First-Order Logic (Predicate Calculus)

Summary of Definitions and Main Results

CS 245

1 Syntax of First-Order Logic

Definition 1 (Alphabet of First-Order Terms and Formulæ)
The *alphabet* of the language of first-order logic consists of the following symbols.

1. Constant Symbols: $c, d, c_1, c_2, \ldots, d_1, d_2, \ldots$;
2. Function Symbols: $f, g, h, f_1, f_2, \ldots, g_1, g_2, \ldots$;
3. Variables: $x, y, z, x_1, x_2, \ldots, y_1, y_2, \ldots$;
4. Predicate (Relational) Symbols: $P, Q, P_1, P_2, \ldots, Q_1, Q_2, \ldots$;
5. Logical Connectives: $\neg, \wedge, \vee, \rightarrow$;
6. Quantifiers: $\forall$ (read "for all" or "for each") and $\exists$ (read "there exists"); and
7. Punctuation: "(", ")", "," and ".".

Each predicate symbol P and each function symbol f is associated with a natural number called its *arity*, written $\mathsf{ar}(P)$ and $\mathsf{ar}(f)$, respectively.

Predicate and function symbols with arity 1 (2, 3) are called *unary* (*binary*, *ternary*, respectively).

The constant, functional, and predicate symbols are called the *non-logical symbols* (or *parameters*).

Note: Predicate symbols with arity 0 are essentially propositional symbols, and function symbols with arity 0 are essentially constants (and thus we could omit constants in Definition 1).

Definition 2 (Terms)
Let CS be a set of constant symbols, FS a set of function symbols, and VS a set of variables. We define the *set of terms* TS inductively as follows.

1. $\mathsf{CS} \subseteq \mathsf{TS}$,
2. $\mathsf{VS} \subseteq \mathsf{TS}$, and
3. if $f \in \mathsf{FS}$ and $t_1, \ldots, t_n \in \mathsf{TS}$, then $f(t_1, \ldots, t_n) \in \mathsf{TS}$, where $n = \mathsf{ar}(f)$;

no other strings are terms.

Definition 3 (Well-Formed Formulæ (WFFs))
Let PS be a set of predicate symbols, TS a set of terms, and VS a set of variables. We define the *set of formulæ of first-order logic (*WFF*)* inductively as follows.

1. if $P \in \mathsf{PS}$ and $t_1, \ldots, t_n \in \mathsf{TS}$, where $n = \mathsf{ar}(P)$, then $P(t_1, \ldots, t_n) \in \mathsf{WFF}$;
2. if $\varphi \in \mathsf{WFF}$, then $(\neg\varphi) \in \mathsf{WFF}$;
3. if $\varphi, \psi \in \mathsf{WFF}$, then $(\varphi \star \psi) \in \mathsf{WFF}$ for each $\star \in \{\wedge, \vee, \rightarrow\}$;
4. if $x \in \mathsf{VS}$ and $\varphi \in \mathsf{WFF}$, then $(\forall x.\varphi) \in \mathsf{WFF}$ and $(\exists x.\varphi) \in \mathsf{WFF}$;

and no other strings are elements of WFF.

In the following we assume that $\varphi \vee \psi$ is a shorthand for $(\neg\varphi) \rightarrow \psi$, $\varphi \wedge \psi$ for $\neg(\varphi \rightarrow (\neg\psi))$, and $(\exists x.\varphi)$ for $\neg(\forall x.(\neg\varphi))$.

Definition 4 (Free and Bound Variables)
Let $\varphi \in \mathsf{WFF}$. We define the *set of free variables of* φ, denoted $\mathsf{FV}(\varphi)$, as follows.

1. If $\varphi = P(t_1, \ldots, t_{\mathsf{ar}(P)})$, then $\mathsf{FV}(\varphi) = \{x \mid x$ appears in t_i for some $0 < i \leq \mathsf{ar}(P)\}$;
2. if $\varphi = (\neg\psi)$, then $\mathsf{FV}(\varphi) = \mathsf{FV}(\psi)$;
3. if $\varphi = (\psi \rightarrow \eta)$, then $\mathsf{FV}(\varphi) = \mathsf{FV}(\psi) \cup \mathsf{FV}(\eta)$; and
4. if $\varphi = (\forall x.\psi)$, then $\mathsf{FV}(\varphi) = \mathsf{FV}(\psi) - \{x\}$.

Variables in the set $\mathsf{FV}(\varphi)$ are called *free (in* φ*)*; other variables that occur in φ are called *bound (in* φ*)*.

For a set of formulæ Σ, we define $\mathsf{FV}(\Sigma) = \bigcup_{\varphi \in \Sigma} \mathsf{FV}(\varphi)$.

Definition 5 (Closed Formulæ (Sentences))
A first-order formula $\varphi \in \mathsf{WFF}$ is *closed* (or a *sentence*) iff $\mathsf{FV}(\varphi) = \emptyset$.

2 Interpretations and Models

Definition 6 (First-Order Interpretations (Structures))
A *first-order interpretation* (or *structure*) I is a pair $(D, (.)^I)$ where

- D is a non-empty set, called the *domain* (or *universe*) and
- $(.)^I$ is an *interpretation function* that maps
  - constant symbols $c \in \mathsf{CS}$ to individuals $(c)^I \in D$,
  - function symbols $f \in \mathsf{FS}$ to functions $(f)^I : D^{\mathsf{ar}(f)} \rightarrow D$, and
  - predicate symbols $P \in \mathsf{PS}$ to relations $(P)^I \subseteq D^{\mathsf{ar}(P)}$.

For a fixed selection L of constant, function, and predicate symbols, called the *vocabulary* (or the signature, or—slightly abusing terminology—the language), we define *L-structures* to be those interpretations restricted to symbols in L.

Definition 7 (Valuation)
Let D be a domain and VS a set of variables. A *valuation* (or *assignment*) is a mapping $\theta : \mathsf{VS} \rightarrow D$.

For a valuation θ, a variable x and a term v, the valuation $\theta[x/v]$ is defined by

$$\theta[x/v](y) = \begin{cases} v & \text{if } x = y, \\ \theta(y) & \text{otherwise.} \end{cases}$$

Definition 8 (Meaning of Terms)
Let I be a first-order interpretation and θ a valuation. For a term $t \in \mathsf{TS}$, we define the *interpretation* of t, denoted $(t)^{I,\theta}$, as follows.

1. $(c)^{I,\theta} = (c)^I$ for $t \in \mathsf{CS}$ (i.e., t is a constant),
2. $(x)^{I,\theta} = \theta(x)$ for $t \in \mathsf{VS}$ (i.e., t is a variable), and
3. $(f(t_1, \ldots, t_{\mathsf{ar}(f)}))^{I,\theta} = (f)^I((t_1)^{I,\theta}, \ldots, (t_{\mathsf{ar}(f)})^{I,\theta})$ otherwise (i.e., for t a functional term).

Definition 9 (Satisfaction Relation)
The *satisfaction relation* $\models$ between a first-order interpretation I, a valuation θ, and a formula $\varphi \in \mathsf{WFF}$, written $I, \theta \models \varphi$, is defined as follows.

- $I, \theta \models P(t_1, \ldots, t_{\mathsf{ar}(P)})$ iff $((t_1)^{I,\theta}, \ldots, (t_{\mathsf{ar}(P)})^{I,\theta}) \in (P)^I$ for $P \in \mathsf{PS}$;
- $I, \theta \models (\neg\varphi)$ iff $I, \theta \not\models \varphi$;
- $I, \theta \models (\varphi \to \psi)$ iff whenever $I, \theta \models \varphi$ then also $I, \theta \models \psi$.
- $I, \theta \models (\forall x.\varphi)$ iff $I, \theta[x/v] \models \varphi$ for all $v \in D$.

A pair (I, θ) such that $I, \theta \models \varphi$ is called a *(pointed) model of* φ. We define $\mathsf{mod}(\varphi)$ to be the set of models of φ: $\mathsf{mod}(\varphi) = \{ (I, \theta) \mid (I, \theta) \models \varphi \}$.

Note: in many presentations the term *model* is used for an *interpretation* which makes a formula true (for all valuations); such a terminology, however, makes defining validity, satisfiability, and logical implication cumbersome. The two uses coincide for *sentences*—a particular case of the following lemma.

Lemma 10 (Relevance)
Let L be the set of all non-logical symbols in $\varphi \in \mathsf{WFF}$, and let

1. I_1 and I_2 be two interpretations such that $I_1(s) = I_2(s)$ for all $s \in L$ and
2. θ_1 and θ_2 be two valuations such that $\theta_1(x) = \theta_2(x)$ for all $x \in \mathsf{FV}(\varphi)$.

Then $I_1, \theta_1 \models \varphi$ if and only if $I_2, \theta_2 \models \varphi$.

The above lemma allows us to consider only L-structures for an appropriately chosen set L of non-logical parameters.

Definition 11 (Satisfiability and Validity)
A modal formula φ is

- *valid* iff $I, \theta \models \varphi$ for all interpretations I and all valuations θ (i.e., true in all models),
- *satisfiable* iff $I, \theta \models \varphi$ for some interpretation I and some valuation θ (i.e., has a model), and
- *unsatisfiable* otherwise.

Definitions of *logical implication ($\Sigma \models \varphi$)* and *equivalence* and their properties are now the same as for propositional logic.

3 Hilbert Proof System

Definition 12 (Substitution)
A *(syntactic) substitution* of a term t for a variable x, written $(.)\,^x_t : \mathsf{WFF} \to \mathsf{WFF}$, is a mapping of terms to terms and formulæ to formulæ, customarily written as a post-fix operator (i.e., $\varphi\,^x_t$ stands for applying the substitution $(.)\,^x_t$ to φ). It is defined as follows.

1. For a term t_1, $(t_1)\,^x_t$ is t_1 with each occurrence of the variable x replaced by the term t.
2. For $\varphi = P(t_1, \ldots, t_{\mathsf{ar}(P)})$, $(\varphi)\,^x_t = P\big((t_1)\,^x_t, \ldots, (t_{\mathsf{ar}(P)})\,^x_t\big)$.
3. For $\varphi = (\neg\psi)$, $(\varphi)\,^x_t = (\neg(\psi)\,^x_t)$;
4. For $\varphi = (\psi \to \eta)$, $(\varphi)\,^x_t = ((\psi)\,^x_t \to (\eta)\,^x_t)$, and
5. for $\varphi = (\forall y.\psi)$, there are two cases:
   - if x is y, then $(\varphi)\,^x_t = \varphi = (\forall y.\psi)$, and
   - otherwise, then $(\varphi)\,^x_t = (\forall z.(\psi\,^y_z)\,^x_t)$, where z is any variable that is not free in t or in φ.

Note: in the last case above, the additional substitution $(.)\,^y_z$ (i.e., renaming the variable y to z in ψ) is needed in order to avoid an accidental *capture of a variable* by the quantifier (i.e., capture of any y that is possibly free in t).

Lemma 13 (Substitution)
Let I be an interpretation, θ a valuation, t a term, x a variable. Then $I, \theta \models \varphi\,^x_t$ if and only if $I, \theta[x/(t)^{I,\theta}] \models \varphi$ for all $\varphi \in \mathsf{WFF}$.

Definition 14 (Hibert System)
The *First-Order Hilbert System* is a deduction system for first-order logic defined by the tuples generated by the following schemes:

Ax1	$\langle \forall^*(\varphi \to (\psi \to \varphi)) \rangle$;	
Ax2	$\langle \forall^*((\varphi \to (\psi \to \eta)) \to ((\varphi \to \psi) \to (\varphi \to \eta))) \rangle$;	
Ax3	$\langle \forall^*(((\neg\varphi) \to (\neg\psi)) \to (\psi \to \varphi)) \rangle$;	
Ax4	$\langle \forall^*(\forall x.(\varphi \to \psi)) \to ((\forall x.\varphi) \to (\forall x.\psi)) \rangle$;	
Ax5	$\langle \forall^*(\forall x.\varphi) \to \varphi\,^x_t \rangle$	for $t \in \mathsf{TS}$ a term;
Ax6	$\langle \forall^*(\varphi \to \forall x.\varphi) \rangle$	for $x \notin \mathsf{FV}(\varphi)$; and
MP	$\langle \varphi, (\varphi \to \psi), \psi \rangle$.	

where $\forall^*$ is a finite sequence of universal quantifiers (e.g., $\forall x_1.\forall y.\forall x$).

Theorem 15
Hilbert system is sound ($\Sigma \vdash \varphi$ then $\Sigma \models \varphi$) and complete ($\Sigma \models \varphi$ then $\Sigma \vdash \varphi$).

Lemma 16 (Generalization)
Let $\Sigma \vdash \varphi$ and $x \notin \mathsf{FV}(\Sigma)$. Then $\Sigma \vdash \forall x.\varphi$.

4 Equality

Definition 17 (Axioms of Equality)
Let $\approx$ be a binary predicate symbol (written in infix). We define the *First-Order Axioms of Equality* as follows:

EqId	$\langle \forall x.(x \approx x) \rangle$;
EqCong	$\langle \forall x.\forall y.(x \approx y) \to (\varphi\,{}^{z}_{x} \to \varphi\,{}^{z}_{y}) \rangle$;

Theorem 18 (Gödel 1930)
Hilbert system *with (axiomatized) equality* is sound ($\Sigma \vdash \varphi$ then $\Sigma \models \varphi$) and complete ($\Sigma \models \varphi$ then $\Sigma \vdash \varphi$) with respect to first-order logic *with (true) equality*.

5 Definability

Definition 19 (Definability in an Interpretation)
Let $I = (D, (.)^I)$ be a first-order interpretation and φ a first-order formula. A set S of k-tuples over D, $S \subseteq D^k$, is *defined by the formula* φ if $S = \{(\theta(x_1), \ldots, \theta(x_k)) \mid I, \theta \models \varphi\}$.

A set S is *definable in first-order logic* if it is defined by some first-order formula φ.

Definition 20 (Definability of a Set of Interpretations)
Let Σ be a set of first-order sentences and $\mathcal{K}$ a set of interpretations. We say that Σ *defines* $\mathcal{K}$ if

$$I \in \mathcal{K} \text{ if and only if } I \models \Sigma.$$

A set $\mathcal{K}$ is *(strongly) definable* if it is defined by a (finite) set of first-order formulæ Σ.

Theorem 21 (Compactness)
A set Σ is consistent if and only if every finite $\Sigma_0 \subseteq \Sigma$ is consistent.

Corollary 22
The class of interpretations with finite domain is not definable in first-order logic.