Propositional Logic

Summary of Definitions and Main Results

CS 245

1 Syntax of Propositional Logic

Statements of propositional logic come from a formal "language", denoted $\mathcal{L}^p$. A formal language consists of strings over an alphabet.

Definition 1 (The Alphabet of Propositional Logic)
The *alphabet* of the language of propositional logic consists of the following symbols:

1. Propositional Symbols: p, q, r, p_1, p_2, $\ldots$, q_1, q_2, $\ldots$;
2. Logical Connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$; and
3. Punctuation: "(" and ")".

(Textbook: 2.2, p. 21 ff.)

Definition 2 (Well-Formed Formulae (WFFs) of Propositional Logic)
Let P be a set of propositional symbols. We define the *set* WFF *of well-formed formulae over* P inductively as follows.

1. $P \subseteq WFF$ (that is, every propositional symbol is a WFF by itself, called an *atom*);
2. if $\varphi \in WFF$, then $(\neg\varphi) \in WFF$; and
3. if $\varphi \in WFF$, $\psi \in WFF$, and $\star \in \{\wedge, \vee, \rightarrow\}$, then $(\varphi \star \psi) \in WFF$.

Nothing else is an element of WFF.

The set WFF is also denoted by $Form(\mathcal{L}^p)$. The set of "atoms"—well-formed formulae of length 1—is denoted $Atom(\mathcal{L}^p)$. (Textbook: Defs 2.2.1, 2.2.2, p. 23)

Normally, we say simply "formula" to mean "well-formed formula". Similarly, we may sometimes omit to write down parentheses, if it's clear where they should go. Nevertheless, a truly well-formed formula has all of the required parentheses.

Notational Conventions

We normally reserve certain kinds of symbols (often with subscripts) to denote certain classes of objects.

$p, q, r, \ldots$	(lowercase Roman letters)	propositional symbols (or atomic formulae)
$\varphi, \psi, \eta, \ldots$	(lowercase Greek letters)	well-formed formulae (formulae for short)
$\Sigma, \Gamma, \Delta, \ldots$	(uppercase Greek letters)	sets of well-formed formulae
$I, J, \ldots$	(uppercase Roman letters)	interpretations (i.e., truth assignments)

2 Interpretations and Models

The truth or falsity of a proposition depends on how we interpret it. Let P be a set of propositional symbols—the set of basic facts.

Definition 3 (Interpretations)
An *interpretation* is a function $I : P \rightarrow \{0, 1\}$ that assigns either *true* (1) or *false* (0) to every propositional symbol in P. (Textbook: 2.4, p. 33 ff.)

Definition 4 (The Satisfaction Relation)
For every interpretation I and every well-formed formula φ, either I *satisfies* φ, denoted $I \models \varphi$, or I does not satisfy φ, denoted $I \not\models \varphi$. This relation is defined by the following inductive rules.

- If φ is an atom p, then $I \models \varphi$ if and only if $I(p) = 1$.
- If φ is a negation $\neg\psi$, then $I \models \varphi$ if and only if $I \not\models \psi$;
- If φ is a conjunction $(\psi_1 \wedge \psi_2)$, then $I \models \varphi$ if and only if both $I \models \psi_1$ and $I \models \psi_2$;
- If φ is a disjunction $(\psi_1 \vee \psi_2)$, then $I \models \varphi$ if and only if either $I \models \psi_1$ or $I \models \psi_2$ (or both); and
- If φ is an implication $(\psi_1 \rightarrow \psi_2)$, then $I \models \varphi$ if and only if either $I \not\models \psi_1$ or $I \models \psi_2$ (or both).

An interpretation I such that $I \models \varphi$ is called a *model* of φ. We define $\mathsf{mod}(\varphi)$ to be the set of all models of φ, i.e., $\mathsf{mod}(\varphi) = \{\, I \mid I \models \varphi \,\}$. (Textbook: 2.2.4, 2.4.5, p. 36–37)

Lemma 5 (Relevance)
Let L be the set of all propositional symbols in $\varphi \in \mathit{WFF}$ and let I_1 and I_2 be two interpretations such that $I_1(p) = I_2(p)$ for all $p \in L$. Then $I_1 \models \varphi$ if and only if $I_2 \models \varphi$.

Definition 6 (Adequate Set of Connectives)
A set of propositional connectives S is *adequate* if for an arbitrary k-ary Boolean function f there is a well formed formula φ_f that uses only logical connectives in S and propositional symbols $p_1, \ldots, p_k$, such that

$$I \models \varphi_f \text{ iff and only if } f(I(p_1), \ldots, I(p_k)) = 1$$

for all interpretations I. (Textbook: 2.8, p. 65–6)

Theorem 7
The sets $\{\wedge, \neg\}$, $\{\vee, \neg\}$, and $\{\rightarrow, \neg\}$ are adequate sets of propositional connectives.
(Textbook: 2.8.1, 2.8.2, p. 67)

Definition 8 (Satisfiability and Validity)
A well-formed formula φ is

- *valid* (or a *tautology*) if $I \models \varphi$ for all interpretations I (i.e., if $\mathsf{mod}(\varphi)$ is the set of all interpretations);
- *satisfiable* if $I \models \varphi$ for some interpretation I (i.e., φ has a model); and
- *unsatisfiable* otherwise (i.e., φ does not have a model).

(Textbook: 2.4.4,2.4.5, p. 36–37)

Definition 9 (Sets of Formulae: Lifting)
Let $\Sigma \subseteq \mathit{WFF}$ be a set of formulae.

We define $I \models \Sigma$ *(I is a model of Σ)* if there is an interpretation I such that $I \models \varphi$ for all $\varphi \in \Sigma$.

Thus $\mathsf{mod}(\Sigma)$, the set of all models of Σ, has the property $\mathsf{mod}(\Sigma) = \bigcap_{\varphi\in\Sigma} \mathsf{mod}(\varphi)$.

We define that Σ *is satisfiable (resp. unsatisfiable)* if Σ has (resp. does not have) a model.

Definition 10 (Logical Implication and Equivalence)
Let $\Sigma \subseteq \mathit{WFF}$ and $\varphi, \psi \in \mathit{WFF}$.

We say that Σ *logically implies* (or *entails*) φ, written $\Sigma \models \varphi$, if $\mathsf{mod}(\Sigma) \subseteq \mathsf{mod}(\varphi)$.

We say that φ *is (logically) equivalent to* ψ, written $\varphi \equiv \psi$, if $\mathsf{mod}(\varphi) = \mathsf{mod}(\psi)$.

Theorem 11 (Logical Implication and Satisfiability)
Let $\Sigma \subseteq \mathit{WFF}$ and $\varphi \in \mathit{WFF}$. Then

$$\Sigma \models \varphi \text{ if and only if } \Sigma \cup \{(\neg\varphi)\} \text{ is unsatisfiable.}$$

3 Proof Systems and Their Properties

Definition 12 (Deduction Rules; Deduction Systems; Proofs)
A *(deduction) rule* is a tuple $\langle\psi_1, \ldots, \psi_k\rangle$ where $k \geq 1$ and $\psi_i \in \mathit{WFF}$ for all k. If $k = 1$, the rule is called an *axiom*; if $k > 1$, the rule is called an *inference rule*.

A *deduction system* S (for the language of well-formed formulae) is a set of deduction rules.

A *proof* (or *formal deduction*) in S is a sequence $\langle\varphi_0, \ldots, \varphi_n\rangle$, where $\varphi_i \in \mathit{WFF}$ for each i and $n \geq 1$, such that for every φ_i in the proof there is a rule $\langle\psi_1, \ldots, \psi_{k-1}, \varphi_i\rangle \in S$ with $\{\psi_1, \ldots, \psi_{k-1}\} \subseteq \{\varphi_1, \ldots, \varphi_{i-1}\}$.

When a proof $\langle\varphi_0, \ldots, \varphi_n, \varphi\rangle$ in S exists, we say that φ is a *theorem of* S, written $\vdash_S \varphi$.

Given a set $\Sigma \subseteq \mathit{WFF}$ of formulae and a formula φ, we define that φ *is deducible from* Σ in S, written $\Sigma \vdash_S \varphi$, if φ is a theorem of $S \cup \{\, \langle\psi\rangle \mid \psi \in \Sigma \,\}$. (Textbook: 4.4.1 is a particular instance of the above), p. 110)

Definition 13 (The Hilbert System)
The *Hilbert System (H)* is a deduction system for propositional logic defined by the tuples

$$\begin{array}{rl}
\text{Ax1:} & \langle\, (\varphi \to (\psi \to \varphi)) \,\rangle; \\
\text{Ax2:} & \langle\, ((\varphi \to (\psi \to \eta)) \to ((\varphi \to \psi) \to (\varphi \to \eta))) \,\rangle; \\
\text{Ax3:} & \langle\, (((\neg\varphi) \to (\neg\psi)) \to (\psi \to \varphi)) \,\rangle; \text{ and} \\
\text{MP:} & \langle\, \varphi, (\varphi \to \psi), \psi \,\rangle.
\end{array}$$

for all $\varphi, \psi, \eta \in WFF$. ("MP" stands for *modus ponens*.)

(Textbook: 4.1, using only Ax1–3 and R1 (MP), p. 109–10)

Theorem 14 (The Deduction Theorem)
$\Sigma \cup \{\varphi\} \vdash_H \psi$ if and only if $\Sigma \vdash_H (\varphi \to \psi)$.

Definition 15 (Consistency)
A set of formulae $\Sigma \subseteq WFF$ is (syntactically) *consistent* if there is a formula $\varphi \in WFF$ such that $\Sigma \nvdash_H \varphi$. Otherwise, we say that Σ is *inconsistent* (or *contradictory*). (Textbook: 5.2.2, p. 126)

Theorem 16 (Deducibility and Consistency)
$\Sigma \vdash_H \varphi$ if and only if $\Sigma \cup \{(\neg\varphi)\}$ is inconsistent.

Theorem 17 (Soundness and Completeness of the Hilbert System)
If $\Sigma \vdash_H \varphi$, then $\Sigma \models \varphi$. In words, H is *sound*.
If $\Sigma \models \varphi$, then $\Sigma \vdash_H \varphi$. In words, H is *complete*. (Textbook: 5.2.3, 5.3.7, p. 126, 131)

Definition 18 (Compactness)
$\Sigma \subseteq WFF$ is satisfiable if and only if for every finite $\Sigma_0 \subseteq \Sigma$, Σ_0 is satisfiable. (Textbook: 6.1.1, p. 147)

Definition 19 (A Conservative Extension of the Hilbert System)
A conservative extension of H to enable handling of conjunctions and disjunctions: in addition to tuples in H, include the following.

$$\begin{aligned}
\wedge\text{-introduction:} &\quad \langle(\varphi \to (\psi \to (\varphi \wedge \psi)))\rangle, \\
\wedge\text{-elimination:} &\quad \langle(((\varphi \wedge \psi) \to \varphi))\rangle, \langle(((\varphi \wedge \psi) \to \psi))\rangle, \\
\vee\text{-introduction:} &\quad \langle(\varphi \to (\varphi \vee \psi))\rangle, \langle(\psi \to (\varphi \vee \psi))\rangle, \text{ and} \\
\vee\text{-elimination:} &\quad \langle(((\varphi \to \eta) \to ((\psi \to \eta) \to ((\varphi \vee \psi) \to \eta))))\rangle.
\end{aligned}$$

(Textbook: 4.1, Ax4–9, p. 109–10)