000_2 \updownarrow 0=000_2$

$1=001_2 \updownarrow 4=100_2$

$2=010_2 \updownarrow 2=010_2$

$3=011_2 \updownarrow 6=110_2$

$4=100_2 \updownarrow 1=001_2$

$5=101_2 \updownarrow 5=101_2$

$6=110_2 \updownarrow 3=011_2$

$7=111_2 \updownarrow 7=111_2$

Problem: Powers of ω are not accessed in order

Possible solutions:

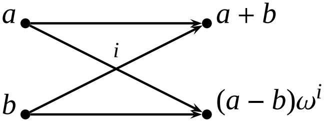
- Precompute all powers of ω — too much space
- Perform steps out of order — terrible for cache
- Permute input before computing — costly

Alternate Formulation of FFT

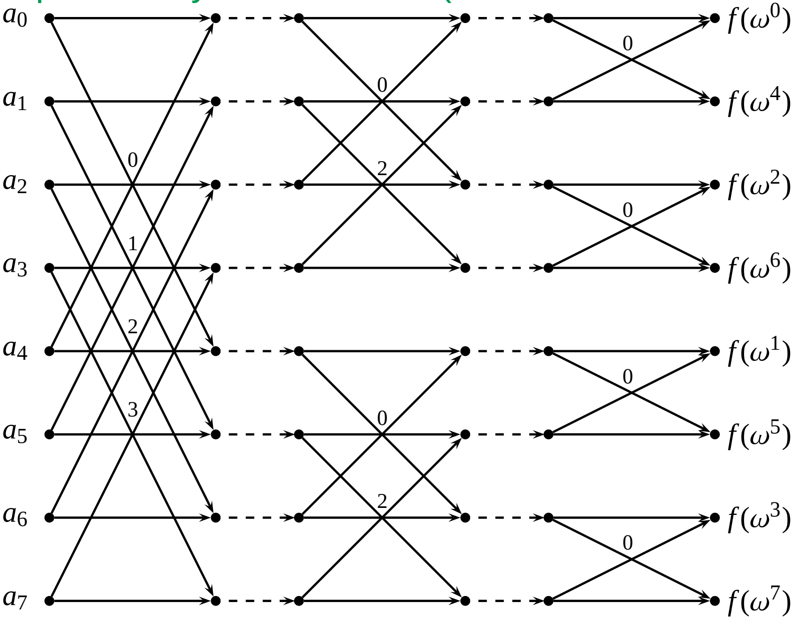
Perform $\frac{n}{2}$ 2-DFTs followed by two $\frac{n}{2}$-DFTs

- Write $f = f_{\text{low}} + x^{n/2} \cdot f_{\text{high}}$
- Compute $f_0 = f_{\text{low}} + f_{\text{high}}$ and $f_1 = f_{\text{low}}(\omega x) - f_{\text{high}}(\omega x)$
- Compute each $f(\omega^{2i}) = f_0(\omega^{2i})$ and $f(\omega^{2i+1}) = f_1(\omega^{2i})$

Modified “butterfly circuit”:



Example: 8-Way In-Place FFT (Alternate Formulation)



Folded Polynomials

Recall the basis for the “alternate” FFT formulation:

$$\begin{aligned} f_0 &= f_{\text{low}} + f_{\text{high}} \\ f_1 &= f_{\text{low}}(\omega x) - f_{\text{high}}(\omega x) \end{aligned}$$

A generalization (recalling that $n = 2^k$):

Definition (Folded Polynomials)

$$f_i = f(\omega^{2^{i-1}} x) \quad \text{rem}\, x^{2^{k-i}} - 1$$

Theorem

$$f\left(\omega^{2^i(2j+1)}\right) = f_{i+1}\left(\omega^{2^{i+1}j}\right)$$

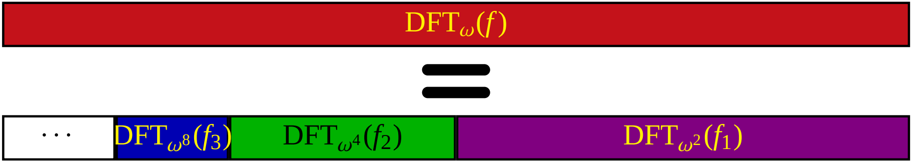
So by computing each f_i at all powers of ω^i,
we get the values of f at all powers of ω.

Recursively Applying the Alternate Formulation

Example (Reverted Binary Ordering of $0, 1, \ldots, 15$)

0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15

$\mathrm{DFT}_{\omega}(f)$ in binary reversed order
can be computed by DFTs of f_is:



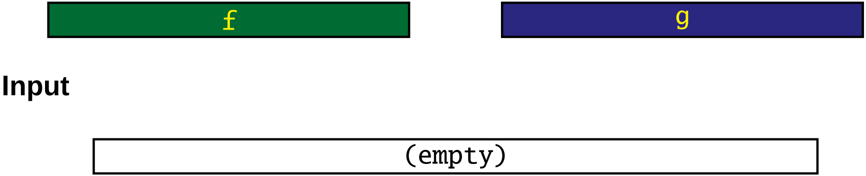
$\mathrm{DFT}_{\omega}(f)$

$=$

$\cdots$ $\mathrm{DFT}_{\omega^8}(f_3)$ $\mathrm{DFT}_{\omega^4}(f_2)$ $\mathrm{DFT}_{\omega^2}(f_1)$

FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration


`f`

`g`

Input

`(empty)`

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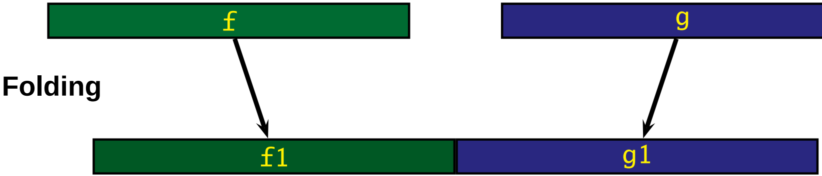


FFT-Based Multiplication without Extra Space

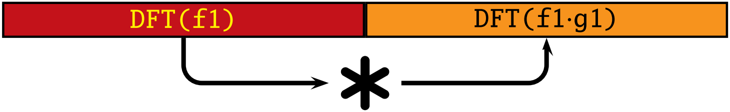
Idea: Solve half of remaining problem at each iteration

In-Place FFTs (alternate formulation)

FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration

Pointwise Multiplication



FFT-Based Multiplication without Extra Space

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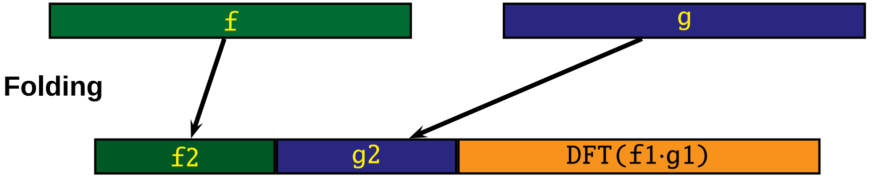


FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration

In-Place FFTs (alternate formulation)

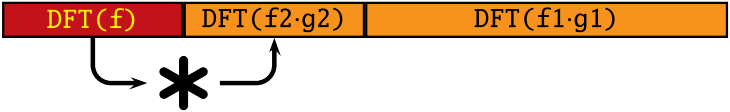
FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration

f

g

Pointwise Multiplication

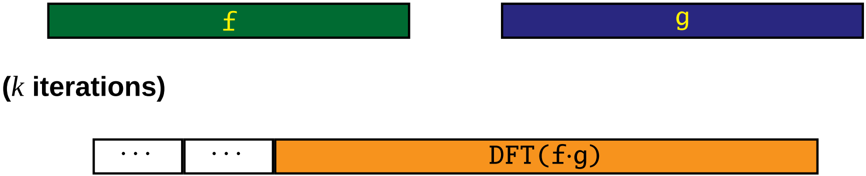


DFT(f) DFT(f2·g2) DFT(f1·g1)

$*$

FFT-Based Multiplication without Extra Space

Idea: Solve half of remaining problem at each iteration



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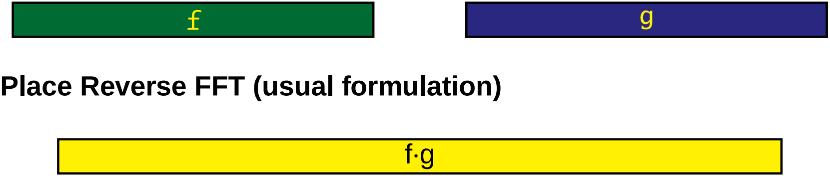


In-Place Reverse FFT (usual formulation)

Analysis

Time cost of the various stages:

- **Folding**: $O(n)$ cost times $\log n$ folds = $O(n \log n)$
- **FFTs**: $O(m \log m)$ for $m = n, n/2, n/4, \ldots, 1$ = $O(n \log n)$
- **Multiplications**: $n/2 + n/4 + \cdots + 1$ = $O(n)$

Total cost: $O(n \log n)$ time and $O(1)$ extra space
when the following conditions hold:

- $n = \deg f + \deg g + 1$ is a power of 2
- R contains an n-PRU ω

Modular Arithmetic

Use floating-point Barrett reduction (from NTL):

- Pre-compute an approximation of $1/p$
- Given $a, b \in \mathbb{Z}_p$, compute an approximation of $q = \lfloor a \cdot b \cdot (1/p) \rfloor$
- Then $ab - qp$ equals $ab \operatorname{rem} p$ plus or minus p.

The cost of this method:

- 2 `double` multiplications
- 2 `int` multiplications
- 1 `int` subtraction
- 3 conversions between `int` and `double`
- 2 “correction” steps to get exact result
  $\hookrightarrow$ not necessary until the very end!

Implementation Benchmarking

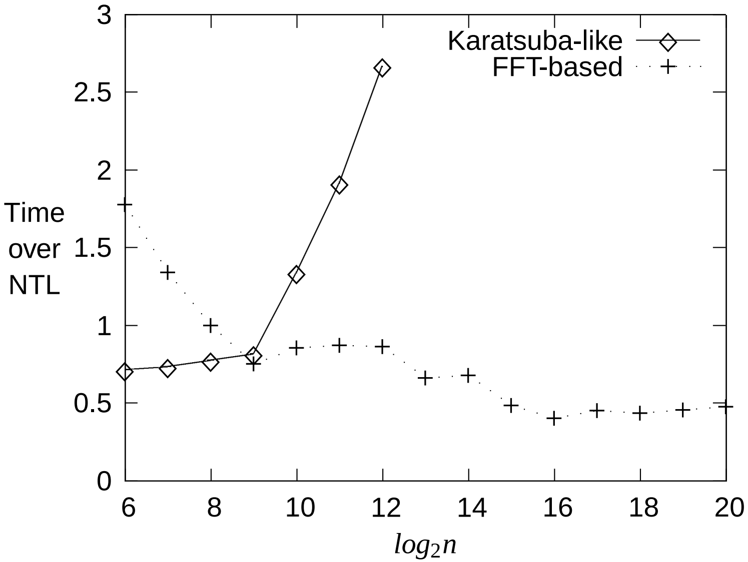
Details of tests:

- 2.5 GHz 64-bit Athalon, 256KB L1, 1MB L2, 2GB RAM
- $p = 167772161$ (28 bits)
- Comparing CPU time (in seconds) for the computation

Disclaimer

We are comparing apples to oranges.

Timing Benchmarks



Future Directions

- Efficient implementation over $\mathbb{Z}$ (GMP)
- Similar results for Toom-Cook 3-way or k-way
- What modulus bit restriction is “best”?
- Is completely in-place (overwriting input) possible?