

Classical Sequent Calculus (LK)

for Propositional Logic

CS 245

LK stands for “Logistischer klassischer Kalkül” (Logical classical Calculus) by Gerhard Gentzen.

Idea: make a proof system that manipulates *assumptions* as well as the formula that is being proven.

Definition 1 (Sequent)

Let Γ and Δ be sets of formulae. We call $\Gamma \vdash \Delta$ a sequent.

Notation: In sequents we write Γ, φ for $\Gamma \cup \{\varphi\}$ and Γ, Δ for $\Gamma \cup \Delta$.

Definition 2 (System LK)

Identity Rules:

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta} \text{ (Axiom)} \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta} \text{ (Cut)}$$

Logical Rules:

$$\begin{array}{cc} \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, (\neg\varphi) \vdash \Delta} (\neg L) & \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash (\neg\varphi), \Delta} (\neg R) \\[10pt] \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, (\varphi \rightarrow \psi) \vdash \Delta} (\rightarrow L) & \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash (\varphi \rightarrow \psi), \Delta} (\rightarrow R) \end{array}$$

The formulae $(\neg\varphi)$ and $(\varphi \rightarrow \psi)$ in the $\neg L$, $\neg R$, $\rightarrow L$, and $\rightarrow R$ rules are called *principal formulae* of the inference, φ is called the *cut formula* in the *Cut* inference.

Note: In many (more formal) definitions of LK, sequents are formed by two *sequences* of formulae; then additional *structural inference rules*, such as exchange (allowing to swap formulae in the sequence), contraction (allowing to remove adjacent duplicate formulae), etc., are needed.

Example 3 (LK Proofs of Hilbert Axioms 1-3)

Axiom 1:

$$\frac{\frac{\frac{}{p, q \vdash p} \text{ (Ax)}}{p \vdash q \rightarrow p} (\rightarrow R)}{\vdash p \rightarrow (q \rightarrow p)} (\rightarrow R)$$

Axiom 2:

$$\begin{array}{c}
\frac{}{p \rightarrow (q \rightarrow r), p \vdash p, r} (Ax) \quad \frac{}{p, q \vdash p, r} (Ax) \quad \frac{}{p, q \vdash q, r} (Ax) \quad \frac{}{p, q, r \vdash r} (Ax) \\
\frac{}{p \rightarrow (q \rightarrow r), p, q \vdash r} (\rightarrow L) \\
\frac{}{p \rightarrow (q \rightarrow r), p \rightarrow q, p \vdash r} (\rightarrow R) \\
\frac{}{p \rightarrow (q \rightarrow r), p \rightarrow q \vdash p \rightarrow r} (\rightarrow R) \\
\frac{}{p \rightarrow (q \rightarrow r) \vdash (p \rightarrow q) \rightarrow (p \rightarrow r)} (\rightarrow R) \\
\vdash (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))
\end{array}$$

Axiom 3:

$$\begin{array}{c}
\frac{}{q, p \vdash p} (Ax) \quad \frac{}{q \vdash q, p} (Ax) \\
\frac{}{q \vdash (\neg p), p} (\neg R) \quad \frac{}{q, (\neg q) \vdash p} (\neg L) \\
\frac{}{\neg p \rightarrow \neg q, q \vdash p} (\rightarrow R) \\
\frac{}{\neg p \rightarrow \neg q \vdash q \rightarrow p} (\rightarrow R) \\
\vdash (\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)
\end{array}$$

We use $\perp$ as an abbreviation for $\neg(p \rightarrow p)$ in LK proofs.

Example 4 (Thm $\perp$)

$$\begin{array}{c}
\frac{}{p \vdash p, \varphi} (Ax) \\
\frac{}{\vdash p \rightarrow p, \varphi} (\rightarrow R) \\
\frac{}{\neg(p \rightarrow p) \vdash \varphi} (\neg L)
\end{array}$$

Hence $\perp \vdash_{LK} \varphi$ for any formula φ .

Theorem 5 (Soundness)

If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Proof: By induction on the derivation of $\Gamma \vdash \Delta$ using the inductive hypothesis

$$\Gamma \vdash \Delta \text{ implies } \text{mod}(\Gamma) \subseteq \text{MOD}(\Delta),$$

where $\text{MOD}(\Delta) = \bigcup_{\varphi \in \Delta} \text{mod}(\varphi)$.

Base Case:

$\psi \in \Gamma \cap \Delta$, i.e., $\Gamma \vdash \Delta$ is an LK-axiom. Hence, $\text{mod}(\Gamma) \subseteq \text{mod}(\psi) \subseteq \text{MOD}(\Delta)$.

Induction:

(a) the *Cut* inference rule:

By inductive hypothesis we have $\text{mod}(\Gamma) \subseteq \text{mod}(\varphi) \cup \text{MOD}(\Delta)$ and $\text{mod}(\Gamma) \cap \text{mod}(\varphi) \subseteq \text{MOD}(\Delta)$. Thus $\text{mod}(\Gamma) \subseteq \text{MOD}(\Delta)$.

(b) the $\rightarrow L$ inference rule:

By inductive hypothesis we have $\text{mod}(\Gamma) \subseteq \text{mod}(\varphi) \cup \text{MOD}(\Delta)$ and $\text{mod}(\Gamma) \cap \text{mod}(\psi) \subseteq \text{MOD}(\Delta)$. Thus $\text{mod}(\Gamma) - \text{mod}(\varphi) \subseteq \text{MOD}(\Delta)$ and $\text{mod}(\Gamma) \cap \text{mod}(\psi) \subseteq \text{MOD}(\Delta)$.

Together, $\text{mod}(\Gamma) \cap \text{mod}(\varphi \rightarrow \psi) \subseteq \text{MOD}(\Delta)$.

(c-e) the $\rightarrow R$, $\neg L$, and $\neg R$ inference rules: similar.

Together, $\Gamma \vdash \varphi$ implies $\text{mod}(\Gamma) \subseteq \text{MOD}(\{\varphi\}) = \text{mod}(\varphi)$, Q.E.D.

Lemma 6 (Weakening)

If $\Gamma \vdash_{LK} \Delta$ then also $\Gamma, \Gamma' \vdash_{LK} \Delta, \Delta'$ for any sets of formulae Γ' and Δ' .

Proof: By straightforward induction on the structure of $\Gamma \vdash_{LK} \Delta$.

The sequent $\Gamma, \Gamma' \vdash_{LK} \Delta, \Delta'$ is called *the weakening* of $\Gamma \vdash_{LK} \Delta$.

Example 7

If $\Gamma \vdash (\varphi \rightarrow \psi), \Delta$ then also $\Gamma, \varphi \vdash \psi, \Delta$. We get $\Gamma, \varphi \vdash \varphi \rightarrow \psi, \psi, \Delta$ by weakening from our assumption; then

$$\frac{\begin{array}{c} \vdots \\ \hline \Gamma, \varphi \vdash \varphi \rightarrow \psi, \psi, \Delta \end{array} \quad (Wkn) \quad \frac{\frac{\Gamma, \varphi \vdash \varphi, \psi, \Delta}{\Gamma, \varphi, \varphi \rightarrow \psi \vdash \psi, \Delta} \quad (Ax) \quad \frac{\Gamma, \varphi, \psi \vdash \psi, \Delta}{\Gamma, \varphi, \varphi \rightarrow \psi \vdash \psi, \Delta} \quad (Ax)}{\Gamma, \varphi \vdash \psi, \Delta} \quad (\rightarrow L) \quad (cut)$$

Theorem 8 (Completeness: reduction from Hilbert System)

$\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Proof: By reducing a Hilbert System proof to LK proof and then appealing to completeness of the Hilbert system.

Let $\Gamma \vdash_H \varphi$ be a Hilbert system proof. By induction on its structure:

Base Cases:

- (a) Instances of Hilbert axioms have LK-proofs given in Example 3; and
- (b) The sequent $\Gamma \vdash \varphi$ is an LK-axiom as $\varphi \in \Gamma$.

Induction:

Assume that the proof of $\Gamma \vdash_H \varphi$ ends in an application of the MP rule on sub-proofs of the form $\Gamma \vdash_H \psi$ and $\Gamma \vdash_H (\psi \rightarrow \varphi)$. Then by induction hypothesis applied twice we get proofs of $\Gamma \vdash_{LK} \psi$ and $\Gamma \vdash_{LK} (\psi \rightarrow \varphi)$. Then by Weakening we get $\Gamma \vdash_{LK} \psi, \varphi$ and using the construction in Example 7 we also have we have $\Gamma, \psi \vdash_{LK} \varphi$. Adding

$$\frac{\Gamma \vdash_{LK} \psi, \varphi \quad \Gamma, \psi \vdash_{LK} \varphi}{\Gamma \vdash_{LK} \varphi} \quad (Cut)$$

then completes the proof.

Theorem 9 (Completeness: reduction from Natural Deduction)

$\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Proof: By reducing a Natural Deduction proof to LK proof and then appealing to completeness of Natural Deduction.

Let $\Gamma \vdash_{ND} \varphi$ be a Natural Deduction proof. By induction on its structure:

Base Case:

The sequent $\Gamma \vdash \varphi$ is also an LK-axiom as $\varphi \in \Gamma$.

Induction:

(a) Assume that the proof ends in an application of the ND $\rightarrow E$ on sub-proofs of the form $\Gamma \vdash_{ND} \psi$ and $\Gamma \vdash_{ND} (\psi \rightarrow \varphi)$. Then by induction hypothesis applied twice we get proofs of $\Gamma \vdash_{LK} \psi$ and $\Gamma \vdash_{LK} (\psi \rightarrow \varphi)$. Using weakening we get $\Gamma \vdash_{LK} \psi, \varphi$ and using the construction in Example 7 we also have $\Gamma, \psi \vdash_{LK} \varphi$. Then

$$\frac{\Gamma \vdash_{LK} \psi, \varphi \quad \Gamma, \psi \vdash_{LK} \varphi}{\Gamma \vdash_{LK} \varphi} (Cut)$$

completes the case;

(b) Assume that the proof ends in an application of the ND $\rightarrow I$ rule on a sub-proof of the form $\Gamma, \varphi \vdash_{ND} \psi$. Applying IH and the LK $\rightarrow R$ rule yields $\Gamma \vdash_{LK} \varphi \rightarrow \psi$ as required;

(c) Assume that the proof ends in an application of the ND RAA rule on a sub-proof of the sequent $\Gamma, (\varphi \rightarrow \perp) \vdash_{ND} \perp$. Then by IH, Weakening, and *Thm* $\perp$ we have

$$\frac{\frac{\Gamma, (\varphi \rightarrow \perp) \vdash_{LK} \perp, \varphi}{\Gamma \vdash_{LK} (\varphi \rightarrow \perp) \rightarrow \perp, \varphi} (\rightarrow R) \quad \frac{\frac{\vdots}{\Gamma, \perp \vdash_{LK} \varphi} (Thm\perp) \quad \frac{\overline{\Gamma, \varphi \vdash_{LK} \perp, \varphi}}{\Gamma \vdash_{LK} (\varphi \rightarrow \perp), \varphi} (Ax)}{\Gamma, (\varphi \rightarrow \perp) \rightarrow \perp \vdash_{LK} \varphi} (\rightarrow L)}{\Gamma \vdash_{LK} \varphi} (cut)$$

This completes the proof.

Theorem 10 (Cut Elimination (“Hauptsatz”))

For every proof of a sequent $\Gamma \vdash \Delta$ in LK there is also proof of the same sequent in LK – {Cut}.

Proof: By induction on the structure of the derivation of $\Gamma \vdash \Delta$ and the complexity of the *Cut* formula in the restricted case where the last inference in the proof is a *Cut* and where this is the only use of the *Cut* inference rule and is of the form

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta} (Cut)$$

Base Cases: one or both of the premises are LK-axioms

- (a) $\varphi \in \Gamma$: then $\Gamma, \varphi \vdash \Delta$ is the same as $\Gamma \vdash \Delta$;
- (b) $\varphi \in \Delta$: then $\Gamma \vdash \varphi, \Delta$ is the same as $\Gamma \vdash \Delta$; or
- (c) $\psi \in \Gamma \cap \Delta$: then $\Gamma \vdash \Delta$ is an LK-axiom; in all cases the resulting proof is *Cut*-free.

Induction:

(a) φ is the *principal* formula in both $\Gamma \vdash \varphi, \Delta$ and $\Gamma, \varphi \vdash \Delta$. By cases analysis:

(a1) implication:

$$\frac{\frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash (\varphi \rightarrow \psi), \Delta} (\rightarrow R) \quad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, (\varphi \rightarrow \psi) \vdash \Delta} (\rightarrow L)}{\Gamma \vdash \Delta} (Cut)$$

is rewritten to

$$\frac{\Gamma \vdash \varphi, \Delta \quad \frac{\Gamma, \varphi \vdash \psi, \Delta \quad \Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \vdash \Delta} (Cut)}{\Gamma \vdash \Delta} (Cut)$$

where $\Gamma, \varphi, \psi \vdash \Delta$ is a weakening of $\Gamma, \psi \vdash \Delta$; the claim then follows by induction.

(a2) negation: (similar)

(b) φ is not the principal formula in one of the premises. By cases analysis:

(b1) the principal formula on the right side of the right premise is $\psi \rightarrow \eta$:

$$\frac{\Gamma \vdash \varphi, (\psi \rightarrow \eta), \Delta \quad \frac{\Gamma, \varphi, \psi \vdash \eta, \Delta}{\Gamma, \varphi \vdash (\psi \rightarrow \eta), \Delta} (\rightarrow R)}{\Gamma \vdash (\psi \rightarrow \eta), \Delta} (Cut)$$

is rewritten to

$$\frac{\Gamma, \psi \vdash \varphi, (\psi \rightarrow \eta), \Delta \quad \frac{\Gamma, \varphi, \psi \vdash \eta, (\psi \rightarrow \eta), \Delta}{\Gamma, \psi \vdash \eta, (\psi \rightarrow \eta), \Delta} (Cut)}{\Gamma \vdash (\psi \rightarrow \eta), \Delta} (\rightarrow R)$$

where $\Gamma, \psi \vdash \varphi, (\psi \rightarrow \eta), \Delta$ and $\Gamma, \varphi, \psi \vdash \eta, (\psi \rightarrow \eta), \Delta$ are weakenings of $\Gamma \vdash \varphi, (\psi \rightarrow \eta), \Delta$ and $\Gamma, \varphi, \psi \vdash \eta, \Delta$, respectively; the claim then follows by induction.

(b2-8) implication and negation on left/right side of both premises: (similar)

Observing that repeated application of this construction on every inner-most instance of the *Cut* inference, while one exists, in the proof of $\Gamma \vdash \Delta$ yields a *Cut*-free LK-proof.