Semiregular patterns on surfaces

Craig S. Kaplan*
Computer Graphics Lab,
David R. Cheriton School of Computer Science,
University of Waterloo

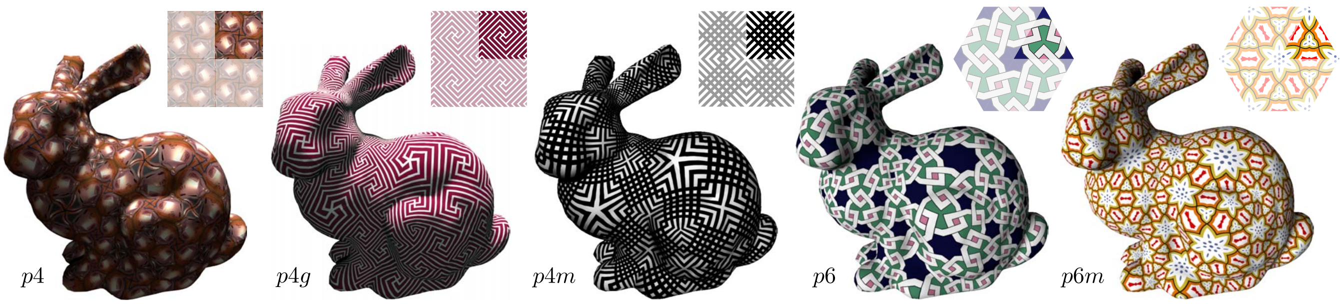


Figure 1: *A sampling of bunnies decorated with different repeating patterns, one from each of the five symmetry types supported in this paper. From left to right: Escherized teapots, A Celtic key pattern, A Japanese stencil pattern, an Islamic interlacement, and a Persian tile design.*

Abstract

Inspired by recent advances in high-quality mesh parameterization, I present a technique for decorating surfaces with seamless ornamental patterns. The patterns are transferred from planar drawings with wallpaper symmetry. I show that when the original drawing belongs to one of a few specific symmetry groups, then it can easily be rendered with low distortion on a suitably parameterized mesh. The result is not symmetric, but retains most of the structure of the original drawing.

Keywords: Ornamental design, Mesh parameterization, Remeshing, Pattern synthesis, Symmetry, Tilings

1 Introduction

A popular topic in computer graphics is the problem of synthesizing textures on surfaces. Usually, the goal is to ensure that the surface is textured in an irregular way, so that any perceived order and repetition are suppressed.

It is also worthwhile to consider the *cultivation* of this order. How might we cover arbitrary surfaces seamlessly with repeating decorative patterns? In the plane, we have an established mathematical theory of symmetry that can account for the structure of many ornamental patterns, together with a rich, ongoing artistic tradition from which to draw inspiration. In general, surfaces do not have enough structure to support a meaningful theory of symmetry, but perhaps planar patterns could be transferred to surfaces in a way that preserves the spirit of the original patterns.

In this paper, I describe a simple solution to the problem of covering arbitrary surfaces seamlessly with repeating decorative patterns belonging to certain symmetry types. My solution does not introduce any new algorithms or equations; it is based entirely on texture mapping of appropriately parameterized meshes. Recent advances in semiregular mesh parameterization have made it worthwhile to demonstrate the artistic possibilities of surface decoration. Several examples are shown in Figure 1.

My main contribution is an explanation of which planar patterns can easily be transferred to surfaces. I also show how to select a representative texture from a pattern. Finally, this paper greatly simplifies the previous remeshing-based approach of Kaplan et al. [2004], while simultaneously producing higher-quality results.

I present my method in the following section, and discuss it in the context of related work by Kaplan et al. and others in Section 3.

2 Transferring planar patterns to surfaces

Symmetry theory provides the most natural basis for understanding and classifying patterns in the plane. A *symmetry* of a figure in the plane is a rigid motion that maps the figure onto itself. The symmetries of a figure naturally form a group, called the figure's *symmetry group*, under composition of transformations. A figure whose symmetries include translations in two non-parallel directions is called a *wallpaper pattern* or (in the context of this paper) simply a *pattern*, and its symmetry group is called a *wallpaper group*. It has been known since the late 19th century that there are exactly 17 distinct types of wallpaper patterns [Conway et al. 2008].

Consider the pattern shown in Figure 2(a). This pattern has symmetry type *p*4. In addition to two directions of translational symmetry, it has two inequivalent families of fourfold rotations, and a family of halfturns. One representative from each of these families is depicted in the figure. In any *p*4 pattern we can extract a square region with the following properties:

*e-mail: `csk@uwaterloo.ca`

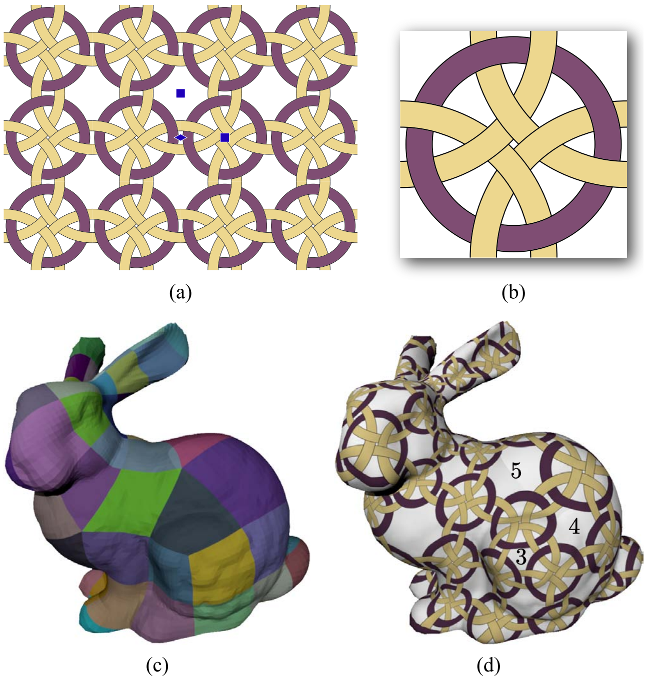


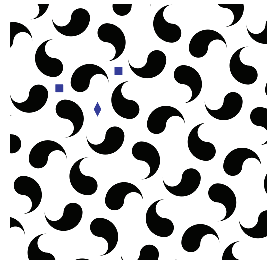
Figure 2: *A demonstration of how a tile may be extracted from a wallpaper pattern of type p4 and mapped onto a surface. A source pattern is shown in (a). A sample of a centre of twofold rotation is marked with a rhomb; two inequivalent centres of fourfold rotation are marked with squares. Given a source pattern, a single tile can be identified that repeats purely by translation (b). Then, given a surface parameterized into quadrilateral domains (c), a copy of the tile can be mapped into each domain to produce a final decorated surface (d). Different numbers of tiles meet around vertices in the coarse mesh; in (d), examples where three, four, and five tiles meet are indicated.*

- The region itself has fourfold rotational symmetry;
- The region's corners are centers of fourfold rotation; and
- The entire pattern can be formed from translated copies of the region.

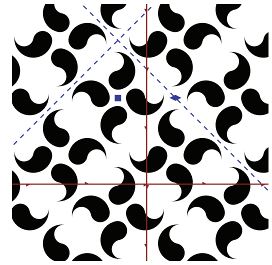
I refer to any such region as a "tile". An example of a tile is shown in Figure 2(b).

Suppose we are now given a surface (i.e., a triangle mesh) equipped with a semiregular parameterization into square domains, as in Spectral Surface Quadrangulation [Dong et al. 2006] or PolyCube-maps [Tarini et al. 2004]. These techniques do not parameterize a mesh surface to a simple domain such as a planar polygon or a sphere, but to a coarse mesh of quadrilateral domains with the same overall topology as the original surface. Each domain has a simple *uv* parameterization that maps it to a square with low angle distortion. Thus a square texture (or collection of textures) can easily be mapped onto the surface one domain at a time. Figure 2(c) shows a parameterization of the bunny into quadrilateral domains.

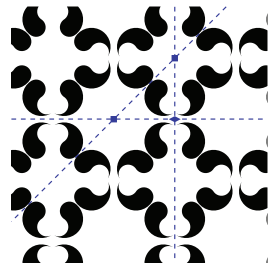
Using this parameterization, we can simply fill every domain with a copy of the tile from Figure 2(b) (or from any other *p*4 pattern). Obviously, the resulting textured surface is faithful to the planar pattern within each domain. The pattern is also continuous across boundaries between adjacent domains, just as it is for two adjacent tiles in the plane. The only potential difficulty is at the corners



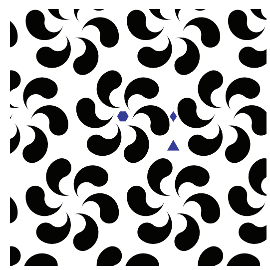
p4: Two distinct centres of fourfold rotation, one centre of twofold rotation, no reflections.



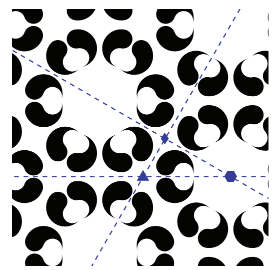
p4g: Differently-oriented systems of reflections and glide reflections laid out in square grid arrangements.



p4m: Reflections in the sides of a 45-degree right triangle.



p6: Centres of twofold, threefold and sixfold rotation, no reflections.



p6m: Reflections in the sides of a 30-degree right triangle.

Figure 3: *Visualizations of wallpaper patterns belonging to each of the five groups that can be mapped seamlessly onto surfaces. Each pattern shows a selection of centres of two-, three-, four- and sixfold rotation using rhombs, triangles, squares and hexagons respectively. Dashed lines indicate reflections and notched red lines indicate glide reflections. Underneath each pattern is the group name and a short description of its distinguishing features.*

where parameterization domains meet. In theory, any number of domains greater than two might meet at such a corner; this number is always four in the plane. Although the surface decoration cannot match the original pattern perfectly at these corners, it achieves a graceful re-interpretation of the original. The symmetries of the tile guarantee that any number of tiles can meet around a point, with the effect of locally growing or shrinking the local order of rotational symmetry around that point.

Figure 2(d) shows a bunny textured using the *p*4 tile. The center of every yellow ring is a corner where domains meet. Examples are shown where three, four, and five tiles meet. Interestingly, this local change in symmetry offers a rough visualization of the surface's Gaussian curvature. The number of purple rings linked by a yellow ring will be less than, equal to, or greater than four depending on whether the Gaussian curvature is locally positive, zero, or negative.

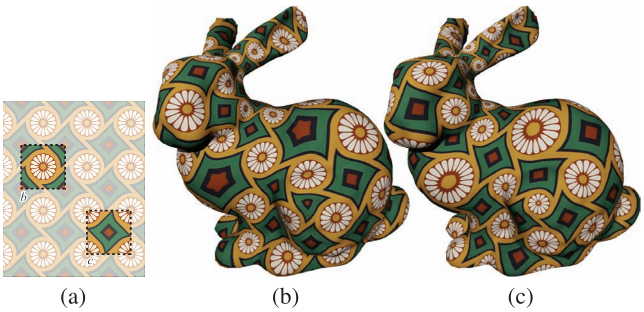


Figure 4: *An example of the use of inequivalent tiles for planar patterns with fourfold symmetry. The two eligible square tiles are shown in (a), together with surfaces using those tiles in (b) and (c). In (b), red squares become regular polygons with different numbers of sides; in (c), flowers have different numbers of petals.*

Two other symmetry types are compatible with a parameterization into square domains: $p4g$ and $p4m$. This result follows immediately from the fact that these two groups contain $p4$ as a subgroup, and must therefore have similar sets of fourfold rotations and half-turns. Because no other symmetry types are guaranteed to be decomposable into squares, the three types given here are the only ones compatible with a parameterization based on square domains.

We can also consider parameterizations by coarse domains that map to equilateral triangles, as in Globally Smooth Parameterization [Khodakovsky et al. 2003]. Here, we must begin with a planar pattern from which we can extract an equilateral triangular tile. By analogy with the square case, the tile must have threefold rotational symmetry and sixfold rotational centres at its vertices. The only two symmetry types that fulfill these requirements are $p6$ and $p6m$. Tiles from patterns of these types can be mapped into the triangular domains of a parameterized surface, producing results similar to the square case.

Based on the preceding discussion, we find that of the 17 wallpaper types, five can easily be applied to parameterized surfaces. Sample patterns belonging to these five groups are shown in Figure 3, together with brief descriptions of the groups' distinguishing features. Washburne and Crowe [1992] present flowcharts for identifying any given wallpaper pattern's group. Ostromoukhov [1998] offers a variety of algorithmic and mathematical tools for analyzing and synthesizing wallpaper patterns.

The focus on five types out of 17 may seem like a severe limitation, but in fact it leaves us with a large body of tradition to work from. These types encompass patterns derived from the symmetries of the three regular tilings of the plane (squares, hexagons, and equilateral triangles), and are therefore very popular in historical and contemporary ornamental design. For example, in a collection of 350 Islamic star patterns catalogued by Abas and Salman [1995], more than half belonged to *p4m* or *p6m*.

2.1 Choosing a tile for fourfold patterns

Note that when a source pattern is of type $p4$ or $p4m$, there are two ways to extract an appropriate square tile. There are two inequivalent families of fourfold rotations. A tile may be centred on a member of either family, with the other family occupying the tile's corners. In the plane, it makes no difference which tile is chosen to represent the pattern. When mapped onto a surface, however, the two possibilities become visually distinct, as shown in Figure 4.

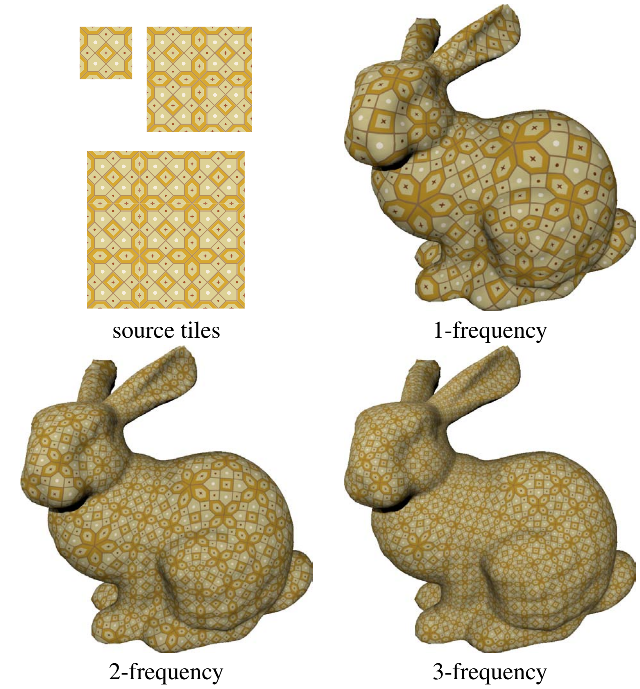


Figure 5: *The use of frequency to create variations with more repetitions of a tile. The leftmost image shows a source tile alone and in 2×2 and 3×3 configurations. For each frequency, a rendered surface is shown. (The word "frequency" is borrowed from its use in describing geodesic domes.)*

The center of the tile will continue to be a fourfold rotation on the surface, but the orders at the corners will vary. This difference does not arise with patterns of type $p4g$, because the two families of rotations are equivalent to each other via a reflection. In that case, the two possible tiles will produce left-handed and right-handed versions of the same decoration.

2.2 High frequencies

The symmetry group $p4$ contains copies of itself as subgroups. These subgroups correspond to $k \times k$ square arrangements of the original pattern's tile, for all $k > 1$. Looked at another way, square arrangements of this tile form a larger tile with all the symmetry properties of the original. The larger tile can therefore serve just as easily as a basis for texturing. Figure 5 demonstrates this flexibility. Given a coarse mesh, it is always possible to decorate it with increasingly fine patterns that have the same structure as the planar version except at the isolated domain corners. The same argument applies to the other two square pattern types, as well the two triangular types.

2.3 3D shells

The combination of a mesh parameterization with a symmetric planar pattern need not be used only for texturing, of course. Another possibility is to generate shell geometry near the surface. In Figure 6, I use the pattern tile as a stencil to be cut out of the mesh surface. I join the surface to a concentric copy to create a thin shell that interprets the original pattern. A similar approach could be

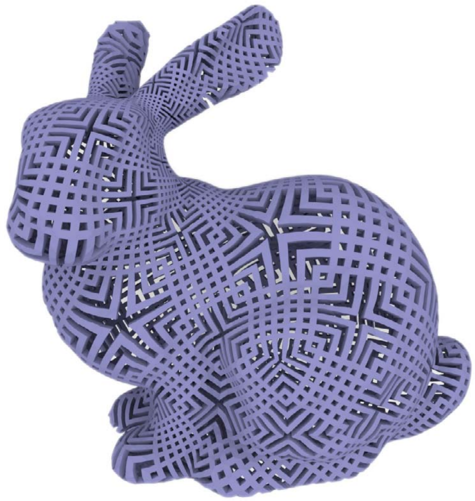


Figure 6: *A bunny constructed by cutting the stencil pattern from the center of Figure 1 out of the original surface.*

Figure 7: *A comparison of two bunnies decorated with the same Islamic star pattern. The bunny on the left is decorated using the remeshing technique of Kaplan et al. [2004], and exhibits considerable pattern distortion in some areas (for example, on the ears). The bunny on the right uses the technique described in this paper.*

used to implement a variation of mesh quilting [Zhou et al. 2006] in which the shell geometry is intended to form a regular pattern.

3 Discussion

I have produced many textured surfaces based on semiregular mesh parameterizations. The square parameterizations use meshes processed with the Spectral Surface Quadrangulation method of Dong et al. [2006], and the triangular parameterizations use the Globally Smooth Parameterization method of Khodakovsky et al. [2003]. In both cases, I rely on the underlying parameterization methods to ensure reasonable continuity across domain boundaries. Additional examples are shown in Figure 8. All images were rendered using pbrt [Pharr and Humphreys 2004].

This work was originally inspired by research on synthesizing irregular patterns on surfaces, specifically the triangular technique described by Neyret and Cani [1999]. Much later, Fu and Leung [2005] presented their square technique based on Wang tiles. These two approaches are very similar—tiles are chosen to fill domains at random, subject to certain constraints on legal adjacencies between tiles. All such constraints are trivially satisfied in this work, where there is a single tile, every edge of which is compatible with every other.

As I mentioned in the introduction, there is not a lot of research that specifically addresses the question of covering surfaces with regular patterns. The work that exists seems to focus not on texturing, but on remeshing. For example, Akleman et al. [2005] show how to subdivide a mesh to create analogues to the semiregular planar tessellations.

The most closely related work is the pattern oriented remeshing technique used by Kaplan et al. [2004] in the construction of Celtic knots on surfaces. They parameterize a genus-0 surface using a geometry image. They can then superimpose an arbitrary tiling on the square domain of the geometry image and use the projection of that tiling onto the original surface as a basis for remeshing.

Their approach produces attractive results, but suffers from several deficiencies. First, a geometry image can exhibit significant distortion, which in turn causes distortion in the remeshed tiles. Figure 7 compares one of their results with the analogous pattern (of type $p4m$) rendered as described in this paper. Second, a tiling may be fundamentally incompatible with the boundaries of the geometry image, yielding seams when the surface is decorated. Kaplan et al. address this problem by stitching together tiles to close gaps along the boundary, but this operation does not preserve the topology of the source tiling. Third, they are unable to render patterns based on images, since their method relies on a tiling as a base for remeshing.

The potential advantage of the technique of Kaplan et al. is that it is not restricted to the five symmetry types discussed in the previous section. In theory it can support any tiling at all, symmetric or otherwise. On the other hand, all but one of the examples in their paper belongs to one of the five symmetry groups discussed here, and the anomalous case could easily be transformed into one compatible with my technique via a simple non-uniform scaling operation.

Ultimately, the approach of Kaplan et al. might be more useful for meshes with boundaries, where there is no need to stitch tiles together. It might then be better to use the recent method of Springborn et al. [2008] to produce a planar parameterization domain upon which an arbitrary tiling may be superimposed. Indeed, this latter paper shows several examples of surfaces decorated with patterns of type $p4$.

A different but related problem is that of transferring planar patterns to the sphere and the hyperbolic plane. Dunham [1986; 1999] has constructed many examples of hyperbolic interpretations of Euclidean patterns. In a recent paper, von Gagern and Richter-Gebert [2009] offer a principled, general solution to the problem of creating hyperbolic versions of Euclidean ornament. They map fundamental regions of Euclidean patterns conformally to their curved analogues in the Poincaré model of the hyperbolic plane. They minimize distortion in the mapping process by using a discrete conformal mapping, as described by Springborn et al. [2008].

In the future, it would be interesting to investigate ways to transfer information that is embedded in a pattern at a higher level than its symmetries. For example, in many tilings the tiles are coloured in a way that breaks the (uncoloured) tiling's symmetries. Perhaps an uncoloured version of the tiling could be mapped onto a surface, and then the colours regenerated approximately using rules derived by noting adjacent colours in the original.

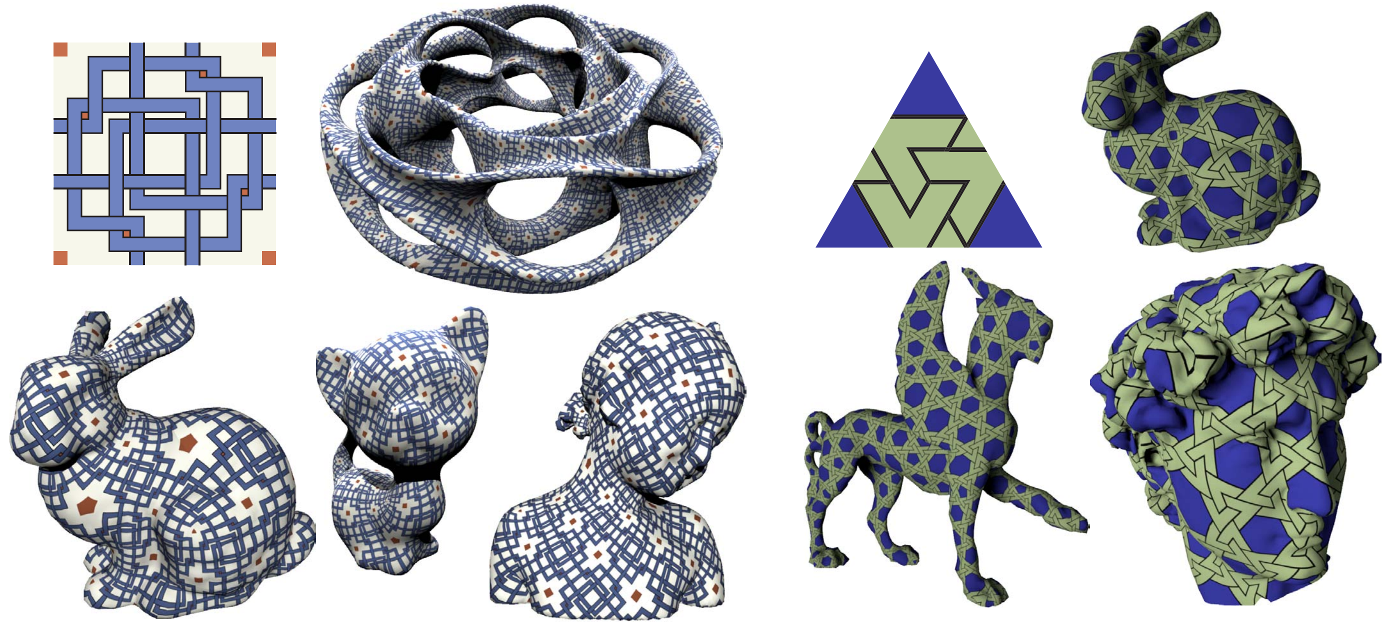


Figure 8: *A sampling of surfaces decorated with patterns of type* p4 *(left) and* p6 *(right).*

Acknowledgments

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