Dihedral Escherization

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Abstract

"Escherization" [9] is a process that finds an Escher-like tiling of the plane from tiles that resemble a user-supplied goal shape. We show how the original Escherization algorithm can be adapted to the *dihedral* case, producing tilings with two distinct shapes. We use a form of the adapted algorithm to create drawings in the style of Escher's print *Sky and Water*. Finally, we develop an Escherization algorithm for the very different case of Penrose's aperiodic tilings.

Key words: Tilings, tessellations, Escher, aperiodic, Penrose tilings

1 Introduction

The Dutch artist M.C. Escher had a singular gift for incorporating mathematics into art. His work continues to delight and fascinate, a mix of paradox and harmony, of whimsy and order. In particular, he produced a large collection of ingenious tessellations [14], made from motifs resembling people, animals, and fantasy creatures. The search for interlocking forms was for Escher a lifelong pursuit and source of both frustration and inspiration.

In the SIGGRAPH 2000 conference, Kaplan and Salesin presented "Escherization" [9], a method for automatically discovering Escher-like tessellations with tiles that resemble arbitrary user-supplied shapes. Given a goal shape S, their system uses continuous optimization to hunt through a parameterized space of tilings in search of a tile shape T that best approximates S. The quality of the approximation is determined via an efficient polygon comparison metric [1].

The Escherization algorithm they describe is able to find Escher-like tilings that are *monohedral*, *i.e.*, made up of copies of a single motif. Escher also created many tessellations featuring two or (less frequently) more motifs. *Dihedral* (two-motif) tessellations are particularly important in his work. Some of his most famous prints (for example, *Sky and Water*, *Verbum*, and *Metamorphosis II*) make use of one or more dihedral tilings [2]. Furthermore, the use of multiple motifs agrees with Escher's predisposition to imbue his work with narrative structure. The interplay between different motifs in a single design provides an opportunity for contrasts, for harmony or discord, for interaction and drama.

In this paper we extend the Escherization algorithm of Kaplan and Salesin to the dihedral case. We may begin by analogy with their work, formulating a *dihedral Escherization problem*:

> **Problem** ("DIHEDRAL ESCHERIZATION"): Given closed plane figures S_1 and S_2 (the "goal shapes"), find new closed figures T_1 and T_2 such that:
>
> 1. T_1 and T_2 are as close as possible to S_1 and S_2, respectively; and
> 2. T_1 and T_2 admit a dihedral tiling of the plane.

We present a solution to the dihedral Escherization problem as an extension to the algorithm given by Kaplan and Salesin. As discussed in Section 3, we augment the representation of a tile shape with a curve that splits it into two pieces. We optimize over this new configuration space using an objective function that compares the two pieces with two goal shapes. In Section 4, we show how a restricted version of the general splitting process can produce what Dress calls *Heaven and Hell patterns* [5]. In Section 5, we develop a tool that uses Heaven and Hell patterns to generate drawings in the style of Escher's print *Sky and Water*. In Section 6, we present an alternative formulation of dihedral Escherization that works on Penrose's aperiodic tile sets $P2$ and $P3$. We conclude in Section 7 with a discussion of directions for future work.

2 Mathematical background

In this section we briefly define some of the concepts of tiling theory that are relevant in this work. A definitive treatment of the subject is given by Grünbaum and Shephard [7]; Kaplan and Salesin [9] provide an overview.

A *tiling of the plane* is a countable collection of tiles that cover the plane without any gaps or overlaps. A tiling is k-*hedral* when every tile is congruent to one of k different shapes, called *prototiles*. The cases $k = 1$ and $k = 2$ are referred to as *monohedral* and *dihedral* respectively.

A *tiling vertex* is a point where three or more tiles meet. A tile's boundary can be subdivided into a collection of tiling vertices connected by arcs called *tiling edges*.

The *symmetries* of a tiling are rigid motions of the plane that map the tiling onto itself. A tiling with translational symmetries in two non-parallel directions is called *periodic*. There are many non-periodic tilings, but often the prototiles of such tilings can also be used to construct periodic tilings. A more remarkable situation occurs when every tiling that can be constructed from a set of shapes is non-periodic; such sets are called *aperiodic*, as are the tilings that can be built from them.

For any two congruent tiles in a tiling, there will be a rigid motion of the plane that maps the first tile onto the second. If this motion also maps the entire tiling to itself then the two tiles are said to be *transitively equivalent*. Transitive equivalence partitions the tiles of a tiling into equivalence classes called *transitivity classes*. A tiling with exactly one transitivity class is called *isohedral*; more generally, a k-*isohedral* tiling is one with k transitivity classes. Differently-shaped tiles will necessarily belong to different classes, and so a k-hedral tiling will be at least k-isohedral, though it may possibly have more transitivity classes.

The combinatorial properties of every isohedral tiling can be summarized using a compact string called an *incidence symbol* [7]. By writing down all possible incidence symbols and eliminating those that cannot correspond to legal tilings, Grünbaum and Shephard showed that the isohedral tilings can be partitioned into precisely 93 combinatorial types labeled $\mathrm{IH1}, \ldots, \mathrm{IH93}$. Each type corresponds to a different way that a tile can be surrounded by its neighbours.

3 Split isohedral Escherization

Our search for a useful space of dihedral tilings begins with Escher himself. A meticulous note-taker, he carefully documented his exploration of two-motif systems [13, 14]. In every case, he starts with one of his monohedral systems and draws a path through the prototile to break it into two shapes. When that division is copied to all other tiles, the result is a dihedral tiling.

We can apply a similar process to the isohedral tilings. We augment the description of an isohedral prototile with a "splitting path," a path that starts and ends on the prototile's boundary. A splitting path naturally subdivides a prototile into two shapes T_1 and T_2. When every tile in an isohedral tiling is subdivided by the splitting path, the result is a "split isohedral tiling," a dihedral tiling with prototiles T_1 and T_2. Every such tiling will be 2-isohedral, though there also exist 2-isohedral tilings that cannot be constructed using a splitting path (see Section 7).

In our implementation, the splitting path is stored as a piecewise linear path in a local coordinate system. The isohedral prototile's boundary is parameterized by arclength and the start and end positions of the path are recorded using two real values between 0 and 1. Once coordinates for the start and end positions are determined, the splitting path can be transformed into place.

The splitting process can be applied to prototiles from any of the 93 isohedral types. As in the case of monohedral Escherization, the choice of isohedral type is discrete and cannot be made within the framework of a continuous optimization. Following Kaplan and Salesin, we run multiple per-tiling-type optimizations in parallel, gradually winnowing down the pool of candidates until only the most successful type remains.

The monohedral Escherization algorithm searched a configuration space where each tuple of floating-point values encoded the shape of an isohedral prototile. Some of the values controlled the positions of the tiling vertices; the rest controlled a non-redundant description of the edge shapes. To handle the split isohedral case, we enlarge the search space to include parameters for the position and shape of the splitting path. At each step in the optimization, we construct the split isohedral prototile, extract shapes T_1 and T_2, and compare them with user-supplied goal shapes S_1 and S_2 using the metric of Arkin *et al.* [1]. The two comparisons yield two non-negative real numbers d_1 and d_2; we use $\max(d_1, d_2)$ as an objective function for dihedral Escherization, forcing both tile shapes to resemble their respective goal shapes as closely as possible. As in the monohedral algorithm, we periodically subdivide the splitting path along with the edge shapes, giving the algorithm a chance to pick up finer details in the goal shapes.

Let S_1' and S_2' denote reflections of goal shapes S_1 and S_2. To find the best split isohedral tiling corresponding to the goal shapes, two instances of the Escherization algorithm are required: one with S_1 and S_2, and one with S_1 and S_2' (or S_1' and S_2). Although the shape comparison metric is insensitive to translation and rotation, it does distinguish between a shape and its reflection. It might happen that S_1 and S_2 interact more favourably if one is reflected.[1]

The split isohedral Escherization process is illustrated in Figure 1. Figure 7 shows some results obtained using this process. One might guess that because of the need to match two goal shapes simultaneously, dihedral Escherization would have a lower success rate than monohedral Escherization. We have found that the additional degrees of freedom offered by the splitting path help to compensate for the added complexity of the problem, and that the dihedral and monohedral optimizations have comparable

[1] Note that only the relative parity matters here; the flexibility of the isohedral tilings guarantees that the case (S_1, S_2) is equivalent to (S_1', S_2'), and that (S_1', S_2) is equivalent to (S_1, S_2').

success rates. Speeds are comparable as well; the dihedral optimizer usually converges in 10–20 minutes. An interactive viewer also allows the user to watch the optimization in progress and abort it if no promising solutions seem likely.

4 Heaven and Hell Escherization

Some of Escher's dihedral tilings, such as *Heaven and Hell* [14], have additional geometric structure. Each tile can be given one of two colours so that adjacent tiles never share a colour. Furthermore, each colour is the exclusive domain of one of the two prototiles; in Heaven and Hell, every angel is white and every devil is black. This sort of colouring is possible when every tiling vertex is surrounded by an alternating sequence of A and B tiles, or equivalently, when every A tile shares edges only with B tiles (and vice versa).

Aesthetically, such tilings are particularly effective because each transitivity class of tiles plays the role of ground to the other's figure: the B tiles exactly fill the negative space created by the A tiles. Moreover, the fact that the colours can be unambiguously associated with tile shapes allows them to become part of the personalities of those shapes, as in the white angels and black devils of *Heaven and Hell*. Escher used this particular space of tilings to produce some of his best-known prints.

The class of 2-isohedral tilings with this additional structure were enumerated by Dress [5], who dubbed them "Heaven and Hell patterns." Based on an analysis using Delaney symbols [4], he classified the Heaven and Hell patterns into 37 distinct types.

Of the types enumerated by Dress, 29 can be expressed as specialized versions of split isohedral tilings. The additional structure comes from a careful choice of locations for the endpoints of the splitting path. Dress's classification shows that an endpoint will always be either one of the tiling vertices of the underlying isohedral prototile, or the midpoint of one of its tiling edges. If the isohedral prototile has n tiling vertices, we can enumerate this set of locations as $L = \{1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots, n, n+\frac{1}{2}\}$, where a whole number k refers to a tiling vertex and $k+\frac{1}{2}$ refers to the midpoint of the edge from k to $k+1$. The numbering of the tiling vertices can be taken from the order in which they appear in the enumeration by Grünbaum and Shephard [7]. Each of the 29 types based on splitting can then be given the notation $(\mathrm{IH}m; a, b)$, where $\mathrm{IH}m$ denotes one of the isohedral types, and where a, b are members of L.

We represent a prototile for the Heaven and Hell tiling $(\mathrm{IH}m; a, b)$ by starting with the representation for the split isohedral prototile of the same type, and fixing the endpoints of the splitting path according to the locations a and b. Once the degrees of freedom controlling the endpoints are removed from the configuration space, the remainder of the dihedral Escherization algorithm can be applied as is.

In Dress's paper, some of the types of Heaven and Hell patterns can be seen as subsumed under other types, in the sense that a type with symmetric prototiles is merely a special case of an asymmetric parent. Dress's explicit ordering makes it easy to recognize that the 29 Heaven and Hell tiling types representable as split isohedral tilings can be summarized using twelve *fundamental* types with asymmetric prototiles. Using the notation given above, the twelve types are as follows:

$$\begin{gathered}(\mathrm{IH}1;1,4),\ (\mathrm{IH}2;2,5),\ (\mathrm{IH}3;2,5),\\ (\mathrm{IH}5;1,4),\ (\mathrm{IH}27;1\tfrac{1}{2},4),\ (\mathrm{IH}31;1,3),\\ (\mathrm{IH}33;1,3),\ (\mathrm{IH}41;1,3),\ (\mathrm{IH}43;1,3),\\ (\mathrm{IH}47;2\tfrac{1}{2},4\tfrac{1}{2}),\ (\mathrm{IH}52;1,3),\ (\mathrm{IH}55;2,4)\end{gathered}$$

Eight remaining types in Dress's classification are not accounted for by split isohedral tilings. These tilings have different proportions of A and B tiles. We do not consider Escherization over such tilings here, although we mention them again in Section 7.

Figure 3 gives an example of Heaven and Hell Escherization.

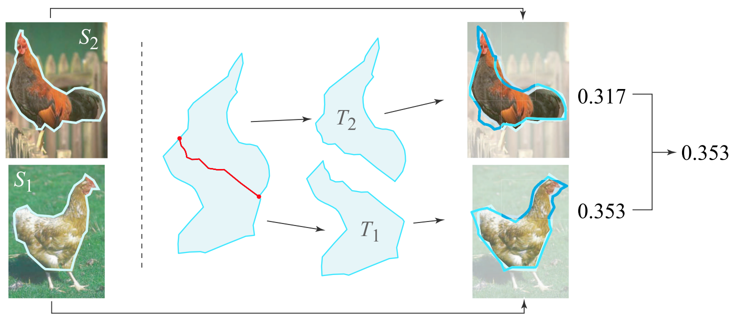
5 Sky and Water designs

Escher's print *Sky and Water* is a very special application of Heaven and Hell patterns. What starts out in the center of the print as a dihedral tiling of stylized fish and birds evolves towards the top and bottom of the print into realistic figures: birds above and fish below. Escher used this device in many prints and sometimes multiple times in a single print (as in *Verbum* and *Metamorphosis II*).

It is critical that the central tiling where the birds and fish meet be a Heaven and Hell tiling. The stylized birds evolve into the background for the realistic fish (and vice versa), and so the tiling needs to be colourable with one colour for each tile shape.

Escherization is especially well suited to the creation of Sky and Water designs because the realistic goal shapes are already part of the process that leads to the stylized tile shapes. To turn a Heaven and Hell tiling into a Sky and Water design, it suffices to gradually blend the tile shape into the goal shape as tiles are placed successively farther from a given "interface line."

We extended the basic Heaven and Hell Escherization algorithm with a suite of interactive tools for constructing Sky and Water designs. One tool lets the user specify an interface line and a set of tiles to draw. Another tool lets the user add decorations to tiles with monochromatic vector-based strokes. Each stroke is a sequence of Bézier



Funky Chickens (IH51)

Figure 1: A summary of our process for split isohedral Escherization. On the left, two goal shapes S_1 and S_2 are traced from images. Next, the isohedral tile and splitting path are shown at a late stage in the optimization. The quality of this configuration is judged by breaking the tile into two shapes T_1 and T_2, which are then compared with S_1 and S_2. The optimization attempts to minimize the value at the right, the maximum of the two comparisons. A finished example based on these goal shapes appears on the right.

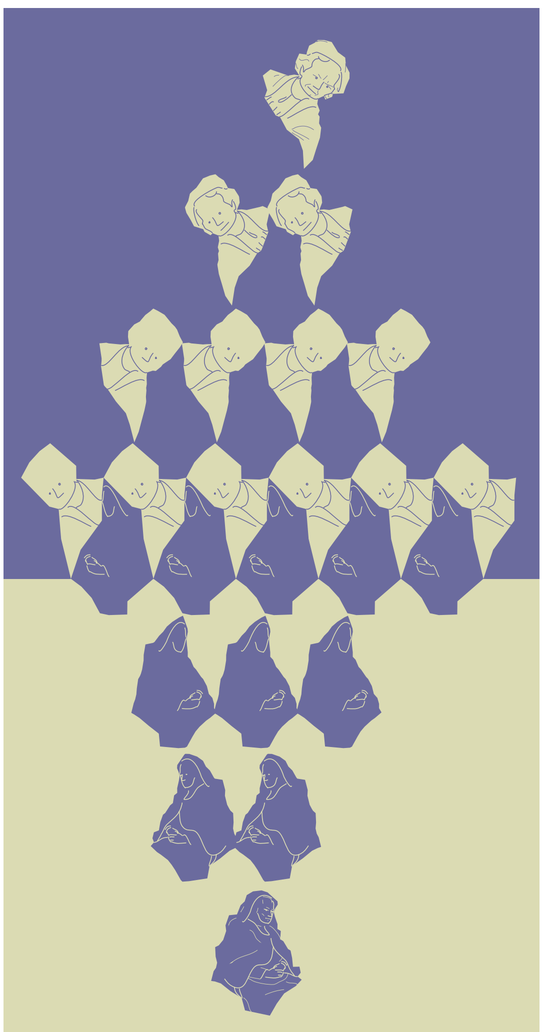
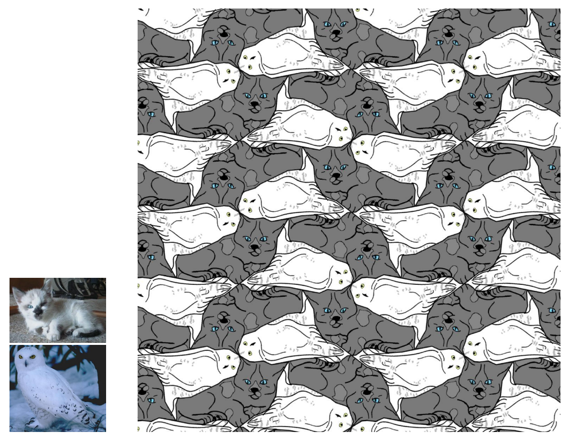
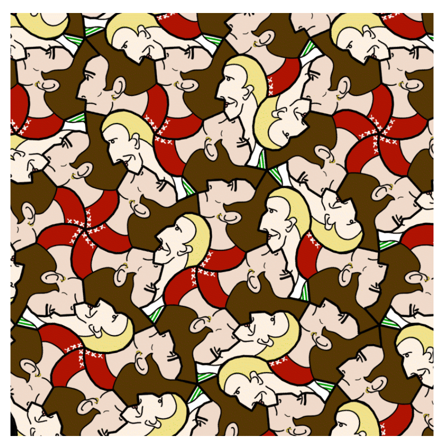


Figure 2: An example of a Sky and Water design, based on the goal shapes of Figure 7d.



The Owl and the Pussycat $(\mathrm{IH}27; 1\frac{1}{2}, 4)$

Figure 3: An example of Heaven and Hell Escherization.



The Pentalateral Commission (P3)

Figure 4: An Escher-like design created by hand, based on Penrose tile set $P3$. The tile shapes were discovered after a few minutes of interactive exploration, and decorated using Adobe Illustrator.

curves with user-specified widths; the curves are fit to the user's drawing gestures using the method of Schneider [15]. Additionally, each stroke is given a "priority" that determines how far from the interface line the tile must be before the stroke is drawn. This approach allows for a prioritized sequence of strokes ordered by their importance in expressing a stylized version of the goal shape.

Finally, a renderer assembles the final drawing, taking the output of the other two tools as input, together with colours for the A and B tile shapes. For every tile, the renderer interpolates that tile with its corresponding goal shape by an amount determined from its distance to the interface line. The interpolation is carried out so that any tiles that touch or cross the line are set to the tile shape, and maximally distant tiles are set to the goal shape. The A tiles are then drawn over a solid background of the B tile colour, and vice versa. Finally, the strokes are warped into place and drawn if they have sufficiently high priorities. Figure 2 shows an example of an Escherized Sky and Water design.

6 Escherization using Penrose tiles

The most widely known aperiodic tilings are those discovered by Penrose [7, 12]. In particular, he demonstrated two tile sets that yield dihedral aperiodic tilings: $P2$, made up of kites and darts, and $P3$, made up of thin and thick rhombs. Like the Mandelbrot set, the Penrose tilings are ambassadors of mathematical beauty to a general audience.

Unfortunately, Escher did not live to see the development of Penrose tilings, and so we can only imagine what sorts of creatures he might have discovered in them. Penrose himself, who corresponded regularly with Escher, expresses his regret at the missed opportunity [11]. He also gives an example of what Escher might have drawn: a modification of $P2$ where the kites and darts have been turned into chickens. Other artists have also created designs based on Penrose tiles [6].

In order to enforce aperiodicity in the tile sets $P2$ and $P3$, the tiles must be augmented with matching conditions that determine the legal ways one tile may be placed next to another. One important way to express these matching conditions is to deform tile edges so that tiles only fit together in prescribed ways. Grünbaum and Shephard give such geometric matching conditions for $P2$ and $P3$ [7]. In both cases, the matching conditions are boiled down to two non-congruent paths and the way they are arranged around the two tiles in the set.

The geometric matching conditions suggest an Escherization algorithm for Penrose tiles. The tiling vertices remain fixed, and the optimization operates on the degrees

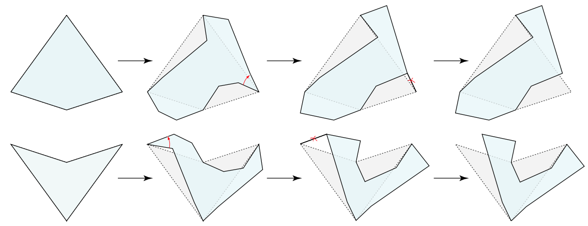


Figure 5: An illustration of how a tiling vertex parameterization can be derived for the Penrose kite and dart. The original edges are modified using Grünbaum and Shephard's edge matching conditions [7]. When adjacent edges coincide, they are removed, displacing the tiling vertices between the edges. The kite and dart each have one unconstrained tiling vertex. The others are all implied by the original matching conditions.

of freedom in the two fundamental edge shapes. These edge shapes are assembled into two tile shapes that are then compared against two goal shapes as usual.

However, this interpretation of the possible shapes of Penrose tiles is limited, as can be seen from Grünbaum and Shephard's reproduction of Penrose's aperiodic chickens [7]. They superimpose the chickens on top of the corresponding unmodified tiling. The registration of these two tilings reveals that the chickens have tiling vertices that are different from those of the original tiling! There are evidently additional degrees of freedom to the Penrose tilings that must be explored and exploited if we are to extend the reach of aperiodic Escherization.

By experimenting with the geometric matching conditions, we have discovered an extended set of points that can be parameterized like the tiling vertices of an isohedral tiling. We call these points the prototile's *quasivertices*. The quasivertices include all the points on a prototile's boundary that act as tiling vertices somewhere in a Penrose tiling, and some additional points that are forced into existence by them.[2]

Figure 5 shows how the ordinary geometric matching conditions yield a new set of parameterizable points for the kite and dart. The fundamental edge shapes are modified so that they partially overlap. The overlapping regions can then be excised from the tiles, producing new tiles that no longer share all of the tiling vertices of the original kite and dart. This process necessarily introduces other vertices into the shapes of the two tiles. Note that a tiling produced from these excised tiles will be visu-

[2] In an isohedral tiling, transitivity guarantees a unique configuration of tiling vertices around every copy of the prototile. No such guarantee exists for $P2$ or $P3$, and we must consider points that are tiling vertices on some instances of a prototile but not others.

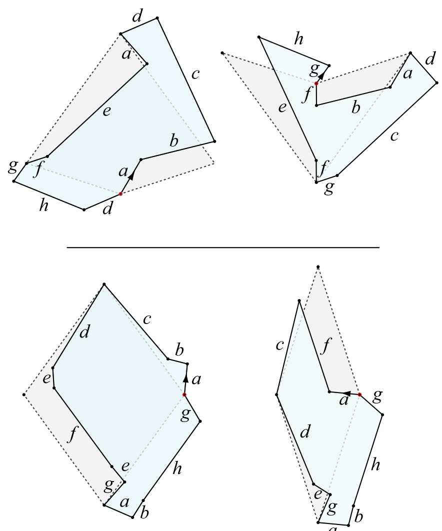


Figure 6: Extended matching rules for Penrose tiling sets $P2$ (above) and $P3$ (below). The edge labels are not related between the two sets. Matching is enforced by identifying pairs of interlocking edges. The interlocking pairs are (a, d), (b, h), (c, e), and (f, g) for $P2$, and (a, g), (b, e), (c, d), and (f, h) for $P3$.

ally indistinguishable from one produced using equivalent tiles with degeneracies. However, they behave differently from the point of view of the shape metric. Our new parameterization of the excised tiles is compatible with the shape metric.

From Figure 5, we conclude that the quasivertices of the kite and dart can be parameterized using four real-valued parameters, determined by the positions of the tiling vertices created at the tips of the two excised regions. Similarly, four parameters suffice to parameterize the Penrose rhombs. Once the free parameters are understood, we can derive explicit formulae for the positions of the quasivertices. See the thesis by Kaplan [8] for details.

Like tiling vertices, quasivertices partition tile boundaries into edges that behave like tiling edges. We must derive new matching rules so that congruent edges are controlled by a single set of parameters within the optimization. By analogy with the incidence symbols used for isohedral tilings, edge shapes can be specified by labeling edges around each tile and indicating adjacency rules for the labels. Figure 6 shows a set of labels and adjacency rules for $P2$ and $P3$.

The edge shapes, combined with the four parameters controlling the tiling vertices, yield a configuration space suitable for Penrose Escherization. Note that this parameterization cannot in general represent both a particular pair of tile shapes and the reflections of those shapes, and that in each of the Penrose sets $P2$ and $P3$ the two prototiles are fundamentally different shapes. For these reasons, given two goal shapes S_1 and S_2 and their reflections S_1' and S_2', we must optimize for the eight combinations (S_1, S_2), (S_1', S_2), (S_1, S_2'), (S_1', S_2'), (S_2, S_1), (S_2', S_1), (S_2, S_1'), and (S_2', S_1').

Rendered designs based on Penrose tilings are given in Figure 8. In general, it is much more difficult to discover satisfactory Escherizations using Penrose tilings. The range of possible tile shapes is limited and peculiar, always having many sharp angles. More obviously, there are fewer "tiling types" than in the split isohedral case; we no longer have the luxury of hunting over many different types for one that happens to be particularly well suited to a given pair of goal shapes. The results are therefore less successful than in the isohedral and 2-isohedral cases, but interesting nevertheless for their connection to the interaction (both mathematical and personal) between Escher and Penrose. On the other hand, an interactive editor for Penrose tiles still allows profitable forward exploration of the space of tilings. Figure 4 shows an example of a tiling that was not Escherized but developed from scratch in a few minutes and decorated in a cartoon style.

7 Discussion and future work

The first dihedral Escherization technique presented here is a natural extension of the isohedral method of Kaplan and Salesin — we subdivide isohedral prototiles with a splitting path. By delving deeper into the tiling theory literature, we can restrict our technique to Dress's Heaven and Hell patterns, and then use those patterns to create Sky and Water designs. We also take a fresh look at Escherization in the context of the aperiodic Penrose tilings. We close this paper by briefly discussing the issues that arise with our technique, and suggesting ideas for future work.

The split isohedral method can be used to construct those 2-isohedral tilings for which the two prototiles occur in equal amounts. All of Escher's two-motif systems have this property, and so we consider the method satisfactory for reproducing his work. However, there are also many 2-isohedral tilings with different relative amounts of the two prototiles. To understand these additional tiling types, we must take a closer look at the mathematics behind 2-isohedral tilings.

Delgado-Friedrichs *et al.* carry out a complete enumeration of the 2-isohedral tilings [3]. They prove a general result that every $(k+1)$-isohedral tiling can be constructed from a k-isohedral tiling through a combination of two operations: SPLIT and GLUE. The SPLIT operation is identical to our use of a splitting path. They show that when the prototiles of a $(k+1)$-isohedral tiling are

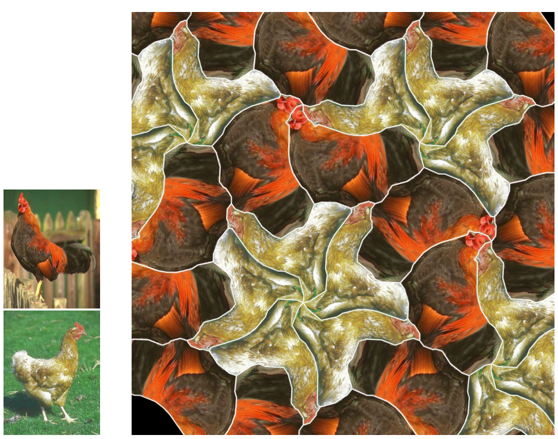
(a) *Strange 'Tractors* (IH28)

(b) *Gödel, Bach (Braided): an Eternal Escherization* (IH2)

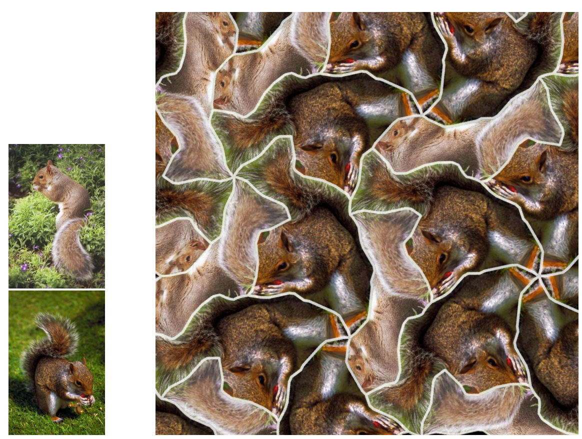
(c) *Pen/Rose Tiling* (IH1)

(d) *Rembrandt and Mrs. van Rijn* (IH1)

Figure 7: Examples of dihedral Escherization using the split isohedral tile method. Many source images appear courtesy of `FreeFoto.com`.



(a) *Busby Berkeley Chickens* (P2)



(b) *A Walk in the Park* (P3)

Figure 8: Examples of dihedral Escherization based on parameterizations of Penrose tile sets $P2$ *and* $P3$.

asymmetric, they can be derived from the prototiles of a k-isohedral tiling via the application of SPLIT to one prototile. The GLUE operation erases the edge between two adjacent tiles, producing tilings with symmetric prototiles. It is through use of GLUE that we can construct 2-isohedral tilings with different relative amounts of the two prototiles.

Because our technique is based only on the SPLIT operation, it does not immediately generalize to all 2-isohedral tilings. A full generalization comes in the form of *combinatorial tiling theory* [4], in which a *Delaney symbol* summarizes the combinatorial structure of a tiling. Delaney symbols can be used to describe k-isohedral tilings for any k; they also generalize to dimensions greater than two and to non-Euclidean geometry.

In principle, we might use Delaney symbols as a basis for parameterizing the shapes of all 2-isohedral tilings, yielding a new dihedral Escherization algorithm. Indeed, given any k user-supplied goal shapes, such a system could discover a k-isohedral tiling that approximates all of them. Unfortunately, Delgado *et al.* show that there are over a thousand 2-isohedral types. In general, the number of k-isohedral types grows very quickly with k. Because the choice of tiling type is discrete, it quickly becomes infeasible to search all possible types by launching independent continuous optimizations. It might be possible to run an optimization on a carefully chosen subset of Delaney symbols that are particularly well-suited to Escherization. As a benchmark for k-isohedral Escherization, consider Escher's symmetry drawing 71, a remarkable periodic tessellation featuring twelve distinct bird motifs.

Taken to the limit, k-hedral Escherization becomes a kind of packing problem. As k grows very large, we become more interested in simply fitting all k goal shapes together plausibly in a kind of puzzle. Such a large block of shapes no longer has as much aesthetic appeal when repeated across the plane. Escher's print *Plane Filling 1* is an example where a number of one-of-a-kind figures are assembled to tile a finite region [2]. As we move from monohedral and dihedral tilings to cases such as this one, the importance of tiling theory is lessened. An approach like that of Jigsaw Image Mosaics [10] might be more appropriate.

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