CS459/698 Privacy, Cryptography, Network and Data Security

Discrete Logarithm, Diffie-Hellman, ElGamal

Spring 2025, Monday/Wednesday 2:30pm-4:00pm

The Discrete Logarithm Problem

$$h = g^x \text{ , find } x$$

It’s supposed to be hard to find x

I bet we can use that

But don’t forget about me

Groups?

Groups - Sets with specific properties

A **group** is a set of elements (usually numbers) that are related to each other according to well-defined operations.

- Consider a multiplicative group Z_p^*
  - This boils down to the set of non-zero integers between 1 and p-1 modulo **p** → A finite group
  - For p = 5, we have group Z_5^* = {1,2,3,4} -> i.e., the order **n** of Z_5^* is 4
  - In this group, operations are carried out mod 5:
    - 3 * 4 = 12 mod 5 = 2
    - 2^3 = 2 * 2 * 2 = 8 mod 5 = 3

Group axioms

To be a group, these sets should respect some axioms

- Closure
- Identity existence
- Associativity
- Inverse existence

- Groups can also be commutative and cyclic (up next)

Let’s take a look at some of these axioms (using multiplication as the operation)

Closure

- For every x,y in the group, x * y is in the group
  - i.e., the multiplication of two group elements falls within the group too
- Example:
  - in Z_5^*, 2* 3 = 6 mod 5 = 1

Identity Existence

- There is an element **e** such that e * x = x * e = x
  - i.e., has an element **e** such that any element times **e** outputs the element itself

- Example:
  - In any Z_p^*, the identity element is 1
  - For Z_5^* : 1 * 3 = 3 mod 5 = 3

Associativity

- For any x, y, z in the group, (x * y) * z = x * (y * z)

- Example:
  - For Z_5^* : (2 * 3) * 4 = 1 * 4 = 2 * (3 * 4) = 2 * 2 = 4

Inverse Existence

- For any **x** in the group, there is a **y** such that x * y = y * x = 1

- Example:
  - For Z_5^* : 2 * 3 = 1 , 3 * 2 = 1 (2 and 3 are inverses)
  - 4 * 4 = 16 mod 5 = 1 (4 is its own inverse)

Abelian Groups

- Abelian groups are groups which are **commutative**
- This means that x * y = y * x for any group elements x and y

- Example:
  - For Z_5^* : 3 * 4 = 2 , 4 * 3 = 2

Cyclic groups

- A group is called **cyclic** if there is at least one element **g** such that its powers (g^1, g^2, g^3, …) mod p span all distinct group elements.
  - **g** is called the "generator" of the group

- Example:
  - For Z_5^*, there are two generators (2 and 3):
    - $2^1 = 2$, $2^2 = 4$, $2^3 = 3$, $2^4 = 1$
    - $3^1 = 3$, $3^2 = 4$, $3^3 = 2$, $3^4 = 1$

Cyclic subgroups

- We can have cyclic **subgroups** within larger finite groups

- Example:
  - The order of any cyclic subgroup of $\boldsymbol{F}^*_{607}$ must divide **n** = $|\boldsymbol{F}^*_{607}|$ = 606
  - Thus, $\boldsymbol{F}^*_{607}$ has subgroups of orders {1, 2, 3, 6, 9, 18, 101, 202, 303, 606}

- Important for later:
  - The subgroup of order 101 is a subset of $\boldsymbol{F}^*_{607}$. All calculations involving its generator **g** must take place in $\boldsymbol{F}^*_{607}$, which uses modulo 607 arithmetic.

  - Even though the subgroup has order n=101, its elements are still numbers in $\boldsymbol{F}^*_{607}$, and their operations are also defined modulo 607.

Discrete Logarithm Problem

The Discrete Logarithm Problem

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The Discrete Logarithm Problem

$$h = g^x \text{ , find } x$$

Discrete: we are dealing with integers instead of real numbers

Logarithm: we are looking for the logarithm of **x** base **g**

- e.g., $\log_2 256 = 8$, since $2^8 = 256$

The Discrete Logarithm Problem

Given $(g,h) \in \mathbf{G} \times \mathbf{G}$, find $x \in \mathbf{Z_p}^*$ such that:

$$h = g^x$$

Here, **G** is a multiplicative group, just like we saw during the examples.

(But **p** is thousands of bits long)

Solutions to the Discrete Logarithm Problem?

If there's one solution, there are infinitely many

(thank you Fermat's little theorem and modular arithmetic "wrap-around")

How to solve DLP in cyclic groups of prime order?

- Is the group cyclic, finite, and abelian?

Has a generator that spans all elements

Has a limited number of elements

Multiplication is commutative

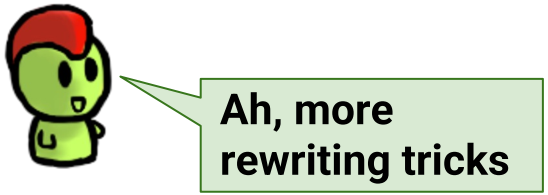
Baby-step/Giant-step algorithms!!!

Baby-Step/Giant-Step Algorithm?

- A cyclic group **G** = <g> which has prime order **n**
- h ∈ G, Goal: find **x** (mod **n**) such that $\mathbf{h = g^x}$

Math exploit!

- Every element **x** ∈ G can be written as: **x** = i + j*⌈sqrt(n)⌉
  - For integers m, i, j satisfying 0 ≤ i, j ≤ m.
  - m = ⌈sqrt(n)⌉



Then:

$$h = g^{i + j*\lceil sqrt(n) \rceil}$$

$$g^i = h \, . \, (g^{-\lceil sqrt(n) \rceil})^j$$

Baby-Step/Giant-Step Algorithm? Notation.

- **$\log_g x \bmod n$** is obtained by comparing two lists:

$$g^i = h \,.\, (g^{-\lceil sqrt(n) \rceil})^j$$

When we find a coincidence, the equality holds and then $x = i + j*\lceil sqrt(n) \rceil$

Can we divide and conquer?

$g^i = h \cdot (g^{-\lceil sqrt(n) \rceil})^j$

Baby-step/Giant-Step Algorithm

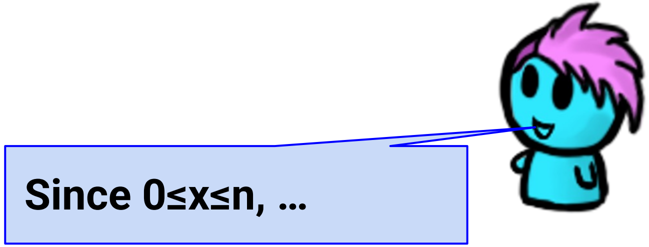
1. $x = i + j*\lceil sqrt(n) \rceil$

Baby-step/Giant-Step Algorithm

$g^i = h \,.\, (g^{-\lceil sqrt(n) \rceil})^j$

1. $x = i + j*\lceil sqrt(n) \rceil$

2. $0 \leq i, j < \lceil sqrt(n) \rceil$



$g^i = h \,.\, (g^{-\lceil sqrt(n) \rceil})^j$

Baby-step/Giant-Step Algorithm

1. $x = i + j * \lceil sqrt(n) \rceil$
2. $0 \leq i, j < \lceil sqrt(n) \rceil$
3. Baby-step: $g_i \longleftarrow g^i$ for $0 \leq i < \lceil sqrt(n) \rceil$

Let's build some tables!

Baby-step/Giant-Step Algorithm

$g^i = h \,.\, (g^{-\lceil sqrt(n) \rceil})^j$

1. $x = i + j*\lceil sqrt(n) \rceil$
2. $0 \leq i, j < \lceil sqrt(n) \rceil$
3. Baby-step: $g_i \longleftarrow g^i$ for $0 \leq i < \lceil sqrt(n) \rceil$

Produces pairs: (g_i, i)

Baby-step/Giant-Step Algorithm

$g^i = h \,.\, (g^{-\lceil sqrt(n) \rceil})^j$

1. $x = i + j * \lceil sqrt(n) \rceil$
2. $0 \leq i, j < \lceil sqrt(n) \rceil$
3. Baby-step: $g_i \longleftarrow g^i$ for $0 \leq i < \lceil sqrt(n) \rceil$
4. Giant-step: $h_j \longleftarrow h * g^{-j \lceil sqrt(n) \rceil}$, for $0 \leq j < \lceil sqrt(n) \rceil$

Produces pairs: (h_j, j)

$g^i = h \, . \, (g^{-\lceil sqrt(n) \rceil})^j$

Baby-step/Giant-Step Algorithm

1. $x = i + j * \lceil sqrt(n) \rceil$
2. $0 \leq i, j < \lceil sqrt(n) \rceil$
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Produces pairs: (h_j, j)

Overall time and space $O(sqrt(n))$

Baby-step/Giant-Step Alg… m

1. x = i + j*[sqrt(n)]

2. 0≤ i, j < [sqrt(n)]

3.

4. Gia… sqrt(n)]

… (h_j, j)

Note: For DLP in group G to be "<u>difficult enough</u>" (e.g., 2^{128} operations), needs prime order subgroup of size greater than 2^{256}

…verall time a… space O(sqrt(n))

DLP Example, $182 = 64^x \pmod{607}$

- Consider the subgroup of prime order 101 (**n** = 101) in F^*_{607}, generated by **g**=64

Take that we know this…

i	64^i (mod 607)	*i*	" "
0		6	
1		7	
2		8	
3		9	
4		10	
5		-	

Focusing on the subgroup **ensures** that every element in the problem is generated by the **known** g=64, making it possible to **solve** the DLP.

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This tells us **x** is in the range **0 ≤ x < 101** because the subgroup has order 101.

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But recall we're operating in mod 607 due to F^*_{607}

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Baby-step: $g_i \longleftarrow g^i$ for $0 \leq i < \lceil \text{sqrt}(n) \rceil$

g = 64
m = $\lceil \text{sqrt}(n) \rceil$ = 11

DLP Example, $182 = 64^x (\text{mod } 607)$

i	64^i (mod 607)	*i*	" "
0	1	6	330
1	64	7	482
2	454	8	498
3	527	9	308
4	343	10	288
5	100	-	

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DLP Example, $182 = 64^x \pmod{607}$

Giant-step: $h_j \longleftarrow h*g^{-j\,[sqrt(n)]}$

$g = 64$
$m = [sqrt(n)] = 11$

i	$182 * 64^{-11*j} \pmod{607}$	*i*	
0		6	
1		7	
2		8	
3		9	
4		10	
5		-	

DLP Example, $182 = 64^x (\text{mod } 607)$

Giant-step: $h_j \longleftarrow h * g^{-j \lceil \text{sqrt}(n) \rceil}$

$g = 64$
$m = \lceil \text{sqrt}(n) \rceil = 11$

i	$182 * 64^{-11*j}$ (mod 607)	*i*	
0	182	6	60
1	143	7	394
2	69	8	483
3	271	9	76
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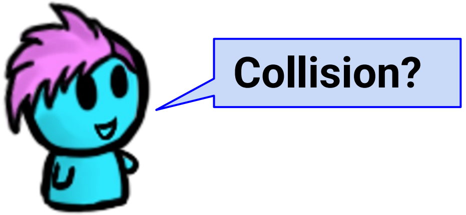

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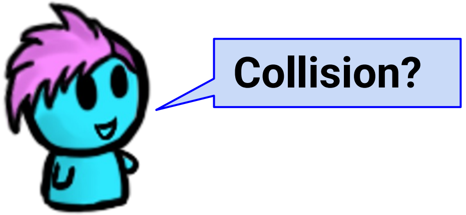

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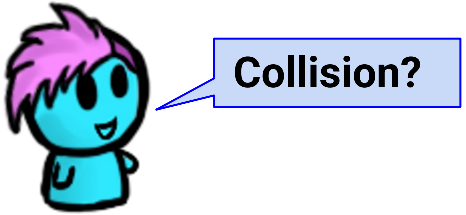

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Match when **i=4** and **j=4**.

(i is not necessarily equal to j, but it happened on this run ¯_(ツ)_/¯

DLP Example, 182 = 64^x(mod 607)

i		i	64^i (mod 607)
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4	343	10	288
5	10		

x = i + j*[sqrt(n)]

Collision?

j		j	182* 64^{-11*j} (mod 607)
0	182	6	60
1	143	7	394
2	69	8	483
3	271	9	76
4	343	10	580

Recall: x = i + j*[sqrt(n)]
So: x = 4 + 4*11 = 48.

DLP Example, 182 = 64^x(mod 607)

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0	1	6	330
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Collision?

j		*j*	182* 64^{-11*j} (mod 607)
0	182	6	60
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2	69	8	483
3	271	9	76

Verify: 64^{48} (mod 607) = 182

Recall: x = i + j*[sqrt(n)]
So: x = 4 + 4*11 = 48.

Diffie-Hellman

Diffie-Hellman Key Exchange

A public-key protocol published in 1976 by Whitfield Diffie and Martin Hellman

Allows two parties that have no prior knowledge of each other to jointly establish a shared secret key over an insecure channel

Key used to encrypt subsequent communications using a symmetric key cipher

Diffie-Hellman Key Exchange

- Used for establishing a <u>shared secret</u> (lacks authentication; we'll see why this is bad)
- Assume as public parameters generator **g** and prime **p**
- Alice (resp. Bob) generates private value **a** (resp. **b**)

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$(A = g^a \bmod p, g, p)$ →

← $(B = g^b \bmod p)$

$B^a = (g^b)^a = g^{ba}$

$A^b = (g^a)^b = g^{ab}$

Alice and Bob can derive the same value by exchanging public values and combining them with their private ones!

Diffie-Hellman Key Exchange

- Used for establishing a <u>shared secret</u> (lacks authentication; we'll see why this is bad)
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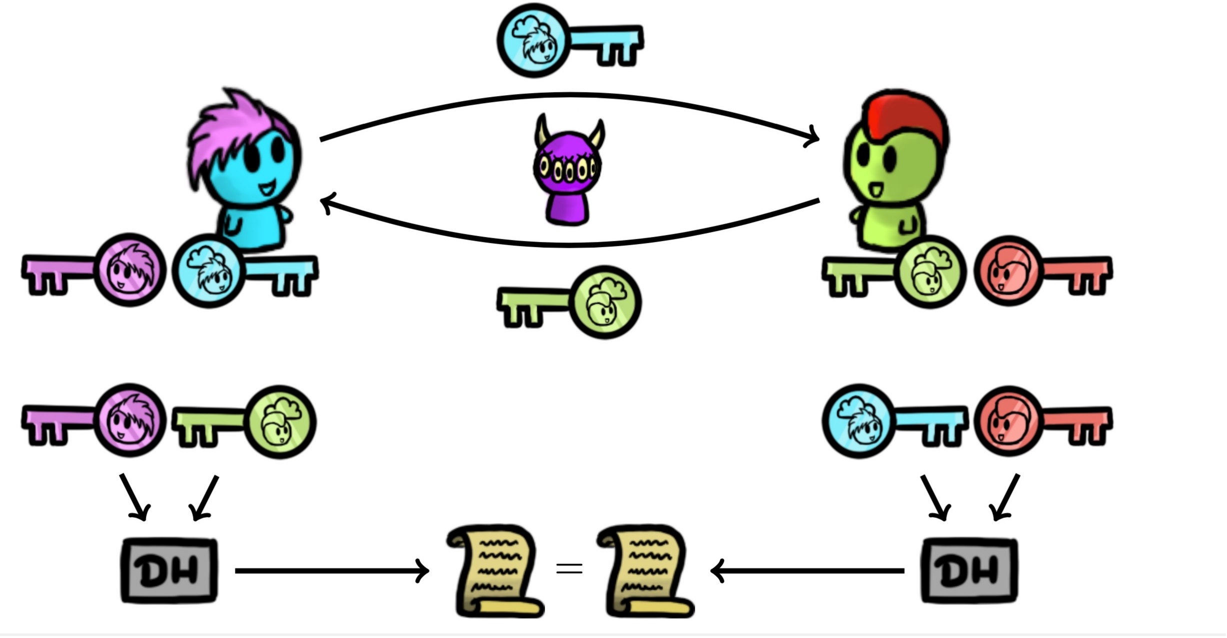
- Alice (resp. Bob) generates private value **a** (resp. **b**)

$B^a = (g^b)^a = g^{ba}$

$A^b = (g^a)^b = g^{ab}$

Resist keying temptation: the shared value should not <u>immediately</u> be used as a key. G^{ab} is a random element inside a group, but not necessarily a random bit string

Diffie-Hellman Key Exchange – Visualization



Diffie-Hellman relies on the DLP

DH can be broken by recovering the private value **a** from the public value $\mathbf{g^a}$

(or **b** from $\mathbf{g^b}$)

The adversary must not be able to solve the DLP

The Decisional Diffie-Hellman Problem

Given $\mathbf{g}$, $\mathbf{g}^a$, $\mathbf{g}^b$ distinguish $\mathbf{g}^{ab}$ from random $\mathbf{g}^c$

- An adversary should NOT be able to learn anything about the secret $\mathbf{g}^{ab}$ after observing public values $\mathbf{g}^a$ and $\mathbf{g}^b$
  - Assume $\mathbf{g}^{ab}$ and $\mathbf{g}^c$ occur with the same probability

The Decisional Diffie-Hellman Problem

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Useful assumption **beyond** DH key exchange! → **ElGamal** relies on the DDH assumption

ElGamal

- **1985 by Taher ElGamal**

ElGamal Public Key Cryptosystem

- Let ***p*** be a prime such that the DLP in $(\mathbf{Z}_p{}^*, .)$ is infeasible
- Let **α** be a generator in $\mathbf{Z}_p{}^*$ and **a** a secret value
- ***PubK*** $=\{(p,\alpha, \beta): \beta \equiv \alpha^a \pmod p)\}$

- For message **m** and secret random **k** in $\mathbf{Z}_{p-1}$:
  - $e_K(m,k) = (y_1, y_2)$, where $\mathbf{y_1} = \alpha^k \bmod p$ and $\mathbf{y_2} = m\beta^k \bmod p$

- For y_1, y_2 in $\mathbf{Z}_p{}^*$:
  - $d_K(y_1, y_2)= y_2(y_1{}^a)^{-1} \bmod p$

ElGamal: The Keys

1. Bob picks a “large” prime **p** and a generator **α**.
   a. Assume message m is an integer $0 < m < p$
2. Bob picks secret integer **a**
3. Bob computes $\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

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4. Bob’s public key is **(p, α, β)**

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2. Bob picks secret integer **a**
3. Bob computes $\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$
4. Bob’s public key is **(p, α, β)** 
5. Bob’s private key is **a** 

ElGamal: Encryption

Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a**

$\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

I choose secret integer **k**

Bob's Pub_K → **(p, α, β)**

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ElGamal: Encryption

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^k \pmod p$

Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a**

β≡α^a (mod p)

ElGamal: Encryption

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^{k}$ (mod p)

Compute $\mathbf{y_2} \equiv \beta^{k}\, m$ (mod p)

Bob's Pub_K → **(p, α, β)**
Bob's $Priv_K$ → **a**
$\beta \equiv \alpha^a \pmod{p}$

ElGamal: Encryption

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^k \pmod{p}$

Compute $\mathbf{y_2} \equiv \beta^k\, m \pmod{p}$

Send $\mathbf{y_1}$ and $\mathbf{y_2}$ to Bob

Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a**

$\beta \equiv \alpha^a \pmod{p}$

ElGamal: Decryption

I choose secret integer **k**

Compute $\mathbf{y_1} \equiv \alpha^{\mathbf{k}} \pmod{p}$

Compute $\mathbf{y_2} \equiv \beta^k m \pmod{p}$

Compute $\mathbf{y_1 y_2^{-a}} \equiv m \pmod{p}$

Send $\mathbf{y_1}$ and $\mathbf{y_2}$ to Bob

Bob's Pub_K → **(p, α, β)**
Bob's $Priv_K$ → **a**
β$\equiv \alpha^a$ (mod p)

ElGamal: Decryption

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Compute $\mathbf{y_1} \equiv \alpha^{\mathbf{k}} \pmod{p}$

Compute $\mathbf{y_2} \equiv \beta^k m \pmod{p}$

Compute $\mathbf{y_1 y_2^{-a}} \equiv m \pmod{p}$

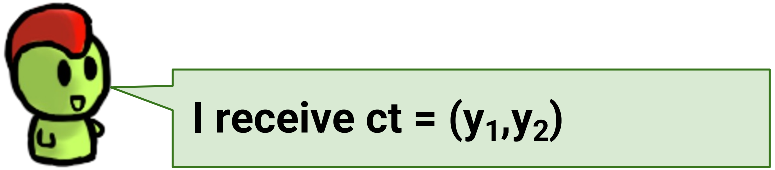
Send $\mathbf{y_1}$ and $\mathbf{y_2}$ to Bob

Bob can decrypt since:

$\mathbf{y_2 y_1^{-a}} \equiv \beta^k m (\alpha^k)^{-a} \equiv (\alpha^a)^k m (\alpha^k)^{-a} \equiv \alpha^{ak} m \alpha^{-ak} \equiv m \pmod{p}$

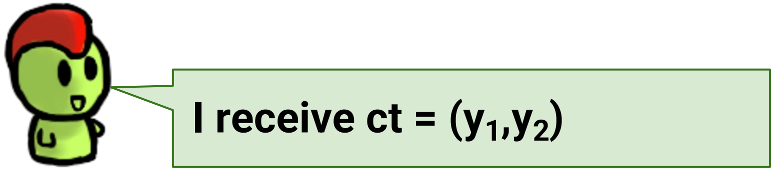
ElGamal Informal Summary

- The plaintext m is “hidden” by multiplying it by β^k to get y_2



ElGamal Informal Summary

- The plaintext m is "hidden" by multiplying it by β^k to get y_2
- The ciphertext includes α^k so that Bob can compute β^k from α^k (because Bob knows a)



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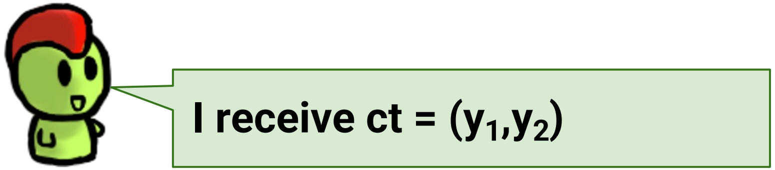
- Thus, Bob can “reveal” m by dividing y_2 by β^k



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I receive ct = (y_1,y_2)

Let's see an example!

Bob's Pub_K → **(p, α, β)**
Bob's $Priv_K$ → **a** = 765
β≡α^a (mod p)

Example

- Let **p**=2579 , **α** = 2 , **β** = 2^{765} mod 2579 = 949

Bob's Pub_K → **(p, α, β)**

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Example

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Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a** = 765

β≡α^a (mod p)

Example

- Let **p**=2579 , **α** = 2 , **β** = 2^{765} mod 2579 = 949

I want to send **m**=1299 to Bob. I choose **k** = 853 for my random integer

$\mathbf{y_1} \equiv \alpha^{\mathbf{k}} \pmod{p}$

$\mathbf{y_2} \equiv \beta^k m \pmod{p}$

Example

Bob's Pub_K → **(p, α, β)**
Bob's $Priv_K$ → **a** = 765
β$\equiv \alpha^a \pmod p$

- Let **p**=2579 , **α** = 2 , **β** = 2^{765} mod 2579 = 949

I want to send **m**=1299 to Bob. I choose **k** = 853 for my random integer

$\mathbf{y_1} \equiv \alpha^k \pmod p$

$\mathbf{y_2} \equiv \beta^k\, m \pmod p$

- $\mathbf{y_1}$ = 2^{853} mod 2579 = 435
- $\mathbf{y_2}$= 949^{853} * 1299 mod 2579 = 2396

Send y_1, y_2 to Bob →

Bob's Pub_K → **(p, α, β)**
Bob's $Priv_K$ → **a** = 765
$\beta \equiv \alpha^a \pmod{p}$

Example

- Bob now has $\mathbf{y_1}$ and $\mathbf{y_2}$
  - $y_1 = 2^{853} \bmod 2579 = 435$
  - $y_2 = 1299 * 949^{853} \bmod 2579 = 2396$

I received **y** = (435, 2396)

Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a** = 765

$\boldsymbol{\beta} \equiv \alpha^a \pmod p$

Example

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  - $y_1 = 2^{853} \bmod 2579 = 435$
  - $y_2 = 1299 * 949^{853} \bmod 2579 = 2396$

$\mathbf{y_2 y_1^{-a}} \equiv \beta^k m (\alpha^k)^{-a} \equiv m \pmod p$

- $m = 2396 * 435^{-765} \bmod 2759 = 1299$

Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a** = 765

$\boldsymbol{\beta} \equiv \alpha^a \pmod p$

Example

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Nice! That's the plaintext I wanted to send.

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Insecure if the adversary can compute **a**$=\log_\alpha \beta$

Nice! That's the plaintext I wanted to send.

Bob's Pub_K → $(\mathbf{p}, \boldsymbol{\alpha}, \boldsymbol{\beta})$

Bob's $Priv_K$ → $\mathbf{a} = 765$

$\boldsymbol{\beta} \equiv \alpha^a \pmod{p}$

Example

- Bob now has $\mathbf{y_1}$ and $\mathbf{y_2}$
  - $y_1 = 2^{853} \bmod 2579 = 435$
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I received $\mathbf{y} = (435, 2396)$

$$\mathbf{y_2 y_1^{-a}} \equiv \beta^k\, m\, (\alpha^k)^{-a} \equiv m \pmod{p}$$

- $m = 2396 * 435^{-765} \bmod 2759 = 1299$

Insecure if the adversary can compute $\mathbf{a} = \log_\alpha \beta$

To be secure, DLP must be infeasible in $\mathbb{Z}_p^*$

Nice! That's the plaintext I wanted to send.

But… We had RSA, why do we need ElGamal?

- Extensions
  - ElGamal supports Elliptic Curve Cryptography (ECC)
  - Stronger security with smaller keys compared to RSA
- Probabilistic Encryption
  - Adds semantic security with randomization (different ciphertexts for the same plaintext).
- Homomorphic properties
  - Additive homomorphism vs. RSA's multiplicative homomorphism

Network Security - Next class