Modal Logic

Summary of Definitions and Main Results

CS 245

1 Syntax of Modal Logic

The alphabet for *modal (propositional) logic* is the same as for propositional logic with the addition of an additional unary connectives "$\Box$" (called *necessarily* or *always*) and "$\Diamond$" (called *possibly* or *eventually*).

Definition 1 (Well formed Formulæ (wff))
Let P be a set of propositional symbols. We define the *set of formulæ of modal logic (*WFMF*)* inductively as follows:

1. $P \subseteq \mathsf{WFMF}$;
2. $\varphi \in \mathsf{WFMF}$ then $(\neg\varphi) \in \mathsf{WFMF}$, $(\Box\varphi) \in \mathsf{WFMF}$, and $(\Diamond\varphi) \in \mathsf{WFMF}$; and
3. $\varphi, \psi \in \mathsf{WFMF}$ then $(\varphi \star \psi) \in \mathsf{WFMF}$ for $\star \in \{\wedge, \vee, \rightarrow\}$;

and no other strings (formed in the alphabet of propositional logic) are elements of WFMF.

In the following we assume that $\varphi \vee \psi$ is a shorthand for $(\neg\varphi) \rightarrow \psi$, $\varphi \wedge \psi$ for $\neg(\varphi \rightarrow (\neg\psi))$, and $\Diamond\varphi$ for $\neg\Box\neg\varphi$.

2 Interpretations and Models

Definition 2 (Modal Interpretations (Kripke Structures))
Let P be a set of propositional symbols. A *modal interpretation* I (or *Kripke structure*, or *interpretation* for short) is a triple (W, R, V) where

- W is a non empty set (whose elements are called *of worlds*);
- $R \subseteq W \times W$ a binary relation over W (called the *visibility* or the *accessibility* relation); and
- $V : W \rightarrow (P \rightarrow \{0, 1\})$ a function that assigns a *propositional interpretation to each world*.

The pair (W, R) is called the *frame* of I.

Definition 3 (Satisfaction Relation)
The *satisfaction relation* $\models$ between a modal interpretation I, a world w, and a formula $\varphi \in \mathsf{WFMF}$, written $I, w \models \varphi$, is defined as follows:

- $I, w \models p$ if $V(w)(p) = 1$ for $p \in P$;
- $I, w \models (\neg\varphi)$ if $I, w \not\models \varphi$;
- $I, w \models (\Box\varphi)$ if $I, v \models \varphi$ for all $v \in W$ such that $(w, v) \in R$;
- $I, w \models (\varphi \to \psi)$ if whenever $I, w \models \varphi$ then also $I, w \models \psi$.

A pair (I, w) such that $I, w \models \varphi$ is called a *(pointed) model of* φ. We define $\mathsf{mod}(\varphi)$ to be the $\mathsf{mod}(\varphi) = \{(I, w) \mid (I, w) \models \varphi\}$.

Note: in many presentations the term *model* and *interpretation* are used as synonyms; such a terminology, however, makes defining validity, satisfiability, and logical implication cumbersome.

Definition 4 (Satisfiability and Validity)
A modal formula φ is

- *valid* if $I, w \models \varphi$ for all interpretations I and all $w \in W$ (i.e., true in all pointed models);
- *satisfiable* if $I, w \models \varphi$ for some interpretation I and $w \in W$ (i.e., has a pointed model); and
- *unsatisfiable* otherwise.

Definitions of *logical implication (*$\Sigma \models \varphi$*)* and *equivalence*, and their properties are now the same as for propositional logic.

2.1 Characterization of Frames/Correspondence Theory

Definition 5
A modal formula φ *characterizes a class of frames* $\mathcal{F}$ if

- $I, w \models \varphi$ for all $I = (W, R, V)$ and $w \in W$ where the frame $(W, R) \in \mathcal{F}$; and
- for every frame $(W, R) \notin \mathcal{F}$ there is $J = (W, R, V)$ and $w \in W$ such that $J, v \not\models \varphi$.

Modal formulae that characterize classes of frames:

$\Box p \to p$	(T)	reflexive	$\Box p \to \Diamond p$	(D)	serial
$\Box p \to \Box\Box p$	(4)	transitive	$\Box\Box p \to \Box p$		dense
$p \to \Box\Diamond p$	(B)	symmetric	$\Diamond p \to \Box p$		unique

In addition $S4$ denotes the formulae $\{K, T, 4\}$ and $S5$ (frames that are equivalence relations) denotes the three formulae $\{K, T, 4, B\}$. Here K is the formula in Ax K below.

3 Proof System K

Definition 6 (System K)
The *System K* is a deduction system for propositional modal logic defined by the tuples

Ax 1–3, MP	Hilbert system axioms and inference rules
Ax K	$\langle \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \rangle$; and
Nec	$\langle \varphi, \Box\varphi \rangle$ only applicable to φ deduced from axioms (i.e., without assumptions).

Theorem 7 (Soundness and Completeness (K and extensions))
$\vdash_K \varphi$ iff $\models \varphi$ (all frames)
$\vdash_{K+T} \varphi$ iff $\models \varphi$ in all reflexive frames
$\vdash_{S4} \varphi$ iff $\models \varphi$ in all transitive frames
$\vdash_{S5} \varphi$ iff $\models \varphi$ in all frames that are equivalence relations