Islamic Star Patterns from Polygons in Contact

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Abstract

We present a simple method for rendering Islamic star patterns based on Hankin's "polygons-in-contact" technique. The method builds star patterns from a tiling of the plane and a small number of intuitive parameters. We show how this method can be adapted to construct Islamic designs reminiscent of Huff's parquet deformations. Finally, we introduce a geometric transformation on tilings that expands the range of patterns accessible using our method. This transformation simplifies construction techniques given in previous work, and clarifies previously unexplained relationships between certain classes of star patterns.

Key words: Islamic star patterns, Islamic art, Tilings, patterns

1 Introduction

Islamic star patterns represent one of the world's great ornamental design traditions [2, 6, 7]. Star patterns are a harmonious fusion of mathematics, art, and spirituality, and expressions of symmetry, balance, and ingenuity.

Star patterns also embody an enduring mathematical mystery. Most of the original design techniques are lost to history, and we are forced to probe the minds of ancient artisans and mathematicians via the patterns they left behind. Many scholars and hobbyists have discovered or rediscovered techniques that produce Islamic patterns [1, 9, 11, 20].

This paper presents a simple technique based on Hankin's "polygons-in-contact" method [12]. Given a tiling of the plane, Hankin's method produces an Islamic star pattern based on that tiling (Section 3). By modifying the construction slightly, we are able to construct designs in the style of Huff's parquet deformations [16, Chapter 10] (Section 3.1). Finally, we show how an operation on tilings called the "rosette transform" can expand the range of patterns available using Hankin's method (Section 4). The rosette transform demonstrates the power of Hankin's method, and formalizes previously unexplained relationships between certain classes of star patterns.

2 Previous work

Our approach follows most directly from recent work by Kaplan and Salesin [18]. They construct designs by filling tiles in a tiling with fragments of star patterns. For tiles that are regular polygons, they give a parameterized space of "design elements" drawn from historical examples. They then complete the pattern by filling irregular polygons using an "inference algorithm". Because their construction is independent of Euclid's parallel axiom, they can draw star patterns seamlessly in Euclidean and non-Euclidean geometry. We simplify their approach by eliminating design elements and applying an inference algorithm uniformly to all tiles. We can still produce complex elements such as rosettes by moving more information into the underlying tiling. There is a large overlap in the designs produced by the two systems – given suitable tilings, Najm could reproduce all the star patterns presented here. But this paper offers insights into the tilings that underlie star patterns, and provides a technique that is simpler and easier to control.

Jay Bonner is an architect who has studied Islamic star patterns extensively. In an unpublished manuscript [3], Bonner gives a systematic presentation of star patterns developed over a vast space of tilings (which he calls "polygonal sub-grids"). Some of the techniques from the book also appear in a recent paper [4]. Bonner's work is intended as a resource for designers, and not specification for software writers. He draws patterns manually using a CAD tool. This paper is in part an attempt to formalize the algorithms that underlie his technique, and to express those algorithms in software for architects, designers, and artists.

3 The polygons-in-contact method

A tiling-based approach to Islamic star patterns seems first to have been articulated in the west by E.H. Hankin in the early part of the twentieth century. In a series of papers [12, 13, 14, 15], he explains his discoveries and gives many examples of how the technique can be used. Hankin's description of his technique provides an excellent starting point for an algorithmic approach (and helps drive contemporary work by Bonner).

> In making such patterns, it is first necessary to cover the surface to be decorated with a network consisting of polygons in contact. Then through the centre of each side of each polygon two lines are drawn. These lines cross each other like a letter X and are continued till they meet other lines of similar origin. This completes the pattern [12, Page 4].

Since that time, scholars such as Lee [19] and Critchlow [8] have referred to Hankin's "polygons-in-contact" technique. This method immediately suggests an algorithm for turning a tiling into an Islamic star pattern. Given a tiling of the plane by polygons (Hankin's "network"), we identify the midpoints of the edges of the tiling as "contact points" where the design will be born. We place a small X at every contact point and "grow" the arms of the X until they encounter lines growing from other contact points. There is one obvious degree of freedom in this process: the angle formed by the arms of the X with the edge from which they emanate. We call this angle the *contact angle* of the pattern. An illustration is given in Figure 1(a).

We can regard this process as growing a small arrangement of lines for each unique tile shape in a tiling. We grow a pair of rays, forming half an X, inward from the midpoints of the tile's edges. We call the arrangement of lines associated with a single tile its *motif*. An implementation of this construction technique should accept a tiling and a contact angle as input, build a motif for each tile shape, and assemble the motifs into a pattern that can then be decorated.

Given an n-sided polygonal tile and a contact angle, we must develop a motif from the $2n$ rays entering that tile through its edge midpoints. A successful motif will partition the rays into pairs, where each matched pair represents a distinct path through the tile. The best possible motif will be the pairing of rays that optimizes a chosen aesthetic goal. We choose the simple goal of minimizing the sum of the lengths of all of the line segments in the motif. This goal reflects the sense of economy and inevitability in Islamic design, and is justified through many historical examples.

Ideally, then, we would iterate over all possible pairings of rays, and find the one that minimizes total length. Unfortunately, this algorithm is not practical – there are $\frac{n!}{(n/2)!2^n}$ ways to partition $2n$ rays into pairs, or over half a billion possibilities for a region with 10 sides.

Instead we use a greedy approach, based on a simplified version of Kaplan and Salesin's *inference algorithm* [18]. We consider all possible pairs of rays. If two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ intersect at a point P, we store that pair in a collection together with a cost equal to the sum of the lengths AP and PD. If the rays are collinear and point towards each other, we store the pair together with the length AD. We can then sort the collection by cost and walk over it in order. For each pair of rays, we incorporate that pair's path into the motif provided neither of the rays has yet been used.

In practice, this algorithm performs well on a wide variety of polygons. It certainly performs perfectly on regular polygons, where it constructs star-shaped motifs. It sometimes produces motifs with unmatched rays, and sometimes paths that venture too far from the underlying tile. In the cases where it fails, it usually does so not because it is greedy, but because the pairing technique is not well-suited to the tile shape in question. In some cases, the inferred motif can be improved by moving the contact points away from the edge midpoints. This adjustment is discussed in greater detail in Section 4.

Figure 2 illustrates the process of growing rays from contact positions. Figure 3 shows some typical designs that can result from using our implementation of Hankin's method.

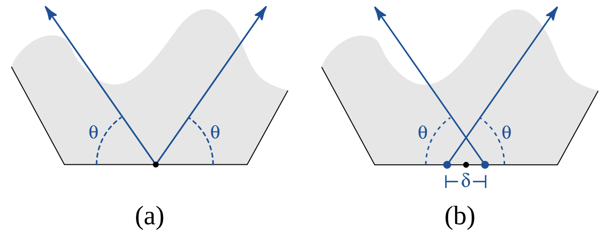


Figure 1: In the first step of Hankin's method, a pair of rays is associated with every contact position on every tile. In (a), a single contact position gets its two rays, each of which forms the contact angle θ with the edge. In (b), we separate the ray origins by distance δ.

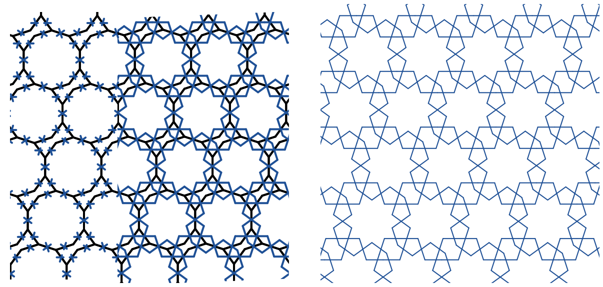


Figure 2: A demonstration of Hankin's method. On the left, contact points sprout an X-shaped arrangement of rays that grow until they meet other rays. When the original tiling is removed, the result is the pattern on the right.

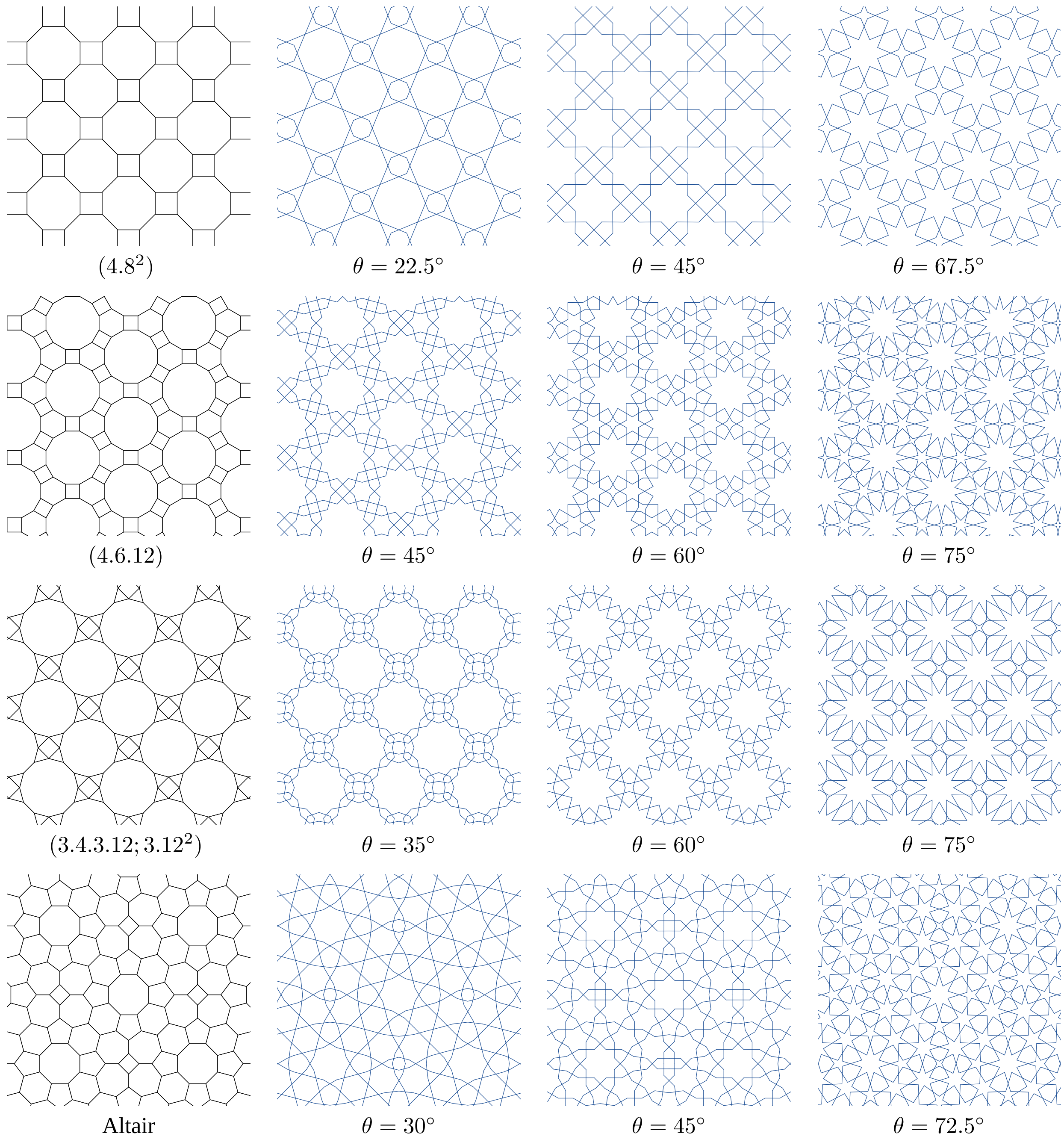


Figure 3: Examples of star patterns constructed using Hankin's method. Each row shows a tiling together with three designs that can be derived from it using three different contact angles. The bottom row features an amusing tiling by nearly regular polygons. It is reproduced from Grünbaum and Shephard [10, Page 64], where it serves as a reminder of the danger of over-reliance on figures. A related design also appears in Bourgoin [6, Plate 163].

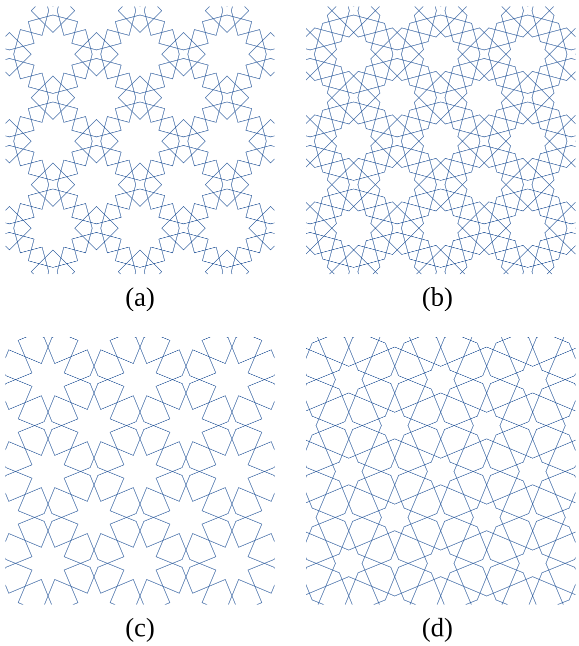


Figure 4: A demonstration of two cases where an extension to the inference algorithm can produce a slightly more attractive motif. In (a), a star pattern is shown with large unfilled areas that were the centers of regular dodecagons in the original tiling. Adding a layer of inferred geometry to the inside of the motif produces the improved design in (b). The process is repeated with a different tiling in (c) and (d).

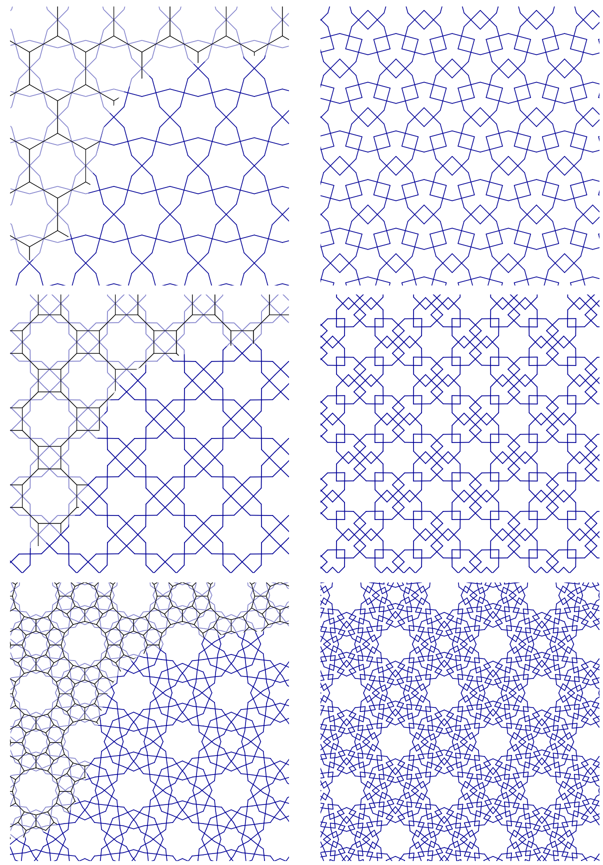


Figure 5: Examples of two-point star patterns constructed using Hankin's method. Each row shows a template tiling, a star pattern with $\delta = 0$, and a related two-point pattern with non-zero δ. The structure of the tiling in the bottom row will be explained in Section 4.

There are some cases where simple modifications to the basic inference algorithm can improve the generated motif. Consider, for example, the star pattern given in Figure 4(a). This pattern contains large regions, derived from regular dodecagons, that are left unfilled. A more attractive motif can be constructed using a second pass of the inference algorithm, building inward from the points where the rays from the first pass meet. The resulting design, shown in Figure 4(b), is more consistent with tradition. In the inference algorithm, it is easy to recognize when the provided tile shape is a regular polygon and to run the second round of inference when specified by the artist. Kaplan and Salesin [18] solve this problem by providing a more explicit parameterization of the range of motifs that can be used to fill regular polygons.

A further enhancement is to allow the contact position to split in two, as shown in Figure 1(b). The split can be accomplished by providing the inference algorithm with a second real-valued parameter δ that specifies the distance between the new starting points of the rays. The parameter δ can vary from zero (giving the original construction) up to the length of the shortest tile edge in the tiling. This modification gives what Bonner calls "two-point patterns," a set of designs that are historically important in Islamic art [3]. Examples of two-point patterns constructed using the δ parameter are shown in Figure 5. The designs corresponding to two-point patterns tend to be made up of very short closed strands, each one forming a loop around a single tiling vertex in the original tiling. The contact angle is typically chosen to be 45°, forming squares around the midpoints of the tiling's edges.

3.1 Islamic parquet deformations

Parquet deformations are a style of ornamental design created by William Huff, and later popularized by Douglas Hofstadter [16, Chapter 10]. They are a kind of "spatial animation," a geometric drawing that makes a smooth transition in space rather than time. Parquet deformations are closely related to M.C. Escher's *Metamorphosis* prints [5, Page 280], though unlike Escher's work they are purely abstract, geometric compositions.

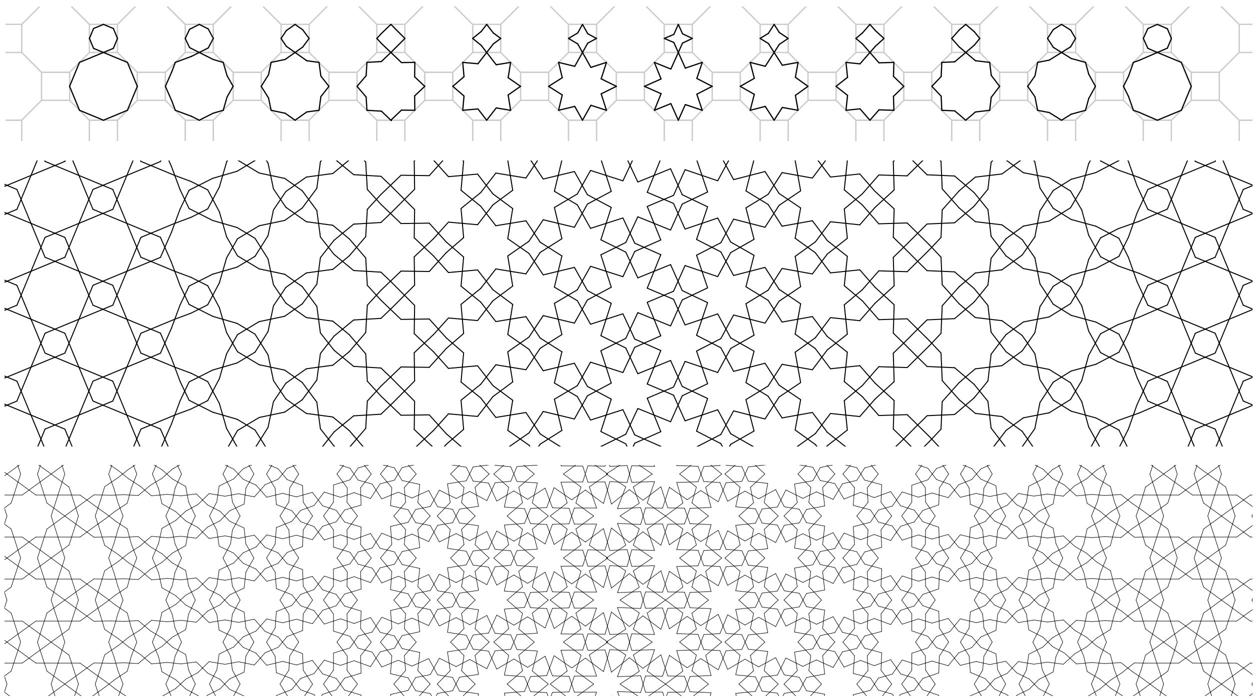


Figure 6: Islamic parquet deformations based on Hankin's method. The top diagram shows the effect of continuously varying the contact angle of a ray depending on the horizontal position of the ray's starting point. When the process is carried to all other tiles, the design in the middle emerges. In this design, the contact angle varies from a minimum of $22.5°$ *at the sides up to* $67.5°$ *in the middle. In the design on the bottom, the contact angle varies from* $36°$ *at the sides up to* $72°$ *in the middle.*

By exploiting Hankin's method, we can introduce a new style of Islamic design that we call "Islamic parquet deformations". We simply modify the inference algorithm so that the contact angle varies along a line. The contact angle for a ray is chosen according to a function of a horizontal position of that ray's start position. In this way, the four rays leaving a given contact position still form an X, even though the contact angle may vary within a single tile.

Smooth variation of the contact angle results in a gently changing geometric design that is still recognizably Islamic (see Figure 6). We believe that these parquet deformations occupy an interesting place in the world of Islamic geometric art. The structure is recognizably in the Islamic tradition, but they would not have been produced historically because very little repetition is involved. The effort of working out the constantly changing shapes by hand and then executing them would have tested the patience of any erstwhile artisan.

4 The rosette transform

A curious property of Hankin's polygons-in-contact method is that different tilings may give rise to the same star pattern under suitable choices of contact angle. The pattern in the top row of Figure 7 is one example. It can be produced from the tiling on the left using a contact angle of $54°$, or from the tiling on the right using a contact angle of $36°$. We might therefore suspect a relationship between the two tilings.

Further evidence for this relationship can be found in the star patterns produced by the Najm system of Kaplan and Salesin [18]. The bottom row of Figure 7 shows a tiling on the left with "rosette" motifs placed in regular decagons. Kaplan and salesin provide an explicit parameterization of these rosettes. But the right hand side of the figure demonstrates that the same pattern can arise via inference alone from the ostensibly related tiling.

Our experience with Hankin's method suggests that there are many pairs of tilings that are related in this way. Referring to Figure 7, we call the tiling on the left a "Najm tiling" and the tiling on the right a "Hankin tiling". The two have similar structure, except that in the Hankin tiling the large regular polygons are separated by rings of potentially irregular polygons, usually pentagons.

In his manuscript [3], Bonner uses both kinds of tilings to create star patterns, and he too observes that a single

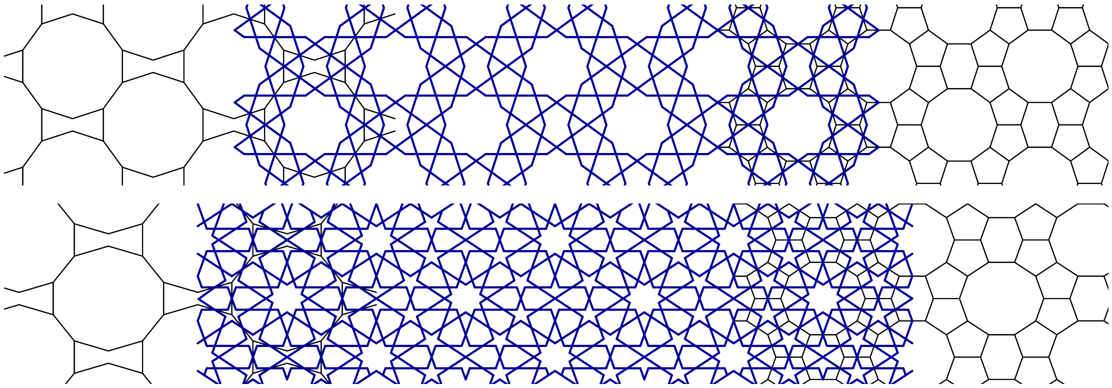


Figure 7: Examples of distinct tilings that can produce the same Islamic design. In each case, the tilings on the left is filled in using a combination of design elements and inference, and the tiling on the right uses inference alone. They meet in the shared design in the center.

design may originate from multiple tilings. In this section we introduce a transformation on tilings called the "rosette transform" that explains the connection between Najm tilings and Hankin tilings. We also compare our method to Najm in terms of the patterns each approach can produce.

The algorithm for the rosette transform is reminiscent of the inference algorithm used in Section 3. Given a tiling, it constructs a planar map for each distinct tile shape. The planar maps are then assembled, this time into a new tiling rather than a final design. It is also a kind of dualization: the most common operation is to erect perpendicular bisectors to edges in the original tiling. The map for each tile shape is constructed in one of two ways.

Regular polygons. If the tile is a regular n-gon $\mathcal{P}$ of radius r with five or more sides, then the map is constructed as in Figure 8. We build a new regular n-gon $\mathcal{P}'$ with radius $r' < r$ and place it concentric with the original polygon but rotated by π/n relative to it. We then add line segments connecting the vertices of $\mathcal{P}'$ to the edge midpoints of $\mathcal{P}$. The inner radius r' is chosen so that the length of each of these new segments is exactly half of the side length of $\mathcal{P}'$. Some trigonometry shows that given n and r, the correct value of r' is given by

$$r' = r\left(\cos\frac{\pi}{n} - \sin\frac{\pi}{n}\tan\left(\frac{\pi(n-2)}{4n}\right)\right)$$

The map returned is $\mathcal{P}'$ together with the segments joining it to the edge midpoints of $\mathcal{P}$.

Irregular polygons. If the tile is a polygon $\mathcal{P}$ that does not satisfy the conditions above, we extend perpendicular bisectors of the sides of $\mathcal{P}$ towards its interior, as shown in Figure 9. The bisectors are truncated where they meet each other. The result becomes the map for this tile.

This step is similar in spirit to running the inference algorithm with a contact angle of $90°$ and is subject to the same pitfalls. We do not expect it to return a meaningful answer for every possible polygon, but in the cases of polygons that occur in Najm tilings, the map it discovers is usually appropriate.

Some heuristics also work here that do not apply in the inference algorithm. One moderately successful heuristic is to consider the intersection points of all pairs of rays and to cluster those points that lie inside the tile. Each cluster can then be averaged down to a single point that all rays contributing to that cluster can use as an endpoint. This adjustment may move some rays away from perpendicularity, introducing "kinks" in the middles of edges of the transformed tiling. We can correct this problem later by detecting the kinks and replacing them with straight line segments.

When this algorithm is run on Najm tilings, it tends to produce Hankin tilings. For instance, the two tiles in Figures 8 and 9 correspond to the Najm tiling on the left hand side of Figure 7. The rosette transform produces the Hankin tiling on the right hand side of the figure. Two additional examples of tilings and their rosette transforms are given in Figure 10.

The rosette transform takes the intelligence out of Kaplan and Salesin's "design elements" and embeds it in the tiling. Suppose that in a given tiling, Najm is used to fill a regular polygon $\mathcal{P}$ with a rosette. In the rosette transform of that tiling, a scaled-down copy $\mathcal{P}'$ of $\mathcal{P}$ will be surrounded by a ring of irregular pentagons. With our method, these pentagons will conspire to form the hexagonal arms of a rosette around a central star constructed inside $\mathcal{P}'$. The rosette transform is motivated by (and named after) the goal of making the pentagons as close as possible to regular, producing rosettes that are nearly ideal in the sense given by Lee [19].

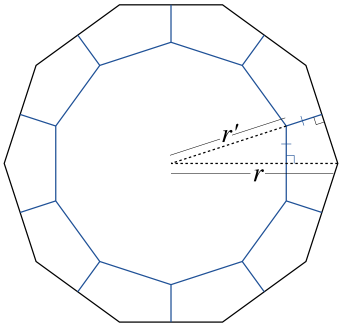


Figure 8: The rosette transform applied to a regular polygon. Here, a regular 10-gon of radius r is filled with a smaller regular 10-gon of radius r' together with segments that join the vertices of the inner polygon to the edge midpoints of the outer one. The inner radius is chosen so that the marked edges have the same length.

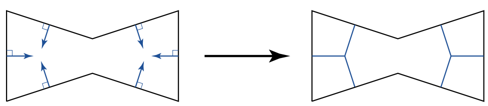


Figure 9: The rosette transform applied to an irregular polygon. On the left, a perpendicular bisector is drawn for every tile edge as a ray pointing to the interior of the tile. The rays are cut off when they meet each other, as with the inference algorithm.

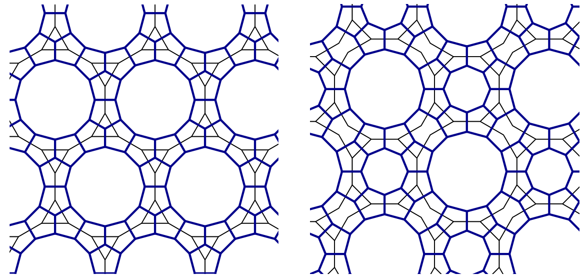
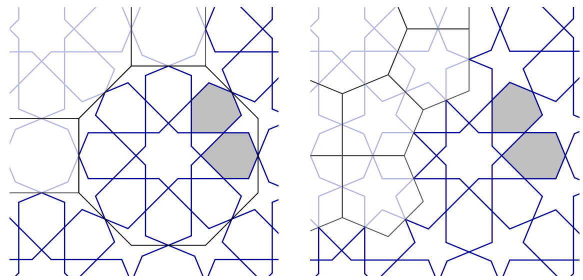


Figure 10: Two demonstrations of the rosette transform. The transformed tiling is shown superimposed in bold over the original.



(a) (b)

Figure 11: An example where Hankin's method can produce imperfect rosettes. In (a), Najm is used to place perfect rosettes inside regular octagons. When Hankin's method is used on the rosette transform in (b), the rosette hexagons (shown shaded) are of two different sizes.

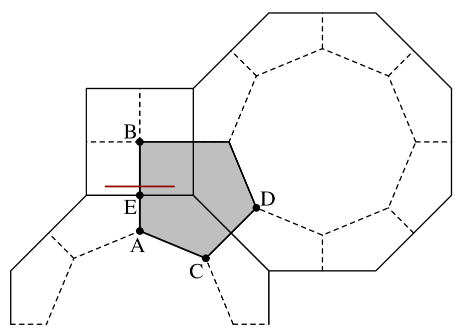


Figure 12: An illustration of the adjustment to contact positions that recovers perfect rosettes in Hankin's method. The horizontal line indicates the midpoint of the edge of the shaded tile in the rosette transform. But the correct location for the contact point is the intersection of that edge with the edge of the original tiling.

In some cases, Hankin's method generates an unsatisfactory pattern when applied to a rosette-transformed tiling. Figure 11 shows an example where Hankin's method discovers the topology of the correct pattern, but produces uneven rosettes. Bonner discusses how to correct this situation by adjusting the contact positions on the pentagons away from the centers when necessary. However, he gives no indication of when or how to carry out this adjustment.

We can appeal to the relationship between tilings and their rosette transforms to resolve this mystery. The design of Figure 11(a) was constructed using the Najm method. We choose rosettes to fill the regular octagonal tiles. By design, the hexagonal arms of the rosettes (two of which are shown shaded in the diagram) contact the edges of their surrounding octagons.

In Figure 11(b), the rosette arms are formed from the ring of pentagons introduced in the rosette transform. The construction of one such pentagon is shown in Figure 12. The problem is that segments AE and BE have different lengths. If we simply place contact positions at edge midpoints, then the contact positions along edges AB and CD will not be equidistant from the center of their adjacent octagon, producing an uneven arrangement of rosette arms. Although the shaded pentagon is topologically dual to the vertex it contains, it is not dually situated (its edges are not bisected by the original tiling). We correct the discrepancy by recording the intersection points of the rosette transform with the original tiling, and using them as contact positions.

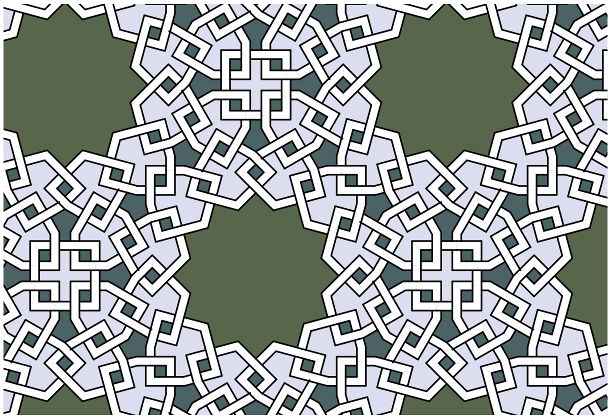


Figure 13: A decorated star pattern produced using the polygons-in-contact method.

Thus there a deep connection between these two kinds of tilings, which can expose the logic behind what may have seemed like an arbitrary but essential adjustment. Furthermore, these revised contact positions can be calculated easily while the rosette transform is computed.

5 Implementation

We have implemented the approach described in this paper as a standalone Java application. The application's interface shows a tiling with a corresponding Islamic star pattern superimposed on it. The user is able to select the tiling to work with and modify the contact angle θ and distance δ used to produce two-point patterns. They can also choose whether to incorporate a second pass of the inference algorithm in large, regular polygons, and whether to adjust the contact positions away from the edge midpoints as described at the end of Section 4. All of these changes are reflected in the design interactively, making it easy and enjoyable to browse a wide range of star patterns. When the user has decided upon a design, they can render it using the decoration styles first developed for Taprats [17]. A decorated pattern is shown in Figure 13.

Our current software handles only periodic tilings (though there is no such limitation in the underlying technique). A tiling is represented by two translation vectors and a collection of untransformed polygons. Each polygon holds a list of transformations that map it to its occurences in a single translational unit. This information is sufficient to cover any region of the plane with a subset of the tiling. In the interactive designer, planar maps representing motifs are associated with the tile shapes and drawn with them. A standalone Java application can be used to construct periodic tilings by hand. The user can create regular polygonal tiles and snap them together, fill in irregular holes with new tiles, and specify translation vectors. Another program accepts a periodic tiling as input, and computes its rosette transform.

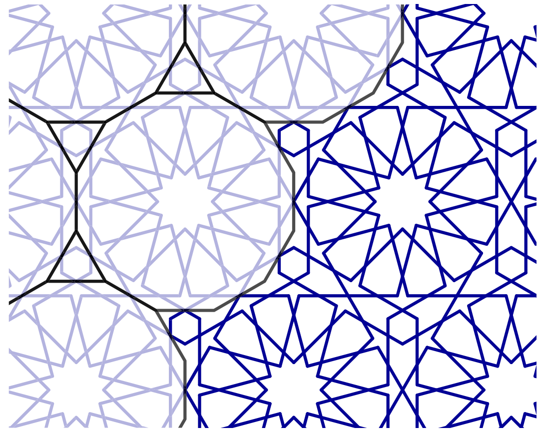


Figure 14: A classic Islamic star pattern that cannot easily be expressed using a combination of Hankin's method and the rosette transform.

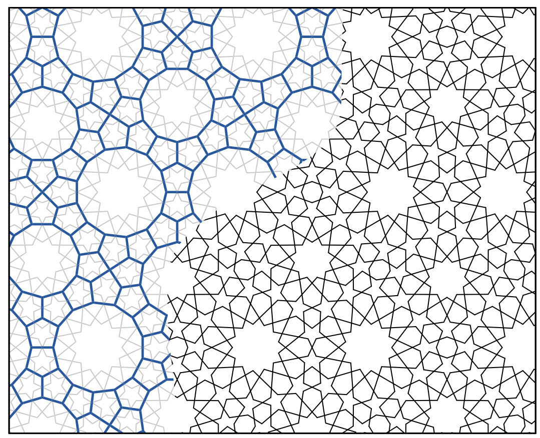


Figure 15: An unusual star pattern, reproduced from Bonner's manuscript, featuring 11- and 13-pointed stars.

The construction of Islamic parquet deformations requires many separate invocations of the inference algorithm, and is currently too slow to run interactively. It is implemented as a command-line application that builds a design from parameters specifying the bounding box and contact angle range.

6 Future work

Despite the power of the Hankin's method combined with the rosette transform, some important historical patterns are still out of reach. One example is given in Figure 14. This pattern is easy to construct from a tiling of dodecagons and triangles using Kaplan and Salesin's Najm. Their technique provides an explicit parameterization of the motifs in the dodecagons, which they call "extended rosettes". It is tempting to assume that we can arrive at the same pattern via two iterations of the rosette transform on this tiling. Closer examination reveals that where lines cross, they form angles of both $30°$ and $60°$. Any inference-based representation of this pattern would need either to allow multiple different contact angles, or non-greedy choices in the inference algorithm. Either way, more work is necessary to discover the principles that govern these choices.

Hankin tilings are better suited than Najm tilings for constructing some patterns. Bonner exhibits several remarkable designs with unusual combinations of motifs; Figure 15 shows a pattern with 11-pointed and 13-pointed rosettes. These remarkable designs are possible because the extra layer of irregular tiles can absorb the error when reconciling the incompatible angles of the regular 11- and 13-gons. Note that the tiling that produces this design is not the rosette transform of any tiling. Hankin tilings can therefore be considered "primitive" in some cases. Another primitive Hankin tiling is the "Altair" tiling in the bottom row of Figure 3. It would be interesting to examine what other unusual combinations of regular polygons could be accommodated in a single pattern in this way, and how to generate the associated Hankin tilings automatically.

One other unexplored direction in this work is its extension to non-Euclidean geometry, as demonstrated by Kaplan and Salesin [18]. This extension would be straightforward. The only change would be a generalization of the formula for scaling regular polygons in the rosette transform. It should be possible to express a general formula using the "absolute trigonometry" given in the appendix of their paper.

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