CS489/698 Privacy, Cryptography, Network and Data Security

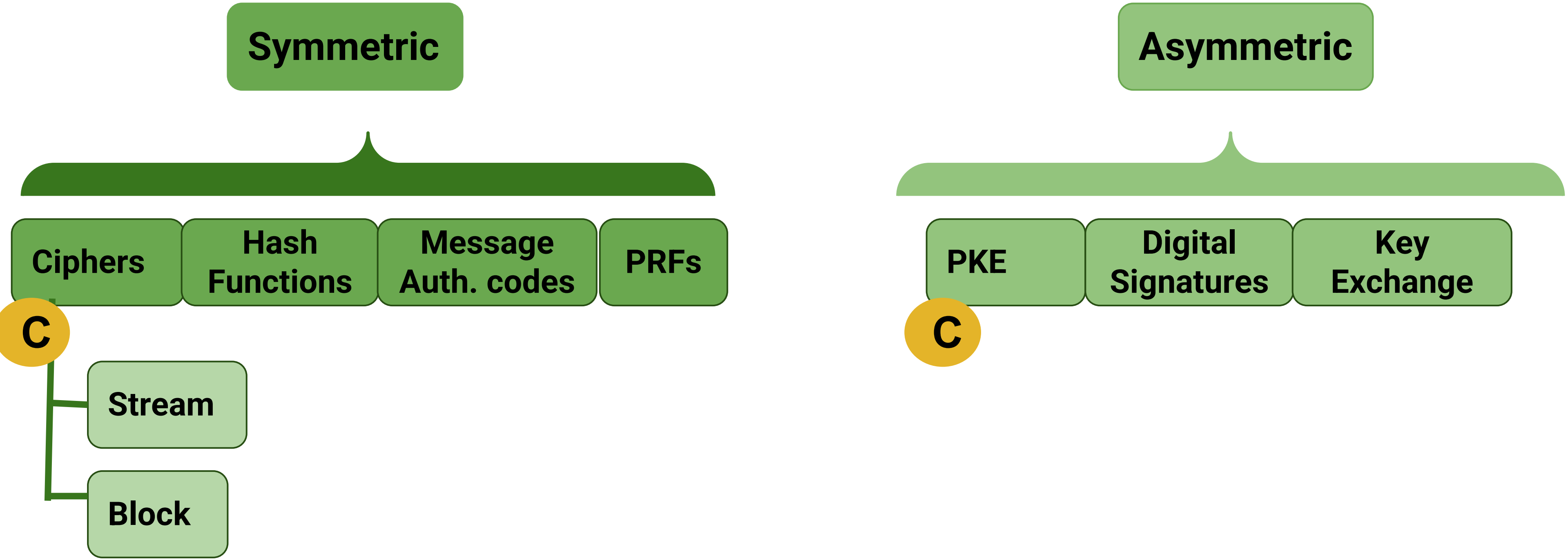
Public Key Cryptography (RSA)

Spring 2024, Monday/Wednesday 11:30am-12:50pm

Assignment One

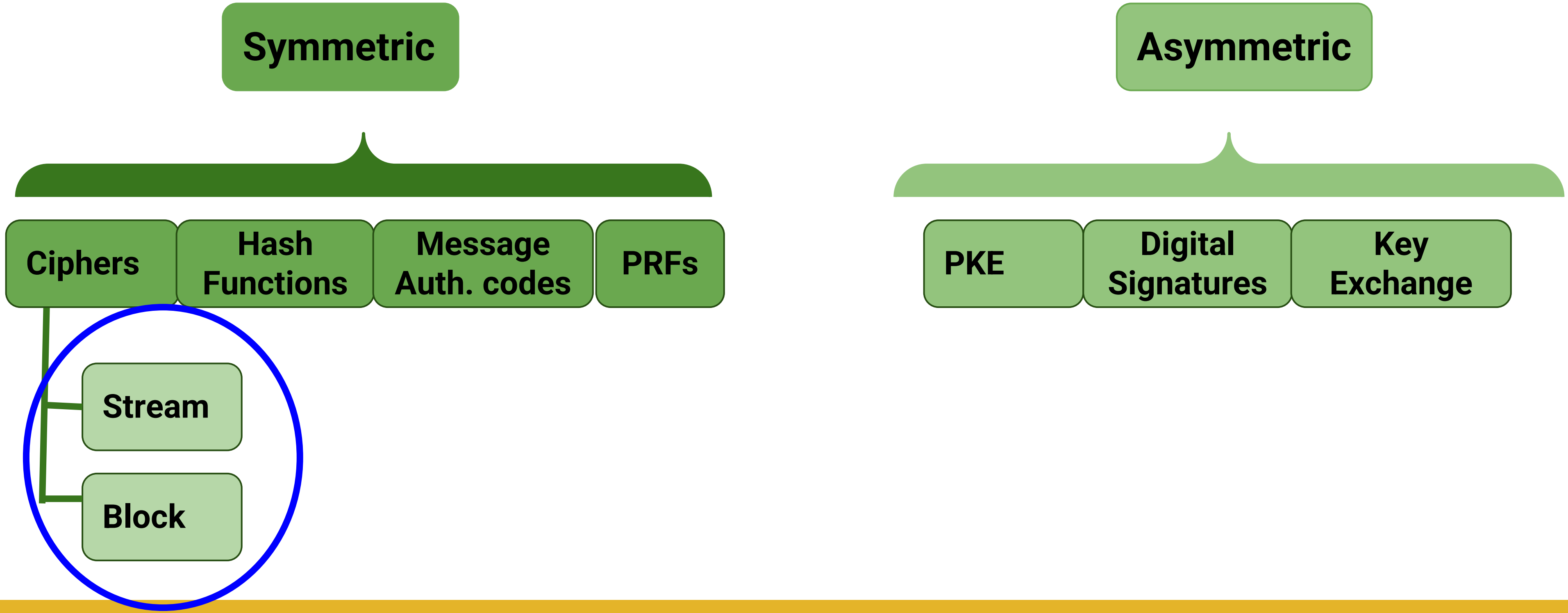
- Available on Learn today at 3pm
- Due **May 29th, 3pm**
- Written and programming

Cryptography Organization

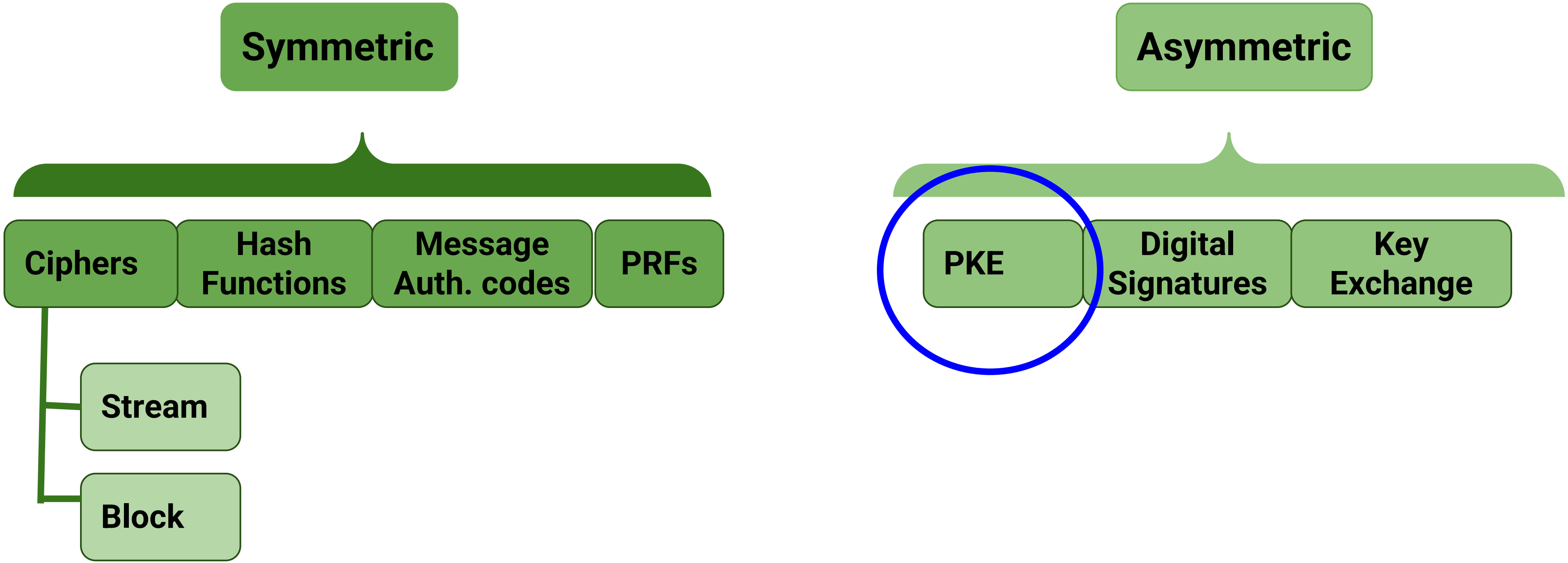


Organization display source: Doug Stebila

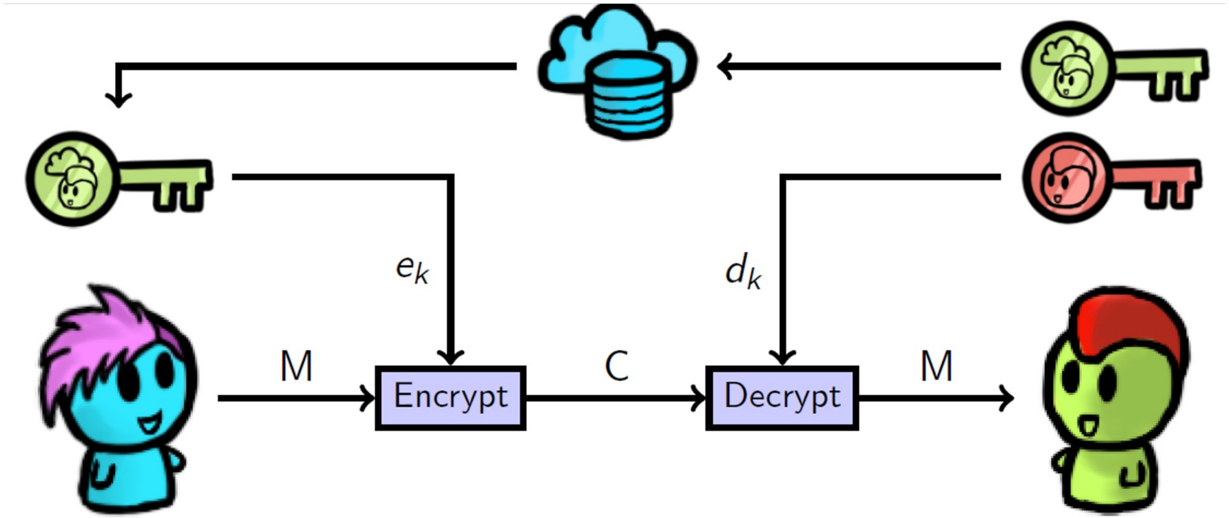
Cryptography Organization



Cryptography Organization



Public Key Cryptography, "1970s"



Examples:

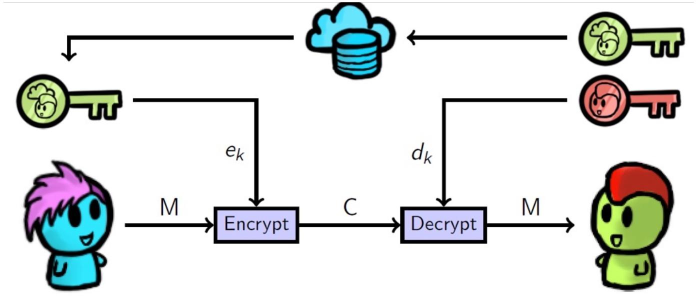
- RSA, ElGamal, ECC, NTRU

Steps for Public Key Cryptography?

1. Bob generates pair

2. Bob gives everyone the public key

3. Alice encrypts m and sends it



4. Bob decrypts using private key

5. Eve and Alice can’t decrypt, only have encryption key

Steps for Public Key Cryptography?

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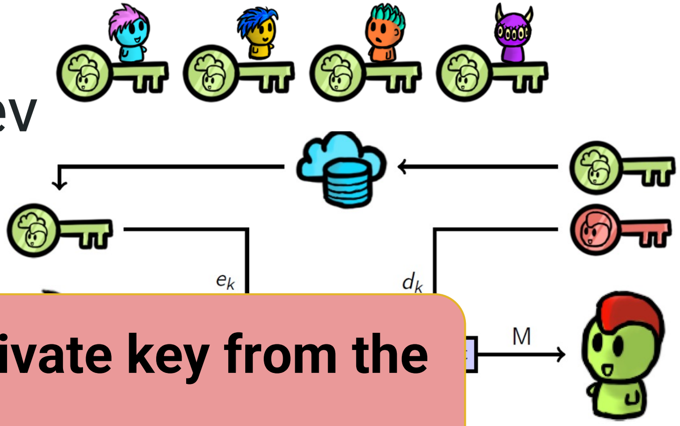

3. Alice encrypts m and sends it

4. Bob d

It must be hard to derive the private key from the public key

5. Eve and Alice can't decrypt, only have encryption key

Requirements for PKE

- The encryption function? Must be easy to compute 
- The inverse, decryption? Must be hard for anyone without the key  vs. 

Thus, we require so called “one-way” functions for this.

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- The encryption function? Must be easy to compute 
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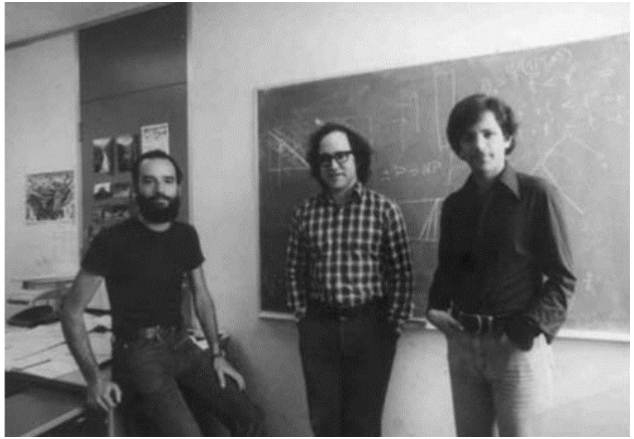
Thus, we require so called “one-way” functions for this.

But because of decryption,
we need a “trapdoor”

Time for Textbook RSA

- A clever arithmetic trick based on a “trapdoor permutation”
- Modular arithmetic: integer numbers that “wrap around”



Left to right: Ron Rivest, Adi Shamir, and Leonard Adleman.

Fun (?) Facts::

- RSA was the first popular public-key encryption method, published in 1977

Prime Numbers

- **Prime:** a natural number that can only be divided by 1 or itself
- **Primes and factorization:** An integer number can be written as a unique product of prime numbers
  - E.g., 1234567 = 127 * 9721

How to know if a number is prime?

○ Run a primality test algorithm (Solovay-Strassen, Miller-Rabin, etc.)

How to discover a number's factors?

○ Run a factorization algorithm (Pollard p-1, etc.)

Time for Textbook RSA

- Overview:

$$(x^e)^d \equiv x \bmod N$$

- Computational difficulty of the **factoring problem**
    - Given two large primes p.q = N, it is very hard to factor N.

Easy for me to pick **e**, **d**, and **n** that satisfy that equation

Ugh. I know **e** and **n** and can’t find **d**!!!

Time for Textbook RSA

- Encryption:

$$y = x^e \bmod N$$

The ciphertext is equal to **x** multiplied by itself **e** times modulo **N**.

Public key is given by **PubK** = (e, N)

Time for Textbook RSA

- Decryption:

$$x = y^d \bmod N = (x^e)^d \bmod N = x^{ed} \bmod N$$

Decryption relies on number **d** such that **e**.**d** = 1 mod **N**, and where $x^{ed} \bmod N = x^1 \bmod N = x$

  - In other words, **d** is the multiplicative inverse of **e** mod **N**

Private key is given by **PrivK** = (d)

Key Generation (how to choose **e** and **d**)

- Pick two random primes **p** and **q**, such that **p**.**q** = **N**
- Generate **φ(N)** = (p-1).(q-1)
  - all relative primes to (p−1)(q−1) form a group with respect to multiplication and are invertible
- Pick **e** as a random prime smaller than φ(N)
  - **e** chosen as relative prime to (p−1)(q−1) to ensure it has a multiplicative inverse mod (p−1)(q−1)
- Generate **d** (the inverse of e mod φ(N))
  - **e**.**d** = 1 mod φ(N)
  - Can be obtained via the extended Euclidean algorithm

If gcd(a,b) = 1, then we say that a and b are **relatively prime** (or coprime).

Extended Euclidean Algorithm

- Given two integers **a** and **b**, the algorithm finds integers **r** and **s** such that **r.a + s.b = gcd(a, b)**. When **a** and **b** are coprime, gcd(a, b) = 1, and r is the modular multiplicative inverse of **a** modulo **b**.
- **Idea:** start with the GCD and recursively work your way backwards.

Say N = 40, e = 7

e.**d** = 1 mod φ(N)

7d = 1 mod 40

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Euclidean Algorithm:

40 = 5 * **7** + <u>5</u>

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Euclidean Algorithm:

40 = 5 * **7** + 5
7 = 1 * **5** + 2

Extended Euclidean Algorithm

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Euclidean Algorithm:

40 = 5 * **7** + 5
7 = 1 * **5** + 2
5 = 2 * **2** + 1

Stop at last non-zero remainder
gcd(7, 40) = 1

Extended Euclidean Algorithm

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Say N = 40, e = 7

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Euclidean Algorithm:

40 = 5 * **7** + 5
7 = 1 * **5** + 2
5 = 2 * **2** + 1   1 = 5 – 2 * 2

Stop at last non-zero remainder
gcd(7, 40) = 1

Extended Euclidean (backtrack):

1 = 5 - 2 * 2

Extended Euclidean Algorithm

- Given two integers **a** and **b**, the algorithm finds integers **r** and **s** such that **r.a + s.b = gcd(a, b)**. When **a** and **b** are coprime, gcd(a, b) = 1, and r is the modular multiplicative inverse of **a** modulo **b**.
- **Idea:** start with the GCD and recursively work your way backwards.

Say N = 40, e = 7

e.**d** = 1 mod φ(N)

7d = 1 mod 40

Euclidean Algorithm:

40 = 5 * **7** + 5
7 = 1 * **5** + 2   2 = 7 – 1 * 5
5 = 2 * **2** + 1

Stop at last non-zero remainder
gcd(7, 40) = 1

Extended Euclidean (backtrack):

1 = 5 - 2 * **2**
1 = 5 - 2 (7 – 1 * 5)
1 = 5 – 2 * 7 + 2 * 5
1 = 3 * 5 – 2 * 7

Extended Euclidean Algorithm

- Given two integers **a** and **b**, the algorithm finds integers **r** and **s** such that **r.a + s.b = gcd(a, b)**. When **a** and **b** are coprime, gcd(a, b) = 1, and r is the modular multiplicative inverse of **a** modulo **b**.
- **Idea:** start with the GCD and recursively work your way backwards.

Say N = 40, e = 7

e.**d** = 1 mod φ(N)

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Euclidean Algorithm:

40 = 5 * **7** + 5   5 = 40 – 5 * 7
7 = 1 * **5** + 2
5 = 2 * **2** + 1

Stop at last non-zero remainder
gcd(7, 40) = 1

Extended Euclidean (backtrack):

1 = 5 - 2 * 2
1 = 5 - 2 (7 – 1 * 5)
1 = 5 – 2 * 7 + 2 * 5
1 = 3 * **5** – 2 * 7
1 = 3 (40 – 5 * 7) – 2 * 7
1 = 3 * 40 - 17 * 7

Extended Euclidean Algorithm

- Given two integers **a** and **b**, the algorithm finds integers **r** and **s** such that **r.a + s.b = gcd(a, b)**. When **a** and **b** are coprime, gcd(a, b) = 1, and r is the modular multiplicative inverse of **a** modulo **b**.
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40 = 5 * **7** + 5
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Stop at last non-zero remainder
gcd(7, 40) = 1

Extended Euclidean (backtrack):

1 = 5 - 2 * 2
1 = 5 - 2 (7 – 1 * 5)
1 = 5 – 2 * 7 + 2 * 5
1 = 3 * 5 – 2 * 7
1 = 3 (40 – 5 * 7) – 2 * 7
1 = ~~3 * 40~~ **- 17** * **7**
d = -17 = 23 mod 40

Textbook RSA (summary)

1. Choose two **“large primes”** *p* and *q* (secretly)
2. Compute n = p*q
3. “Choose” value *e* and find *d* such that $(x^e)^d \equiv x \bmod n$
4. **Public key**: (e, n)
5. **Private key**: d
6. Encryption: $y = x^e \bmod n$
7. Decryption: $y^d \bmod n$

Example (Tiny RSA)

Parameters:

- p=53, q=101, N=5353
- φ(N) = (53-1).(101-1) = 5200
- e=139 (random pick)
- d=1459 (extended Euclidean)

- Message:
- x=20

Encryption: $y = x^e \bmod N$

$y = 20^{139} \bmod 5353 = 5274$

Decryption: $x = y^d \bmod N$

$X = 5274^{1459} \bmod 5353 = \underline{20}$

Nice!

Example (Tiny RSA)

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Decryption: $x = y^d \bmod N$

$X = 5274^{1459} \bmod 5353 = \underline{20}$

Attacking RSA

I know **e** and **N**…
What can I do to find **d**?

Attack idea:

- Factor **N** to obtain **p** and **q**
- Obtain φ**(N)**
- From φ**(N)** and **e**, generate **d** just like Alice would

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- d=1459

- y = 5274

Attacking RSA

W... find **d**?

WARNING: Factoring is hard..but

Attack idea:

- Factor **N** to obta... **p** and...
- Obtain φ**(N)**
- From φ**(N)** and **e**, generate **d** just like Alice would

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- e=139
- ...=1459
- y ... 5274

Factoring and RSA

- You want to factor the public modulus?
- Good news, abundant literature on factoring algorithms
- Bad news, "appropriate" primes will not be defeated

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Bad primes: easily factored

Strawman Approach at Factoring

- Try to divide a number by all numbers smaller than it until you find a number **a** that divides N

- Then, carry on to divide N with **a+1** and so on…

- We end up with a list of factors of N

Way too computationally expensive.

A Smarter Approach at Factoring

- We only need to test prime numbers (not every a < N)

- We only need to test those smaller than $\sqrt{N}$
  - If both p and q are larger than N, then p.q > N, which is impossible

A Smarter Approach at Factoring

- We only need to test prime numbers (not every a < N)

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Still too computationally expensive for large N.

N = 4096 bits requires about 2^{128} operations

Attacking ”bad primes”

- Some primes are not suited to be used for RSA, as they make N easier to factor

- Examples:
  - Either **p** or **q** are small numbers
  - **p** and **q** are too close together
  - **p** and **q** are both close to 2^b, where b is a given bound
  - $N = \mathbf{p}^r.\mathbf{q}^s$ and $r > \log p$
  - …

Let’s dive into an example…

Fermat's Little Theorem

- The theorem states:
  - $a^p \equiv a \bmod p$, for prime **p** and integer **a**
  - Special case when **p** is co-prime with integer **a** → gcd(p,a) = 1
    $a^{p-1} \equiv 1 \bmod p$
  - This is also true for any multiple of p-1 (you keep wrapping around):
    $a^{k(p-1)} \equiv 1 \bmod p$
  - We can rewrite as:
    $a^{k(p-1)} - 1 = \mathbf{p}.\mathbf{r}$

Can we use this to find factors of N?

- Consider we have **N = p.q**

    - Recall:
      $a^{k(p-1)} - 1 = \mathbf{p.r}$

    - Putting this together, we have:
      $\gcd(a^{k(p-1)}-1, N) =$
      $= \gcd(\underline{p}.r, \underline{p}.q) =$
      $= p$

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This allow us to find a factor of **N**!

But how does this help us? We don't know **p**, nor do we have a way of calculating **k**.

The Pollard p-1 Factoring Algorithm

Inputs: Odd integer N and a "bound" b

- We guess **k(p-1)** by bruteforce
- Place **a** to the power of integers with a lot of prime factors. Likely that the factors of p−1 are there.
  → $a^{k!} \bmod N$
- Calculate gcd($a^{k(p-1)}$ -1,N)
- If it is not equal to one, we found a factor

1. a = 2
2. for j =2 to b
   a. Do $a \equiv a^j \bmod N$
3. d = gcd(a-1,N)
4. if 1 < d < N
   a. Then return (d)
   b. Else return ("failure")

The Pollard p-1 Factoring Algorithm

Let's factor N = 713:

a	**d**
$2^1 \equiv 2 \bmod 713$,	$\gcd(1,713)==1$
$2^2 \equiv 4 \bmod 713$,	$\gcd(3,713)==1$
$4^3 \equiv 64 \bmod 713$,	$\gcd(63,713)==1$
$64^4 \equiv 326 \bmod 713$,	$\gcd(325,713)==1$
$326^5 \equiv 311 \bmod 713$,	$\gcd(310,713)==$**31**

1. a = 2
2. for j =2 to B
   a. Do $a \equiv a^j \bmod N$
3. d = gcd(a-1,N)
4. if 1 < d < N
   a. Then return (d)
   b. Else return ("failure")

713/31 = 23

23 * 31 = 713

The case of "smooth" factors

- A prime is deemed smooth if it has multiple small factors

  $\mathbf{p-1} = p_1^{e1} \,.\, p_2^{e2} \ldots, \forall\, p_i^{ei},\ p_i^{ei} \leq B$

  - Pollard p-1 algorithm is useful when **p** is smooth
  - Its iterative approach is more likely to include **p −1** sooner rather than later

So far so good, but…

WARNING: this was textbook RSA, do **not** use!!!

Why not "Textbook RSA"? Example

Example: (Tiny RSA), $p=53$, $q=101$, $e=139$, $d=1459$

Encryption: $y \equiv x^e \pmod{N}$, **Decryption:** $x = y^d \pmod{N}$

- Compute N
- Compute $Y_1 = E_e(1011)$. Verify the decryption works
- Compute $Y_2 = E_e(4)$. Verify the decryption works
- Compute $D_d(Y_1 * Y_2)$. What is happening…and why?

Note::

- The * here indicates multiplication/compute a product

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- Compute $D_d(Y_1 * Y_2)$. What is happening…and why?

A: The decryption is the product of the original plaintexts!!!

Note::

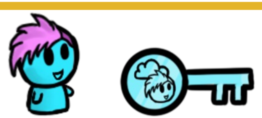
- The * here indicates multiplication, compute a product

Malleability

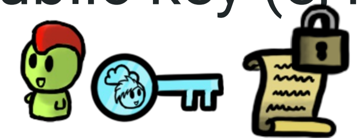
A: $y_1 * y_2 = (x_1)^e * (x_2)^e = (x_1 * x_2)^e$

- It is possible to transform a ciphertext into another ciphertext that decrypts to a related plaintext
- Undesirable (most of the time)

RSA and a Chosen Ciphertext Attack



- Alice is using RSA, public key (e, n)



- Bob sends $y = E_e(x)$

- We are Eve! We snag y.
- Alice…is confident about textbook RSA, will decrypt any ciphertext except y for us

Goal: Ask Alice to decrypt something (other than y) that helps us learn x

Executing CCA on Textbook RSA

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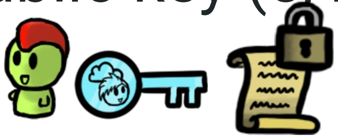

I am so clever mwahaha

- We-Eve ask Alice to decrypt $y_2 = 2^e * y_1$

Q: Decrypts to?

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Executing CCA on Textbook RSA

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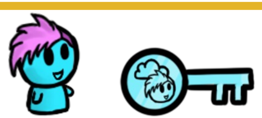

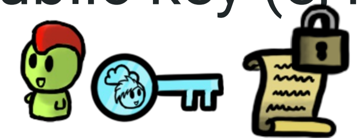

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A: decryption gives $(2^e * y_1)^d \equiv 2x$

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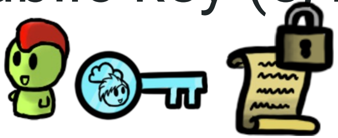

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Textbook RSA: vulnerable to CCA
Note: Can be addressed with padding techniques

Show Naive RSA Encryption is not IND-CPA Secure

1. Eve produces two plaintexts, x_0 and x_1

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4. Sooo, Eve computes $y <- x_1^e$ (mod N)

   If $y^* = y$ then Eve knows $x_b = x_1$
   If $y^* <> y$ then Eve knows $x_b = x_0$

Show Naive RSA Encryption is not IND-CPA Secure

1. Eve produces two plaintexts, x_0 and x_1

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   If $y^* = y$ then Eve knows $x_b = x_1$
   If $y^* \neq y$ then Eve knows $x_b = x_0$

Adversaries and their Goals

You've assumed my goal is the secret/private key...

Adversaries and their Goals

You’ve assumed my goal is the secret/private key…

…but less ambitious goals can be very effective…

Adversaries and their Goals

Goal 1: Total Break

- Win the secret key k or
- Win Bob's private key k_b
- Can decrypt any y_i *for:*

$$y_i = E_k(x) \text{ or } y_i = E_{kb}(x)$$

- All messages using compromised k revealed
- Unless **detected** game over

Goal 2: Partial Break

- Decrypt **a ciphertext** *y* (without the key)
- Learn **some** specific information about a message *x* from *y*

**Need to occur with non-negligible probability.

- **Some (or a)** message revealed

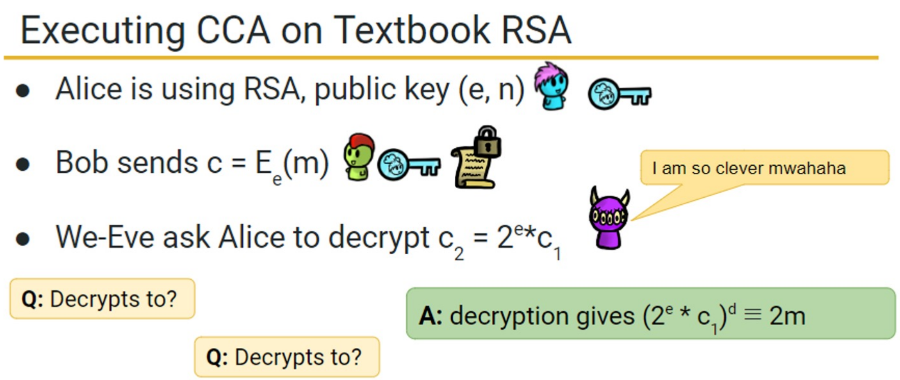
Goal 3: Distinguishable Ciphertexts

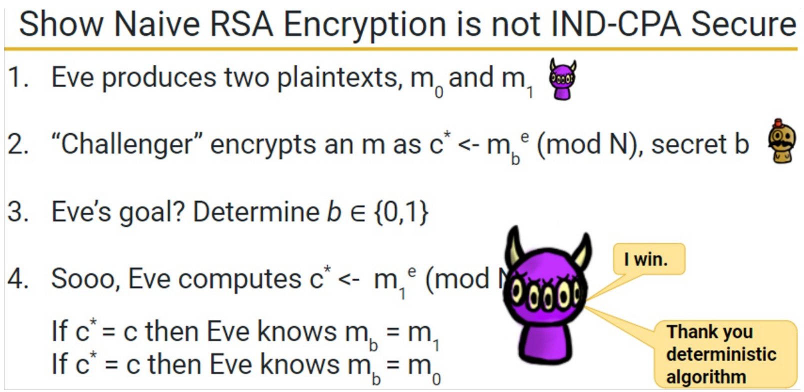
- *P{learn b* ∈ {0,1}} exceeds ½
- Distinguish between $E(x_1)$ and $E(x_2)$ or between E(x) and E(random string)

- The ciphertexts are leaking small/some information…

Semantic Security of RSA

- We saw CCA against Naive RSA
- We showed IND-CPA on Naive RSA





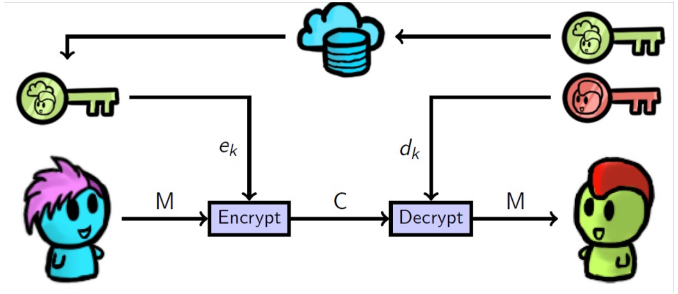
Fix it? Ciphertext Distinguishability

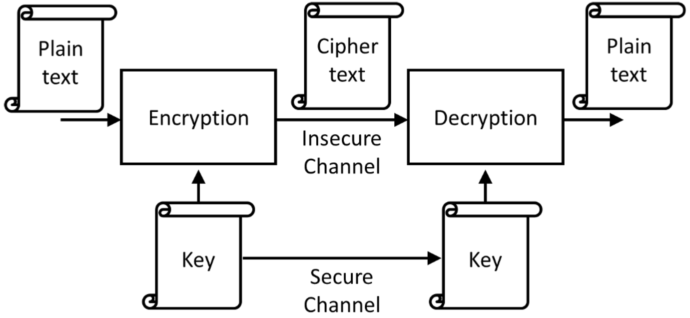
Goal: prove (given comp. assumptions) no information regarding *x* is revealed in polynomial time by examining $y = E(x)$

- If E( ) is deterministic, fail
- Thus, require some randomization

RSA-OAEP: Optimal Asymmetric Encryption Padding

Practicality of Public-Key vs. Symmetric-Key

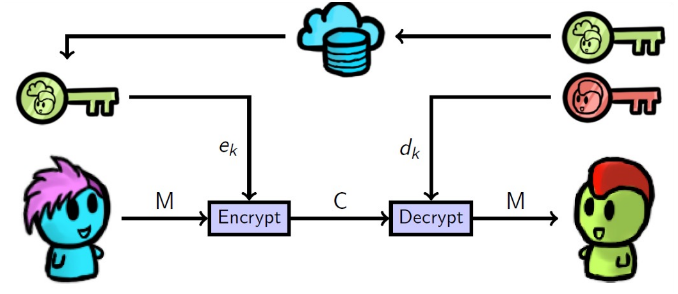




1. Longer keys
2. Slower
3. Different keys for E(x) and D(y)

1. Shorter keys
2. Faster
3. Same key for E(x) and D(y)

Practicality of Public-Key vs. Symmetric-Key



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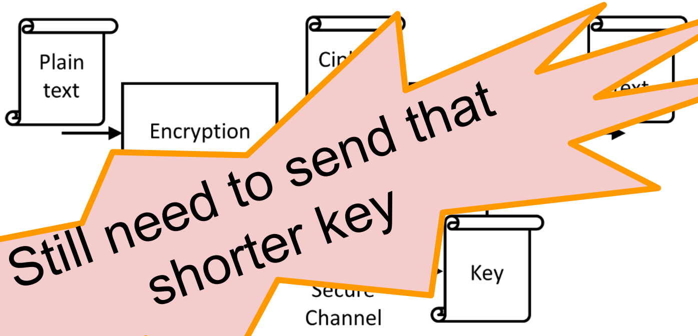

Still need to send that shorter key

1. Shorter keys
2. Faster
3. Same key for E(x) and D(y)

Hybrid Cryptography

- Combine the two!!!!!!!
- Pick a random "128-bit" key K for a symmetric-key system
- Encrypt the large message with the key K (e.g., using AES)

And then…

- Encrypt the key K using a public-key system!
- Send the encrypted message and encrypted key to Bob

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Hybrid cryptography is used in (many) applications on the internet

Just Checking…

Public: (e_A, d_A)

Secret: K

Public: (e_B, d_B)

Secret: ?

- Enc/Dec functions: $E_{key}(*)$, $D_{key}(*)$
- Alice wants to send a **large** message *m* to Bob,

Q: How should Alice build the message efficiently? How does Bob recover m?

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Public: (e_A, d_A)

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Public: (e_B, d_B)

Secret: ?

- Enc/Dec functions: $E_{key}(*)$, $D_{key}(*)$
- Alice wants to send a **large** message *m* to Bob,

Q: How should Alice build the message efficiently? How does Bob recover m?

A: Alice computes $y_1 = E_{eB}(K)$, $y_2 = E_K(x)$ and sends $<y_1||y_2>$
Bob recovers $K = D_{dB}(y_1)$ and then $x = D_K(y_2)$

Up next: More Cryptography…