The Relational Model
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School of Computer Science
University of Waterloo

Databases CS348
How do we ask Questions (and understand Answers)?

Find all pairs of (natural) numbers that add to 5!

Question: \( \{(x, y) \mid x + y = 5\} \)

Answer: \( \{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)\} \)

... but but but why? (explain this to a 6 year old!)
because \((0, 5, 5)\), etc., appear in PLUS!

Find pairs of numbers that add to the same number as they subtract to (i.e., \( x + y = x - y \))!

Question: \( \{(x, y) \mid \exists z. PLUS(x, y, z) \land PLUS(z, y, x)\} \)

Answer: \( \{(0, 0), (1, 0), \ldots, (5, 5)\} \)

... answer depends on the content (instance) of PLUS!

Find the neutral element (of addition)!

Question: \( \{(x) \mid PLUS(x, x, x)\} \)

Answer: \( \{(0)\} \)

Addition Table

<table>
<thead>
<tr>
<th>PLUS</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>5</td>
<td>5</td>
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<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
<td>5</td>
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<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
How do we ask Questions about Employees?

Find **all employees who work for “Bob”!**

**Question:** \{((x, y) | EMP(x, y, Bob))\}

**Answer:** \{((Sue, CS), (Bob, CO))\}

**why?** because \((Sue, CS, Bob), \text{ etc.}, \text{ appear in EMP!} \)

Find pairs of emp-s working for the same boss!

**Q:** \{((x_1, x_2) | \exists y_1, y_2, z.EMP(x_1, y_1, z) \land EMP(x_2, y_2, z))\}

**A:** \{((Sue, Bob), (Fred, John), (Jim, Eve)) \}

← **is that all?**

Find employees who are their own bosses!

**Q:** \{((x) | \exists y.EMP(x, y, x))\}

**A:** \{((Sue), (Bob))\}

---

**Employee Table**

<table>
<thead>
<tr>
<th>EMP</th>
<th>Name</th>
<th>Dept</th>
<th>Boss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sue</td>
<td>CS</td>
<td>Bob</td>
<td></td>
</tr>
<tr>
<td>Bob</td>
<td>CO</td>
<td>Bob</td>
<td></td>
</tr>
<tr>
<td>Fred</td>
<td>PM</td>
<td>Mark</td>
<td></td>
</tr>
<tr>
<td>John</td>
<td>PM</td>
<td>Mark</td>
<td></td>
</tr>
<tr>
<td>Jim</td>
<td>CS</td>
<td>Fred</td>
<td></td>
</tr>
<tr>
<td>Eve</td>
<td>CS</td>
<td>Fred</td>
<td></td>
</tr>
<tr>
<td>Sue</td>
<td>PM</td>
<td>Sue</td>
<td></td>
</tr>
</tbody>
</table>
The Relational Model

Idea

All information is organized in (a finite number of) relations.

Features:

- simple and clean data model
- powerful and declarative query/update languages
- semantic integrity constraints
- data independence
Relational Structures/Databases

Components:

- **Universe**: a set of values \( D \) with equality (\( = \))
- **Relation**: schema: name \( R \), arity \( k \) (the number of attributes)
  - instance: a relation \( R \subseteq D^k \).
- **Database**: schema: finite set of relation schemes
  - instance: a relation \( R_i \) for each \( R_i \)

Notation

**Signature**: \( \rho = (R_1, \ldots, R_n) \)

**Instance**: \( DB = (D, =, R_1, \ldots, R_n) \)
Examples of Relational Structures a.k.a. Databases

- the integer numbers with addition and multiplication:
  \[ \rho = (\text{plus}, \text{times}) \quad \text{DB} = (\mathbb{Z}, =, \text{plus}, \text{times}) \]
- a Bibliography Database
- ...
Example: Bibliography

Relations (signatures) used in examples:

author(aid, name)
wrote(author, publication)
publication(pubid, title)
book(pubid, publisher, year)
journal(pubid, volume, no, year)
proceedings(pubid, year)
article(pubid, crossref, startpage, endpage)

⇒ names of attributes will be important later (for SQL)
Example (sample instance)

\[
\begin{align*}
\text{author} &= \{ (1, \text{John}), (2, \text{Sue}) \} \\
\text{wrote} &= \{ (1, 1), (1, 4), (2, 3) \} \\
\text{publication} &= \{ (1, \text{Mathematical Logic}), \\
&\quad (3, \text{Trans. Databases}), \\
&\quad (2, \text{Principles of DB Syst.}), \\
&\quad (4, \text{Query Languages}) \} \\
\text{book} &= \{ (1, \text{AMS}, 1990) \} \\
\text{journal} &= \{ (3, 35, 1, 1990) \} \\
\text{proceedings} &= \{ (2, 1995) \} \\
\text{article} &= \{ (4, 2, 30, 41) \}
\end{align*}
\]
Example (tabular form)

<table>
<thead>
<tr>
<th>author</th>
<th>wrote</th>
</tr>
</thead>
<tbody>
<tr>
<td>aid</td>
<td>name</td>
</tr>
<tr>
<td>1</td>
<td>John</td>
</tr>
<tr>
<td>2</td>
<td>Sue</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>author</th>
<th>publication</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>publication</th>
</tr>
</thead>
<tbody>
<tr>
<td>pubid</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

⇒ that’s why relations are often called “tables”.

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Simple (Atomic) “Truth”

Idea

*Relationships between objects (tuples) that are present in an instance are true, relationships absent are false.*

In the sample *Bibliography* database instance

- “John” is an *author* with id “1”: 
  \[(1, \text{John}) \in \text{author};\]
- “Mathematical Logic” is a publication: 
  \[(1, \text{Mathematical Logic}) \in \text{publication};\]
  Moreover it is a book published by “AMS” in “1990”: 
  \[(1, \text{AMS}, 1990) \in \text{book};\]
- “John” wrote “Mathematical Logic”: 
  \[(1, 1) \in \text{wrote};\]
- “John” has **NOT** written “Trans. Databases”: 
  \[(1, 3) \notin \text{wrote};\]
- etc.
# Queries

**Idea 1:** use *variables* to collect answers

\[ \text{author}(x, y) \text{ asks for all valuations } [x \mapsto a, y \mapsto b, \ldots] \]

such that the pair \((a, b) \in \text{author}\)

**Idea 2:** build more complex queries from simpler ones using...

**Logical connectives:**
- Conjunction (and):
  \[ \text{author}(x, y) \land \text{wrote}(x, z) \]
- Disjunction (or):
  \[ \text{author}(x, y) \lor \text{publication}(x, y) \]
- Negation (not):
  \[ \neg \text{author}(x, y) \]

**Quantifiers:**
- Existential (there is...):
  \[ \exists x. \text{author}(x, y) \]
Relational Calculus: Syntax

**Idea**

Complex statements about truth can be formulated using the language of first-order logic.

**Definition (Syntax)**

Given a database schema \( \rho = (R_1, \ldots, R_k) \) and a set of variable names \( \{x_1, x_2, \ldots\} \), formulas are defined by

\[
\varphi ::= R_i(x_{i_1}, \ldots, x_{i_k}) \mid x_i = x_j \mid \varphi \land \varphi \mid \exists x_i. \varphi \mid \varphi \lor \varphi \mid \neg \varphi
\]

- conjunctive formulas
- positive formulas
- first-order formulas
First-order Variables and Valuations

How do we interpret variables?

**Definition (Valuation)**

A valuation is a function

\[ \theta : \{ x_1, x_2, \ldots \} \rightarrow D \]

that maps variable names to values in the universe.

**Idea**

*Answers to queries ⇔ valuations to free variables that make the formula true with respect to a database.*
Complete Semantics

Definition

The truth of formulas is defined with respect to
1. a database instance \( \text{DB} = (D, =, R, S, \ldots) \), and
2. a valuation \( \theta : \{x_1, x_2, \ldots\} \rightarrow D \)

as follows:

\[
\begin{align*}
\text{DB}, \theta &\models R(x_{i_1}, \ldots, x_{i_k}) \text{ if } R \in \rho, (\theta(x_{i_1}), \ldots, \theta(x_{i_k})) \in R \\
\text{DB}, \theta &\models x_i = x_j \text{ if } \theta(x_i) = \theta(x_j) \\
\text{DB}, \theta &\models \varphi \land \psi \text{ if } \text{DB}, \theta \models \varphi \text{ and } \text{DB}, \theta \models \psi \\
\text{DB}, \theta &\models \neg \varphi \text{ if } \text{not } \text{DB}, \theta \models \varphi \\
\text{DB}, \theta &\models \exists x_i.\varphi \text{ if } \text{DB}, \theta[x_i \mapsto v] \models \varphi \text{ for some } v \in D
\end{align*}
\]

Definition

An answer to a query \( \{(x_1, \ldots, x_k) \mid \varphi\} \) over \( \text{DB} \) is a relation:

\[
\{(\theta(x_1), \ldots, \theta(x_k)) \mid \text{DB}, \theta \models \varphi\}
\]

where \( \{x_1, \ldots, x_k\} = \text{FV}(\varphi) \).
Example

Find pairs of emp-s working for the same boss!

Q: \{ (x_1, x_2) \mid \exists y_1, y_2, z. EMP(x_1, y_1, z) \land EMP(x_2, y_2, z) \} 
A: \{ (Sue, Fred), \ldots \} 

because:

1. EMP, [x_1 \mapsto Sue, y_1 \mapsto CS, z \mapsto Bob, \ldots] \models EMP(x_1, y_1, z)
2. EMP, [x_2 \mapsto Fred, y_2 \mapsto CO, z \mapsto Bob, \ldots] \models EMP(x_2, y_2, z)
3. EMP, [x_1 \mapsto Sue, y_1 \mapsto CS, x_2 \mapsto Fred, y_2 \mapsto CO, z \mapsto Bob, \ldots] 
   \models EMP(x_1, y_1, z) \land EMP(x_2, y_2, z)
4. EMP, [x_1 \mapsto Sue, x_2 \mapsto Fred, \ldots] 
   \models \exists y_1, y_2, z. EMP(x_1, y_1, z) \land EMP(x_2, y_2, z)
Sample Queries

over numbers (with addition and multiplication):

- list all composite numbers
- list all prime numbers

over the bibliography database:

- list all publications
- list titles of all publications
- list titles of all books
- list all publications without authors
- list (pairs of) coauthor names
- list titles of publications written by a single author
Equivalences and Syntactic Sugar

Boolean Equivalences

- \( \neg (\neg \varphi_1) \equiv \varphi_1 \)
- \( \varphi_1 \lor \varphi_2 \equiv \neg (\neg \varphi_1 \land \neg \varphi_2) \)
- \( \varphi_1 \rightarrow \varphi_2 \equiv \neg \varphi_1 \lor \varphi_2 \)
- \( \varphi_1 \leftrightarrow \varphi_2 \equiv (\varphi_1 \rightarrow \varphi_2) \land (\varphi_2 \rightarrow \varphi_1) \)
- ...  

First-order Equivalences

- \( \forall x. \varphi \equiv \neg \exists x. \neg \varphi \)
How do we ask Questions (and understand Answers)?

Find the neutral element (of addition)!

Question: \( \{ (x) \mid \text{PLUS}(x, x, x) \} \)

Answer: \( \{ (0) \} \)

but shouldn’t the query really be

\[ \{ (x) \mid \forall y. \text{PLUS}(x, y, y) \land \text{PLUS}(y, x, y) \} \tag{\ast} \]

Idea

\( \ast \) is the same as \( \{ (x) \mid \forall y. \text{PLUS}(x, y, y) \} \)

because \( \text{PLUS} \) is commutative

is the same as \( \{ (x) \mid \text{PLUS}(x, x, x) \} \)

because \( \text{PLUS} \) is monotone

⇒ Laws of Arithmetic for Natural Numbers
Laws a.k.a. Integrity Constraints

**Idea**

*What must be always true for the natural numbers (i.e., for PLUS)?*

- Addition is commutative
  \[ \forall x, y, z. \text{PLUS}(x, y, z) \rightarrow \text{PLUS}(y, x, z) \]
  \[ (\neg \exists x, y, z. \text{PLUS}(x, y, z) \land \neg \text{PLUS}(y, x, z)) \]

- Addition is a (relational representation of a) binary function
  \[ \forall x, y, z_1, z_2. \text{PLUS}(x, y, z_1) \land \text{PLUS}(x, y, z_2) \rightarrow z_1 = z_2 \]
  \[ (\neg \exists x, y, z_1, z_2. \text{PLUS}(x, y, z_1) \land \text{PLUS}(x, y, z_2) \land \neg (z_1 = z_2)) \]

- Addition is a total function
  \[ \forall x, y. \exists z. \text{PLUS}(x, y, z) \]

- Addition is monotone in both arguments (harder), etc., etc.
Laws a.k.a. Integrity Constraints for Employees

Idea

Integrity constraints
⇒ yes/no queries that must be true in every valid database instance.

- Every Boss is an Employee
  \[ \forall x, y, z. \text{EMP}(x, y, z) \rightarrow \exists u, w. \text{EMP}(z, u, w) \]

- Every Boss manages a unique Department
  \[ \forall x_1, x_2, y_1, y_2, z. \text{EMP}(x_1, y_1, z) \land \text{EMP}(x_2, y_2, z) \rightarrow y_1 = y_2 \]

- No Boss cannot have another Employee serving as their Boss
  \[ \forall x, y, z. \text{EMP}(x, y, z) \rightarrow \text{EMP}(z, y, z) \]
Integrity Constraints

Relational *signature* captures only the structure of relations.

**Idea**

*Valid database instances satisfy additional integrity constraints.*

- values of a particular attribute belong to a prescribed *data type*.
- values of attributes are unique among tuples in a relation (*keys*).
- values appearing in one relation must also appear in another relation (*referential integrity*).
- values cannot appear simultaneously in certain relations (*disjointness*).
- values in certain relation must appear in at least one of another set of relations (*coverage*).
- ...
Example Revisited (Bibliography)

Typing constraints
- Author id’s are integers.
- Author names are strings.

Uniqueness of values/Keys
- Author id’s are unique and determine author names.
- Publication id’s are unique as well.
- Articles are identified by their id and the id of a collection they have appeared in.

Referential Integrity/Foreign Keys
- “books”, ”journals”, ”proceedings”, and ”articles” are ”publications”.
- The components of a “wrote” tuple must be an “author” and a “publication”.

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Example Revisited (cont.)

Disjointness

- “books” are different from “journals”.
- “books” are different from “proceedings”.

Coverage

- Every “publication” is a “book” or a “journal” or a “proceedings” or an “article”.
- Every “article” appears in a “book” or in a “journal” or in “proceedings”.

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Views and Integrity Constraints

**Idea**

*Answers to queries can be used to define derived relations (views)*

⇒ extension of a DB schema

- subtraction, complement, . . .
- *collection*-style publication, editor, . . .

In general, a view is an integrity constraint of the form

\[ \forall x_1, \ldots, x_k. R(x_1, \ldots, x_k) \leftrightarrow \varphi \]

for \( R \) a new relation name and \( x_1, \ldots, x_k \) free variables of \( \varphi \).
Definition (Database Schema)

Let $\rho$ be a signature. A database schema is a (finite) set of integrity constraints $\Sigma$ over $\rho$.

Definition

A database instance $\text{DB}$ (over a schema $\rho$) conforms to a schema $\Sigma$ (written $\text{DB} \models \Sigma$) if and only if $\text{DB}, \theta \models \varphi$ for any integrity constraint $\varphi \in \Sigma$ and any valuation $\theta$. 
Story so far…

1. databases ⇔ relational structures
2. queries ⇔ set comprehensions with formulas in First-Order logic
3. integrity constraints ⇔ closed formulas in FO logic

… so is there anything new here?

⇒ YES: database instances must be finite
Unsafe Queries

- \( \{ (y) \mid \neg \exists x. \text{author}(x, y) \} \)
- \( \{ (x, y, z) \mid \text{book}(x, y, z) \lor \text{proceedings}(x, y) \} \)
- \( \{ (x, y) \mid x = y \} \)

\( \Rightarrow \) we want only queries with finite answers (over finite databases).

**Definition (Domain-independent Query)**

A query \( \{ (x_1, \ldots, x_k) \mid \varphi \} \) is *domain-independent* if

\[
\text{DB}_1, \theta \models \varphi \iff \text{DB}_2, \theta \models \varphi
\]

for any pair of database instances \( \text{DB}_1 = (D_1, =, R_1, \ldots, R_k) \) and \( \text{DB}_2 = (D_2, =, R_1, \ldots, R_k) \) and all \( \theta \).

**Theorem**

Answers to domain-independent queries contain only values that exist in \( R_1, \ldots, R_k \) (the active domain).

Domain-independent + finite database \( \Rightarrow \) “safe”
Safety and Query Satisfiability

Theorem

*Satisfiability*\(^1\) of first-order formulas is undecidable;
- co-r.e. in general
- r.e for finite databases

Proof.

Reduction from PCP (see Abiteboul *et. al.* book, p.122-126).

\[\begin{align*}
\varphi \text{ is satisfiable} & \iff \{ (x, y) \mid (x = y) \land \varphi \} \text{ is not domain-independent.}
\end{align*}\]

\(^1\) Is there a database for which the answer is non-empty?
Range-restricted Queries

Definition (Range restricted formulas)

A formula $\varphi$ is range restricted when, for $\varphi_i$ that are also range restricted, $\varphi$ has the form

$$R(x_{i_1}, \ldots, x_{i_k}),$$

$$\varphi_1 \land \varphi_2,$$

$$\varphi_1 \land (x_i = x_j) \quad (\{x_i, x_j\} \cap FV(\varphi_1) \neq \emptyset),$$

$$\exists x_i. \varphi_1 \quad (x_i \in FV(\varphi_1)),$$

$$\varphi_1 \lor \varphi_2 \quad (FV(\varphi_1) = FV(\varphi_2)), \text{ or}$$

$$\varphi_1 \land \neg \varphi_2 \quad (FV(\varphi_2) \subseteq FV(\varphi_1)).$$

Theorem

Range-restricted $\Rightarrow$ Domain-independent.
Domain Independent v.s. Range-restricted

Do we lose expressiveness by restricting to Range-restricted queries?

Theorem

*Every domain-independent query can be written equivalently as a range restricted query.*

Proof.

1. restrict every variable in $\varphi$ to *active domain*,
2. express the active domain using a *unary query* over the database instance.
Computational Properties

- Evaluation of every query terminates
  ⇒ relational calculus is not *Turing complete*

- **Data Complexity** in the size of the database, for a *fixed* query.
  ⇒ in PTIME
  ⇒ in LOGSPACE
  ⇒ $AC_0$ (constant time on polynomially many CPUs in parallel)

- **Combined complexity**
  ⇒ in PSPACE
  ⇒ can express NP-hard problems (encode SAT)
Query Evaluation vs. Theorem Proving

Query Evaluation
Given a query \( \{(x_1, \ldots, x_k) \mid \varphi \} \) and a finite database instance \( DB \) find all answers to the query.

Query Satisfiability
Given a query \( \{(x_1, \ldots, x_k) \mid \varphi \} \) determine whether there is a (finite) database instance \( DB \) for which the answer is non-empty.

- much harder (undecidable) problem
- can be solved for fragments of the query language
Query Equivalence and DB Schema

Do we ever need the power of *theorem proving*?

**Definition (Query Subsumption)**

A query \( \{ (x_1, \ldots, x_k) \mid \varphi \} \) *subsumes* a query \( \{ (x_1, \ldots, x_k) \mid \psi \} \) with respect to a database schema \( \Sigma \) if

\[
\{ (\theta(x_1), \ldots, \theta(x_k)) \mid \text{DB}, \theta \models \psi \} \subseteq \{ (\theta(x_1), \ldots, \theta(x_k)) \mid \text{DB}, \theta \models \varphi \}
\]

for every database \( \text{DB} \) such that \( \text{DB} \models \Sigma \).

- *necessary* for query simplification
- equivalent to proving

\[
\left( \bigwedge_{\phi_i \in \Sigma} \phi_i \right) \rightarrow (\forall x_1, \ldots, x_k. \psi \rightarrow \varphi)
\]

- undecidable in general; decidable for fragments of relational calculus
What queries cannot be expressed in RC?

Note

*RC is not Turing-complete*

⇒ there must be computable queries that cannot be written in RC.

Built-in Operations

- ordering, arithmetic, string operations, etc.

Counting/Aggregation

- cardinality of sets (*parity*)

Reachability/Connectivity/…

- *paths in a graph* (*binary relation*)

Model extensions: Incompleteness/Inconsistency

- tuples with *unknown* (but existing) values
- incomplete relations and *open world assumption*
- conflicting information (e.g., from different data sources)