CS459/698
Privacy, Cryptography, Network and Data Security

Homomorphic Encryption

Spring 2025, Monday/Wednesday 2:30pm-3:50pm

What is Homomorphic Encryption?

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- **Definition:** Homomorphic encryption is a cryptographic technique that allows computations to be performed on encrypted data without requiring decryption.

- Raw data can remain fully encrypted while it's being processed, manipulated, and run through various algorithms.

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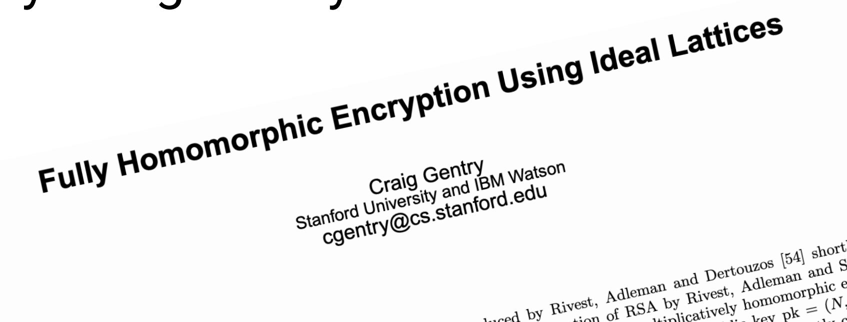
- Idealized in 1978, fully realized in 2009 by Craig Gentry



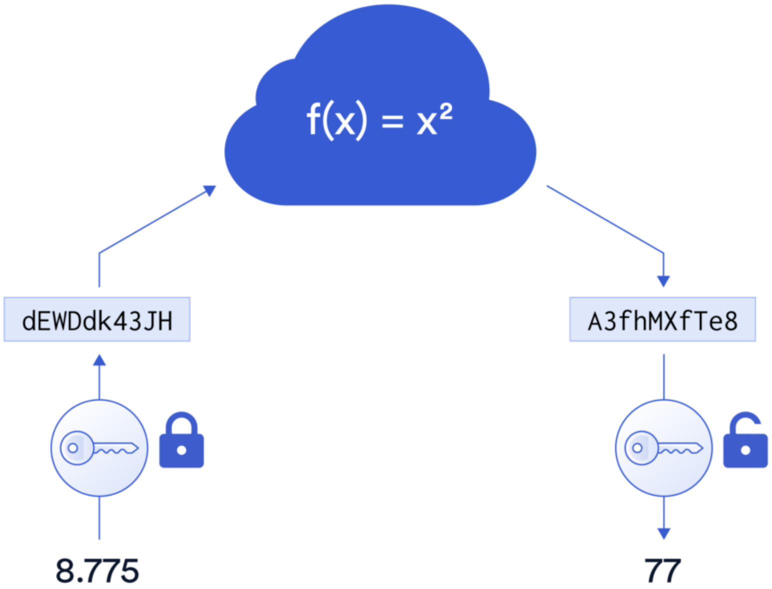
Homomorphic Encryption for Dummies

“Anybody can come and they can stick their hands inside the gloves and manipulate what’s inside the locked box. They can’t pull it out, but they can manipulate it; they can process it… Then they finish and the person with the secret key has to come and open it up—and only they can extract the finished product out of there.”

-- Craig Gentry

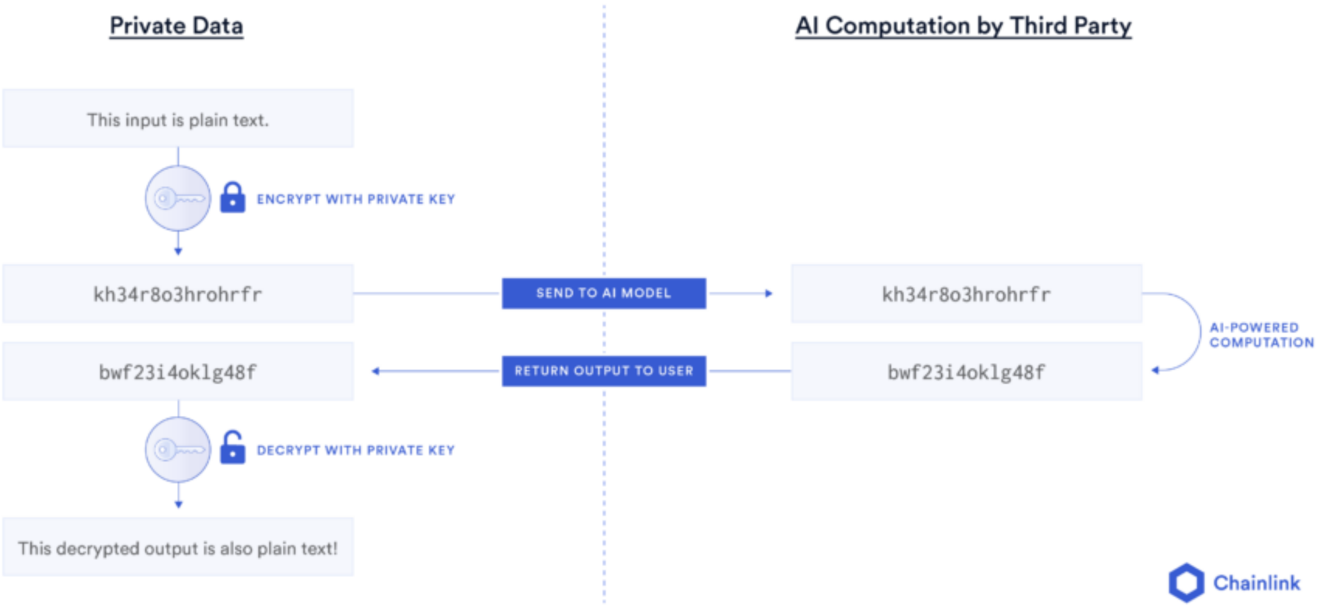
https://www.youtube.com/watch?v=pXb39wj5ShI

Computing on Ciphertexts (Simple Math)



https://chain.link/education-hub/homomorphic-encryption

Computing on Ciphertexts (More sophisticated math)



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Homomorphic Encryption in the Wild

https://dualitytech.com/blog/homomorphic-encryption-making-it-real/

Homomorphic Encryption in the Wild

- Used as a tool in many real-world scenarios:
  - https://www.ibm.com/security/services/homomorphic-encryption
  - https://www.statcan.gc.ca/en/data-science/network/homomorphic-encryption
  - https://www.statcan.gc.ca/en/data-science/network/statistical-analysis-homomorphic-encryption
  - https://www.intel.com/content/www/us/en/developer/tools/homomorphic-encryption/overview.html
  - https://www.microsoft.com/en-us/research/project/microsoft-seal/

E.g., Homomorphic Encryption for Secure Voting

- Microsoft's ElectionGuard

Microsoft | **Microsoft On the Issues** Our Company News and Stories Topics More

Protecting democratic elections through secure, verifiable voting

May 6, 2019 | Tom Burt - Corporate Vice President, Customer Security & Trust

What does ElectionGuard do?

ElectionGuard is a way of checking election results are accurate, and that votes have not been altered, suppressed or tampered with in any way. Individual voters can see that their vote has been accurately recorded, and their choice has been correctly added to the final tally. Anyone who wishes to monitor the election can check all votes have been correctly tallied to produce an accurate and fair result.

So what is this all about?

Homomorphic Encryption

Consider the following:

Two ciphertexts use the same key, **c** = E_K(**x**), **d** = E_K(**y**)

Let **f()** be a function that operates over plaintext **x** and **y**

Homomorphic Encryption

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$$g(c, d) = E_K(f(x, y))$$

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$$g(c, d) = E_K(f(x, y))$$

g() is a **homomorphic function** on the ciphertexts **c**, **d**, …

Partial versus Fully Homomorphic Encryption

The function on the plaintexts is:

…either multiplication or addition **but not both.**

Partial HE

…either multiplication or addition, **or both**

Fully HE

4 Shades of Homomorphic Encryption

Homomorphic Encryption Types				
	Partially	Somewhat	Leveled Fully	Fully
Rating	Simple	Intermediate	Advanced	Most advanced
Computations	Addition or multiplication	Addition and/or multiplication	Complex but limited	Complex and unlimited
Use cases	Sum or product	Basic statistical analysis	AI/ML, MPC	AI/ML, MPC

https://chain.link/education-hub/homomorphic-encryption

4 Shades of Homomorphic Encryption

Only useful for **simpler** operations. Relatively **efficient**.

RSA, ElGamal, Paillier

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4 Shades of Homomorphic Encryption

of operations that can be performed is **bounded** and the accuracy of the computation may **degrade** as more operations are performed.

BGN

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4 Shades of Homomorphic Encryption

BFV, CKKS, TFHE (leveled)

Can perform an **arbitrary** # of computations on encrypted data, if it has a pre-defined set of computations **specified ahead of time**.

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4 Shades of Homomorphic Encryption

**BGV, DGHV,
TFHE (w/bootstrapping)**

Enables **any** # of computations to be performed on encrypted data **without a predefined sequence or limit**. Computationally **expensive**.

Homomorphic Encryption Types				
	Partially	**Somewhat**	**Leveled Fully**	**Fully**
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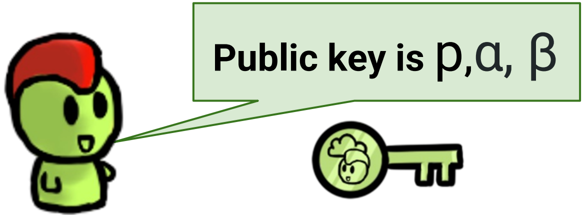
A partial homomorphic encryption scheme based on El Gamal

Recap: ElGamal Public Key Cryptosystem

- Let ***p*** be a prime such that the DLP in $(\mathbf{Z}_p^*, .)$ is infeasible
- Let **α** be a generator in $\mathbf{Z}_p^*$ and **a** a secret value
- **PubK** $=\{(p,\alpha, \beta): \beta \equiv \alpha^a \pmod{p}\}$

- For message **m** and secret random **k** in $\mathbf{Z}_{p-1}$:
  - $e_K(m,k) = (y_1, y_2)$, where $\mathbf{y_1} = \alpha^k \bmod p$ and $\mathbf{y_2} = m\beta^k \bmod p$

- For y_1, y_2 in $\mathbf{Z}_p^*$:
  - $d_K(y_1, y_2) = y_2(y_1^a)^{-1} \bmod p$



Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a**

$y_1 \equiv \alpha^k \pmod{p}$

$y_2 \equiv m\,\beta^k \pmod{p}$

$\beta \equiv \alpha^a \pmod{p}$

Consider Multiplicative HE

f(x, y) = x · y

Private key: a, **public key:** α^a

Instead of k, choose r and s

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β$\equiv \alpha^a \pmod p$

$f(x, y) = x \cdot y$

Private key: a, **public key:** α^a

Instead of k, choose r and s

Goal: show how the multiplication of ciphertexts corresponds to the multiplication of plaintexts.

Consider Multiplicative HE

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$\boldsymbol{\beta} \equiv \alpha^a \pmod p$

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Private key: a, **public key:** α^a

Instead of k, choose r and s

$c_1 = \alpha^r$, $c_2 = \mathbf{x}\, \alpha^{ra}$;

$d_1 = \alpha^s$, $d_2 = \mathbf{y}\, \alpha^{sa}$

Idea: Create ciphertexts for the two different plaintexts

Consider Multiplicative HE

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Bob's $Priv_K \rightarrow$ **a**

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g(c, d):

- $e_1 = c_1 \cdot d_1 = \alpha^r \alpha^s = \alpha^{r+s}$
- $e_2 = c_2 \cdot d_2 = \mathbf{xy}\,\alpha^{ra} \alpha^{sa} = \mathbf{xy}\,\alpha^{a(r+s)}$

Idea: combine ciphertexts of two different plaintexts

Consider Multiplicative HE

Bob's Pub_K → **(p, α, β)**

Bob's $Priv_K$ → **a**

$\mathbf{y_1} \equiv \alpha^k \pmod p$

$\mathbf{y_2} \equiv m\,\beta^k \pmod p$

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$\mathbf{f(x, y) = x \cdot y}$

Private key: a, **public key:** α^a

Instead of k, choose r and s

$c_1 = \alpha^r$, $c_2 = \mathbf{x}\,\alpha^{ra}$;

$d_1 = \alpha^s$, $d_2 = \mathbf{y}\,\alpha^{sa}$

$\mathbf{g(c, d) = xy}\ \alpha^{a(r+s)}$

$\mathbf{xy} = \mathbf{xy}\ \alpha^{a(r+s)} / \alpha^{a(r+s)}$

Idea: decrypt the combined ciphertext

Consider Additive HE

Multiplicative: The math of ElGamal ensures that multiplying the encrypted values corresponds to multiplying the original values.

Additive: Here, we no longer have the same nice properties of how exponents play together.

- **"Crazy" idea:** Something like $g(E_K(\alpha^x), E_K(\alpha^y)) = E_K(\alpha^{x+y})$ could work
  - But we would need to break the discrete log of α^{x+y} to retrieve the sum
    - Only really works for small **x** and **y**

The Paillier Partially Homomorphic Encryption Scheme

Paillier's Encryption Scheme

- Proposed by Pascal Pailler in 1999

- The Paillier cryptosystem is a public-key cryptosystem known for its **additive** homomorphic properties.

- The security of the Paillier cryptosystem is based on the difficulty of the **composite residuosity class problem**
  - This problem involves determining whether a given number is an $\boldsymbol{n}$-th residue modulo $\boldsymbol{n^2}$ for a composite $\boldsymbol{n}$.

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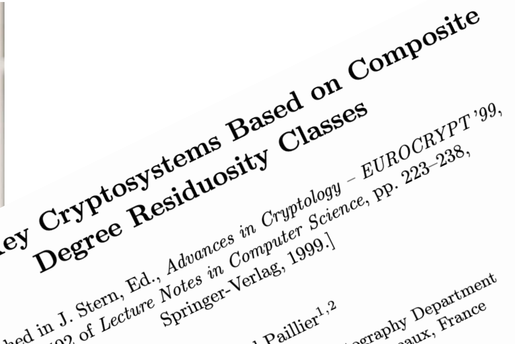
Public-Key Cryptosystems Based on Composite
Degree Residuosity Classes
Advances in Cryptology – EUROCRYPT '99,
Computer Science, pp. 223–238,
Springer-Verlag, 1999.]

Paillier's Encryption Scheme

- Let $\boldsymbol{p}$, $\boldsymbol{q}$ be two large primes; $\boldsymbol{N} = \boldsymbol{pq}$
- Ciphertexts are mod $\boldsymbol{N}^2$

g is a generator

- Choose $\boldsymbol{r}$; plaintext $\boldsymbol{m}$ (mod **p**) is encrypted as $\mathbf{g^m\ r^N\ (mod\ N^2)}$

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From the **product of ciphertexts** to **addition of plaintexts**

- Multiply encryption of $\mathbf{m_1}$ and $\mathbf{m_2}$:

$E(m_1,r_1) \cdot E(m_2,r_2) \bmod N^2 =$

$g^{m1} \cdot g^{m2} \cdot r_1^N \cdot r_2^N \bmod N^2 =$

$g^{m1+m2} \cdot (r_1 \cdot r2)^N \bmod N^2$

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  ⇒ Efficient additive HE, even for large numbers

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$$D(E(m_1, r_1) \cdot E(m_2, r_2) \bmod n^2) = m_1 + m_2 \bmod n.$$

Simplica Numara!

DGHV: A Fully Homomorphic Encryption Scheme

Fully Homomorphic Encryption (FHE)

- Many schemes now, usually abbreviated by the first letters of the last names of the authors
- Different security assumptions (not factoring or discrete log)
  - Lattice problems: Learning with errors, …

Examples:

- First construction by Gentry in 2009
- E.g. FV, BGV, or DGHV (not used in practice)

The **DGHV** Fully Homomorphic Encryption Scheme

- FHE scheme whose security is based on the difficulty of the **approximate greatest common divisor** (AGCD) problem.
  - Finding the greatest common divisor of a set of integers that are close to multiples of a secret integer.

Fully Homomorphic Encryption over the Integers

Marten van Dijk	Craig Gentry	Shai Halevi	Vinod Vaikuntanathan
MIT	IBM Research	IBM Research	IBM Research

June 8, 2010

https://medium.com/@j248360/explaining-the-dghv-encryption-scheme-1acb6cd74dd6
https://www.esat.kuleuven.be/cosic/blog/co6gc-homomorphic-encryption-part-1-computing-with-secrets/
https://github.com/coron/fhe

Consider Simplified DGHV (not used in practice)

- $\boldsymbol{m} \in \{0, 1\}$
- Secret key: prime $\boldsymbol{p}$

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- **Encryption**
  - Choose q, r such that r < p --> r is random noise
  - $c = q.p + 2.r + m$

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- $\boldsymbol{m} \in \{0, 1\}$
- Secret key: prime $\boldsymbol{p}$

- **Encryption**
  - Choose $\boldsymbol{q}$, $\boldsymbol{r}$ such that r < p --> $\boldsymbol{r}$ is random noise
  - $\boldsymbol{c} = \boldsymbol{q}.\boldsymbol{p} + 2.\boldsymbol{r} + \boldsymbol{m}$

- **Decryption**
  - $\boldsymbol{m} = \boldsymbol{c} \bmod 2 \oplus (\lfloor \boldsymbol{c}/\boldsymbol{p} \rfloor \bmod 2)$

Computing with Simplified DGHV

- **Ciphertexts**
  - $c_1 = q_1.p + 2.r_1 + m_1$
  - $c_2 = q_2.p + 2.r_2 + m_2$

Computing with Simplified DGHV

- **Ciphertexts**
  - $c_1 = q_1.p + 2.r_1 + m_1$
  - $c_2 = q_2.p + 2.r_2 + m_2$

- **Addition**
  - $c_1 + c_2 = (q_1+q_2).p + 2.(r_1+r_2) + m_1 + m_2$

Note that noise grows **linearly**

Computing with Simplified DGHV

- **Ciphertexts**
  - $c_1 = q_1.p + 2.r_1 + m_1$
  - $c_2 = q_2.p + 2.r_2 + m_2$
- **Addition**
  - $c_1 + c_2 = (q_1+q_2).p + 2.(r_1+r_2) + m_1 + m_2$
- **Multiplication**
  - $c_1 \cdot c_2 = q'.p + 2.r' + m_1.m_2$
  - $r' = 2.\mathbf{r_1.r_2} + r_1.m_2 + r_2.m_1$
  - $q' = q1 \cdot q2 \cdot p + q1 \cdot m2 + q2 \cdot m1$

Note the increased growth of the noise. (no longer linear). One gets a new ciphertext with noise **roughly twice larger** than in the original ciphertexts c1 and c2.

The bootstrapping problem in FHE

Bootstrapping… in Fully HE Schemes

- If $\boldsymbol{r} > \boldsymbol{p}/2 \Rightarrow$ decryption fails on DGHV
  - Also a problem for other schemes.

- If the noise **grows too much**, it can **corrupt** the encrypted data and make it unusable

- Each operation **increases the noise**, so one must **control** this growth

Bootstrapping… in Fully HE Schemes

- To obtain a FHE scheme, (i.e. unlimited addition and multiplication on ciphertexts), one must **reduce** the amount of noise in a ciphertext

- **Bootstrapping** is a procedure that reduces noise
  - E.g., in DGHV, homomorphically compute the decryption function for obtaining an encryption of **m** with lower noise
  - Still, bootstrapping is **slow** in most fully HE schemes
  - Thus, w/ fully HE, aim to **avoid** subsequent multiplications

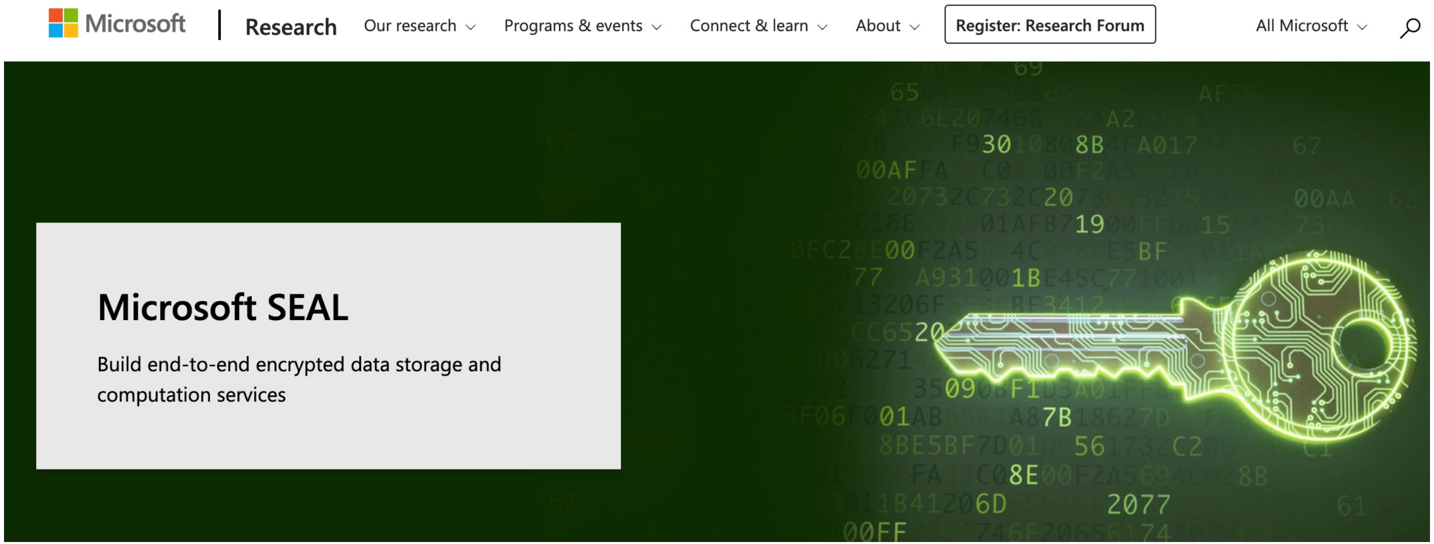
Practical FHE Schemes

- **FV, BGV, BFV, CKKS**
  - Lattice-based encryption schemes
  - Encrypt vectors (usually as polynomials)

- **TFHE**
  - Fully HE over the Torus
  - Usually encrypts bits
  - Very fast bootstrapping (frequently performed)
  - https://tfhe.github.io/tfhe/

Try it… on your own ☺

- Download Microsoft's SEAL library and hack away!
  - https://www.microsoft.com/en-us/research/project/microsoft-seal/



Quick announcement (again ☺)

- **Student Course Perceptions** (https://perceptions.uwaterloo.ca/)
  - Close on Wed, July 30th
  - Did you like it? Did you hate it? Let me know!