Classical Sequent Calculus (LK)

for Propositional Logic

CS 245

Idea: make a proof system that manipulates *assumptions* as well as the formula that is being proven.

Definition 1 (Sequent) Let Γ and Δ be sets of formulae. We call $\Gamma \vdash \Delta$ a *sequent*.

Notation: In sequents we write Γ, φ for $\Gamma \cup \{\varphi\}$ and Γ, Δ for $\Gamma \cup \Delta$..

Definition 2 (System LK)

Identity Rules:

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta}\ (Axiom) \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta}\ (Cut)$$

Logical Rules:

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, (\neg\varphi) \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash (\neg\varphi), \Delta}\ (\neg R)$$

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, (\varphi \to \psi) \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash (\varphi \to \psi), \Delta}\ (\to R)$$

The formulæ $(\neg\varphi)$ and $(\varphi \to \psi)$ in the $\neg L$, $\neg R$, $\to L$, and $\to R$ rules are called *principal formulæ* of the inference, φ is called the *cut formula* in the *Cut* inference.

Note: In many (more formal) definitions of LK, sequents are formed by two *sequences* of formulae; then additional *structural inference rules*, such as exchange (allowing to swap formulae in the sequence), contraction (allowing to remove adjacent duplicate formulae), etc., are needed.

Example 3 (LK Proofs of Hilbert Axioms 1-2) Axiom 1:

$$\cfrac{\cfrac{\cfrac{}{p, q \vdash p}\ (Ax)}{p \vdash q \to p}\ (\to R)}{\vdash p \to (q \to p)}\ (\to R)$$

Axiom 2:

$$\dfrac{\dfrac{\dfrac{}{p \to (q \to r), p \vdash p, r}\,(Ax) \qquad \dfrac{\dfrac{}{p, q \vdash p, r}\,(Ax) \qquad \dfrac{\dfrac{}{p, q \vdash q, r}\,(Ax) \qquad \dfrac{}{p, q, r \vdash r}\,(Ax)}{q \to r, p, q \vdash r}\,(\to L)}{p \to (q \to r), p, q \vdash r}\,(\to L)}{\dfrac{p \to (q \to r), p \to q, p \vdash r}{\dfrac{p \to (q \to r), p \to q \vdash p \to r}{p \to (q \to r) \vdash (p \to q) \to (p \to r)}\,(\to R)}\,(\to R)}\,(\to L)$$

$$\dfrac{p \to (q \to r) \vdash (p \to q) \to (p \to r)}{\vdash (p \to (q \to r)) \to ((p \to q) \to (p \to r))}\,(\to R)$$

Axiom 3:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{q, p \vdash p}\,(Ax)}{q \vdash (\neg p), p}\,(\neg R) \qquad \dfrac{\dfrac{}{q \vdash q, p}\,(Ax)}{q, (\neg q) \vdash p}\,(\neg L)}{\neg p \to \neg q, q \vdash p}\,(\to L)}{\dfrac{\neg p \to \neg q \vdash q \to p}{\vdash (\neg p \to \neg q) \to (q \to p)}\,(\to R)}\,(\to R)$$

Theorem 4 (Soundness) If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

<u>P r o o f:</u> By induction on the derivation of $\Gamma \vdash \Delta$ using the inductive hypothesis

$$\Gamma \vdash \Delta \quad \text{implies} \quad \mathsf{mod}(\Gamma) \subseteq \mathsf{MOD}(\Delta),$$

where $\mathsf{MOD}(\Delta) = \bigcup_{\varphi \in \Delta} \mathsf{mod}(\varphi)$.

Base Case:

$\psi \in \Gamma \cap \Delta$, i.e., $\Gamma \vdash \Delta$ is an LK-axiom. Hence, $\mathsf{mod}(\Gamma) \subseteq \mathsf{mod}(\psi) \subseteq \mathsf{MOD}(\Delta)$.

Induction:

(a) the *Cut* inference rule:
By inductive hypothesis we have $\mathsf{mod}(\Gamma) \subseteq \mathsf{mod}(\varphi) \cup \mathsf{MOD}(\Delta)$ and $\mathsf{mod}(\Gamma) \cap \mathsf{mod}(\varphi) \subseteq \mathsf{MOD}(\Delta)$. Thus $\mathsf{mod}(\Gamma) \subseteq \mathsf{MOD}(\Delta)$.

(b) the $\to L$ inference rule:
By inductive hypothesis we have $\mathsf{mod}(\Gamma) \subseteq \mathsf{mod}(\varphi) \cup \mathsf{MOD}(\Delta)$ and $\mathsf{mod}(\Gamma) \cap \mathsf{mod}(\psi) \subseteq \mathsf{MOD}(\Delta)$. Thus $\mathsf{mod}(\Gamma) - \mathsf{mod}(\varphi) \subseteq \mathsf{MOD}(\Delta)$ and $\mathsf{mod}(\Gamma) \cap \mathsf{mod}(\psi) \subseteq \mathsf{MOD}(\Delta)$. Together, $\mathsf{mod}(\Gamma) \cap \mathsf{mod}(\varphi \to \psi) \subseteq \mathsf{MOD}(\Delta)$.

(c-e) the $\to$*R*, $\neg$*L*, and $\neg$*R* inference rules: similar.

Together, $\Gamma \vdash \varphi$ implies $\mathsf{mod}(\Gamma) \subseteq \mathsf{MOD}(\{\varphi\}) = \mathsf{mod}(\varphi)$, Q.E.D.

Lemma 5 (Weakenning) If $\Gamma \vdash_{LK} \Delta$ then also $\Gamma, \Gamma' \vdash_{LK} \Delta, \Delta'$ for any sets of formulæ Γ' and Δ'.

P r o o f: By straightforward induction on the structure of $\Gamma \vdash_{LK} \Delta$.

The sequent $\Gamma, \Gamma' \vdash_{LK} \Delta, \Delta'$ is called *the weakening* of $\Gamma \vdash_{LK} \Delta$.

Lemma 6 (Deduction Theorem (for LK)**)** If $\Gamma \vdash_{LK} (\varphi \to \psi), \Delta$ then also $\Gamma, \varphi \vdash_{LK} \psi, \Delta$.

P r o o f: By induction on the structure of the proof of $\Gamma \vdash_{LK} (\varphi \to \psi), \Delta$.

Base Cases: $\Gamma(\varphi \to \psi) \vdash_{LK} (\varphi \to \psi), \Delta$ is an LK-axiom. Then, however, $\Gamma, \varphi \vdash_{LK} \varphi, \psi, \Delta$ and $\Gamma, \varphi, \psi \vdash_{LK} \psi, \Delta$ are also LK-axioms and an application of the $\to L$ inference yields $\Gamma, (\varphi \to \psi), \varphi \vdash_{LK} \psi, \Delta$.

Otherwise $\Gamma \vdash_{LK} \Delta$ is an LK-axiom and $\Gamma, \varphi \vdash_{LK} \psi, \Delta$ is a weakening of that axiom.

Induction:

(a) $\varphi \to \psi$ is principal in the last step of the proof of $\Gamma \vdash_{LK} (\varphi \to \psi), \Delta$. Thus the last inference was

$$\frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash (\varphi \to \psi), \Delta}\ (\to R)$$

and the antecedent of this inference is indeed the required sequent;

(b) $\varphi \to \psi$ is not principal in the last step of the proof. Then the same formula appears in all antecedents of the last inference and the claim follows by cases analysis and induction.

Theorem 7 (Completeness) $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

P r o o f: By reducing a Hilbert System proof to LK proof and then appealing to completeness of the Hilbert system.

Let $\Gamma \vdash_H \varphi$ be a Hilbert system proof. By induction on its structure:

Base Cases:

(a) Instances of Hilbert axioms have LK-proofs given in Example 3; and
(b) The sequent $\Gamma \vdash \varphi$ is an LK-axiom for assumptions φ as $\varphi \in \Gamma$.

Induction:

Assume that the proof of $\Gamma \vdash_H \varphi$ ends in an application of the MP rule on sub-proofs of the form $\Gamma \vdash_H \psi$ and $\Gamma \vdash_H (\psi \to \varphi)$. Then by induction hypothesis applied twice we get proofs of $\Gamma \vdash_{LK} \psi$ and $\Gamma \vdash_{LK} (\psi \to \varphi)$. By Weakening Lemma we then have $\Gamma \vdash_{LK} \psi, \varphi$ and by the Deduction Theorem (for LK) we have $\Gamma, \psi \vdash_{LK} \varphi$. Hence

$$\frac{\Gamma \vdash_{LK} \psi, \varphi \qquad \Gamma, \psi \vdash_{LK} \varphi}{\Gamma \vdash_{LK} \varphi}\ (Cut)$$

completes the proof.

Theorem 8 (Cut Elimination) For every proof of a sequent $\Gamma \vdash \Delta$ in LK there is another proof of the same sequent in LK $- \{Cut\}$.

Proof: By induction on the structure of the derivation of $\Gamma \vdash \Delta$ and the complexity of the *Cut* formula in the restricted case where the last inference in the proof is a *Cut* and where this is the only use of the *Cut* inference rule of the form

$$\frac{\Gamma \vdash \varphi, \Delta \qquad\qquad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta}\ (\mathit{Cut})$$

Base Cases: one or both of the premises are LK-axioms

(a) $\varphi \in \Gamma$: then $\Gamma, \varphi \vdash \Delta$ is the same as $\Gamma \vdash \Delta$;

(b) $\varphi \in \Delta$: then $\Gamma \vdash \varphi, \Delta$ is the same as $\Gamma \vdash \Delta$; or

(c) $\psi \in \Gamma \cap \Delta$: then $\Gamma \vdash \Delta$ is an LK-axiom; in all cases the resulting proof is *Cut*-free.

Induction:

(a) φ is the *principal* formula in both $\Gamma \vdash \varphi, \Delta$ and $\Gamma, \varphi \vdash \Delta$. By cases analysis:

(a1) implication:

$$\frac{\dfrac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash (\varphi \to \psi), \Delta}\ (\to\! R) \qquad \dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, (\varphi \to \psi) \vdash \Delta}\ (\to\! L)}{\Gamma \vdash \Delta}\ (\mathit{Cut})$$

is rewritten to

$$\frac{\Gamma \vdash \varphi, \Delta \qquad \dfrac{\Gamma, \varphi \vdash \psi, \Delta \qquad \Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ (\mathit{Cut})}{\Gamma \vdash \Delta}\ (\mathit{Cut})$$

where $\Gamma, \varphi, \psi \vdash \Delta$ is a weakening of $\Gamma, \psi \vdash \Delta$; the claim then follows by induction.

(a2) negation: (similar)

(b) φ is not the principal formula in one of the premises. By cases analysis:

(b1) the principal formula on the right side of the right premise is $\psi \to \eta$:

$$\frac{\Gamma \vdash \varphi, (\psi \to \eta), \Delta \qquad \dfrac{\Gamma, \varphi, \psi \vdash \eta, \Delta}{\Gamma, \varphi \vdash (\psi \to \eta), \Delta}\ (\to\! R)}{\Gamma \vdash (\psi \to \eta), \Delta}\ (\mathit{Cut})$$

is rewritten to

$$\dfrac{\dfrac{\Gamma, \psi \vdash \varphi, (\psi \to \eta), \Delta \qquad \Gamma, \varphi, \psi \vdash \eta, (\psi \to \eta), \Delta}{\Gamma, \psi \vdash \eta, (\psi \to \eta), \Delta}\ (\mathit{Cut})}{\Gamma \vdash (\psi \to \eta), \Delta}\ (\to\! R)$$

where $\Gamma, \psi \vdash \varphi, (\psi \to \eta), \Delta$ and $\Gamma, \varphi, \psi \vdash \eta, (\psi \to \eta), \Delta$ are weakenings of $\Gamma, \varphi, \psi \vdash \eta, \Delta$ and $\Gamma, \varphi, \psi \vdash \eta, \Delta$, respectively; the claim then follows by induction.

(b2-8) implication and negation on left/right side of both premises: (similar)

Observing that repeated application of this construction on every inner-most instance of the *Cut* inference, while one exists, in the proof of $\Gamma \vdash \Delta$ yields a *Cut*-free LK-proof.