# Introduction to Quantum Information Processing

## QIC 710 / CS 667 / PH 767 / CO 681 / AM 871

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# Classical error correcting codes

# Binary symmetric channel

Each bit that goes through it has probability $\varepsilon$ of being flipped

**3-bit repetition code:**

- Encode each bit $b$ as $bbb$
- Decode each received message $b_1 b_2 b_3$ as $\text{majority}(b_1, b_2, b_3)$

This reduces the effective error probability per data bit to $3\varepsilon^2$ at a cost of tripling the message length ("rate" is $1/3$).

**A theorem about "good" classical codes:**

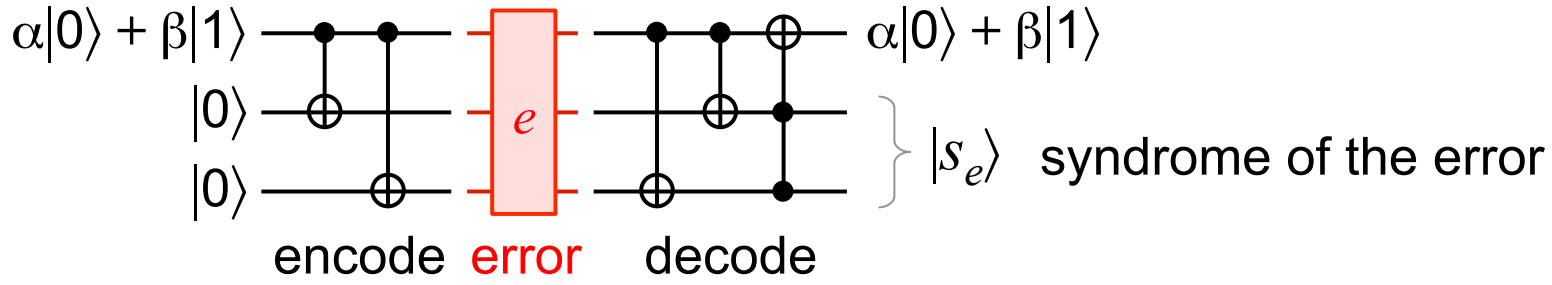
For all $\varepsilon < 1/2$, there exist encoding and decoding functions $E: \{0,1\}^n \to \{0,1\}^m$ and $D: \{0,1\}^m \to \{0,1\}^n$ such that $m/n$ is constant and the probability of ***any*** errors $\to 0$ as $n \to \infty$

$n/m$ is the "rate" of the code (reciprocal of message expansion)

# Shor's 9-qubit code

# 3-qubit code for one $X$-error

The following 3-qubit quantum code protects against up to one error, if the error can only be a quantum bit-flip (an $X$ operation)




Error can be any one of: $I\otimes I\otimes I$ $X\otimes I\otimes I$ $I\otimes X\otimes I$ $I\otimes I\otimes X$

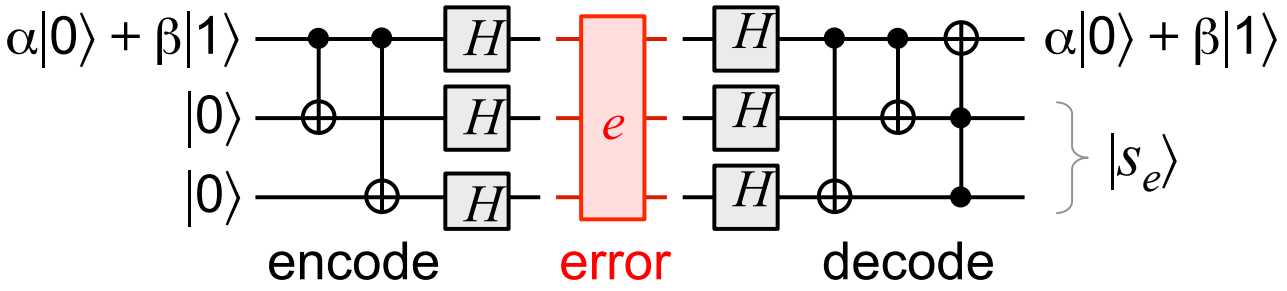
Corresponding syndrome: $|00\rangle$ $|11\rangle$ $|10\rangle$ $|01\rangle$

The essential property is that, in each case, the data $\alpha|0\rangle + \beta|1\rangle$ is shielded from (i.e., unaffected by) the error

What about $Z$ errors? This code leaves them intact …

# 3-qubit code for one $Z$-error

Using the fact that $HZH = X$, one can adapt the previous code to protect against $Z$-errors instead of $X$-errors

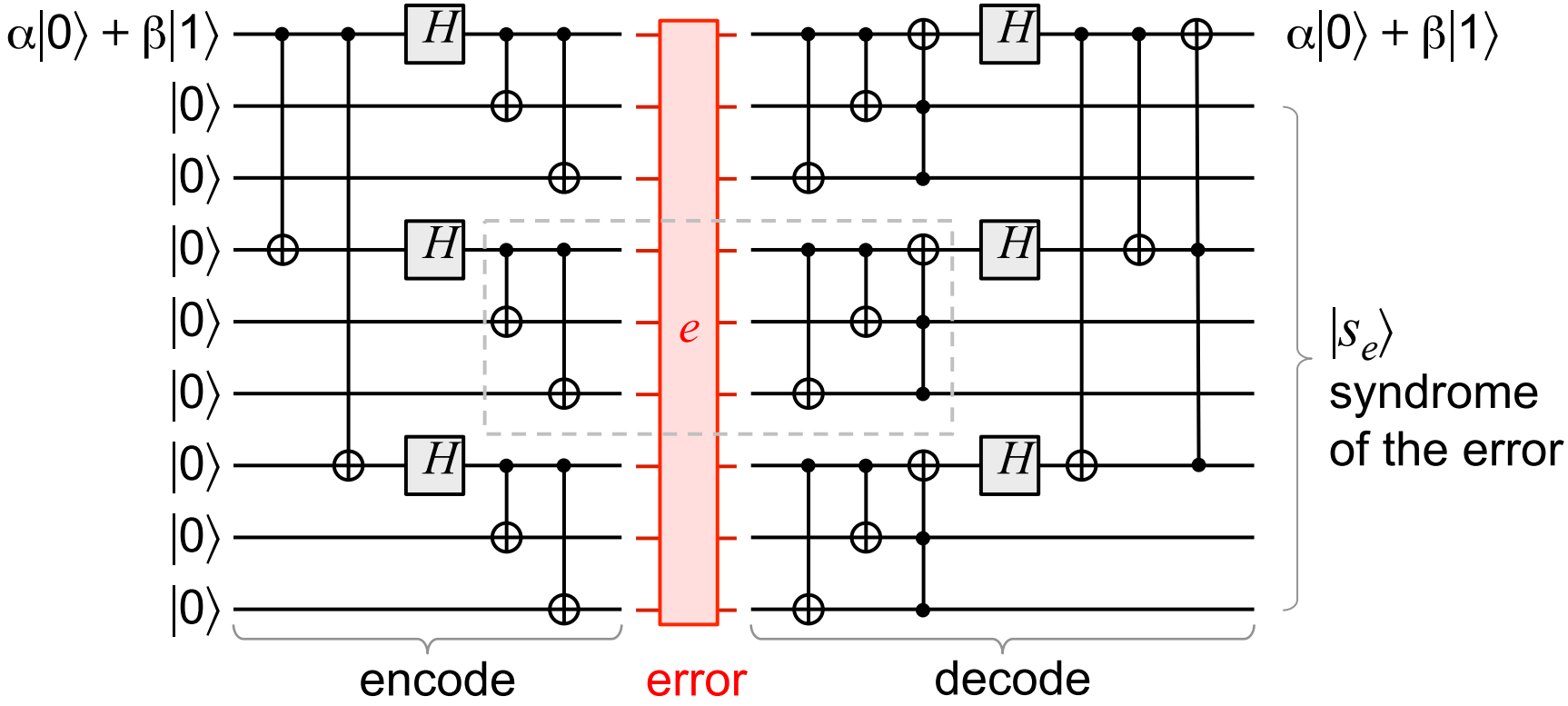



Error can be any one of: $I\otimes I\otimes I$ $Z\otimes I\otimes I$ $I\otimes Z\otimes I$ $I\otimes I\otimes Z$

This code leaves $X$-errors intact

Is there a code that protects against errors that are arbitrary one-qubit unitaries?

# Shor’ 9-qubit quantum code




The “inner” part corrects any single-qubit $X$-error

The “outer” part corrects any single-qubit $Z$-error

Since $Y = iXZ$, single-qubit $Y$-errors are also corrected

# Arbitrary one-qubit errors

Suppose that the error is some arbitrary one-qubit unitary operation $U$

Since there exist scalars $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$, such that

$$U = \lambda_1 I + \lambda_2 X + \lambda_3 Y + \lambda_4 Z$$

a straightforward calculation shows that, when a $U$-error occurs on the $k^{\text{th}}$ qubit, the output of the decoding circuit is

$$(\alpha|0\rangle + \beta|1\rangle)(\lambda_1 |s_{e_1}\rangle + \lambda_2 |s_{e_2}\rangle + \lambda_3 |s_{e_3}\rangle + \lambda_4 |s_{e_4}\rangle)$$

where $s_{e_1}$, $s_{e_2}$, $s_{e_3}$ and $s_{e_4}$ are the syndromes associated with the four errors ($I$, $X$, $Y$ and $Z$) on the $k^{\text{th}}$ qubit

Hence the code actually protects against ***any*** unitary one-qubit error (in fact the error can be any one-qubit quantum operation)

# CSS Codes

# Introduction to CSS codes

CSS codes (named after Calderbank, Shor, and Steane) are quantum error correcting codes that are constructed from classical error-correcting codes with certain properties

A classical ***linear*** code is one whose codewords (a subset of $\{0,1\}^m$) constitute a vector space

In other words, they are closed under linear combinations (here the underlying field is $\{0,1\}$ so the arithmetic is $\mathrm{mod}\ 2$)

# Examples of linear codes

For $m = 7$, consider these codes (which are linear):

basis for space

$C_2 = \{0000000,\ 1010101,\ 0110011,\ 1100110,$
$0001111,\ 1011010,\ 0111100,\ 1101001\}$

$C_1 = \{0000000,\ 1010101,\ 0110011,\ 1100110,$
$0001111,\ 1011010,\ 0111100,\ 1101001,$
$1111111,\ 0101010,\ 1001100,\ 0011001,$
$1110000,\ 0100101,\ 1000011,\ 0010110\}$

Note that the minimum Hamming distance between any pair of codewords is: 4 for $C_2$ and 3 for $C_1$

The minimum distances imply each code can correct one error

# Parity check matrix

Linear codes with maximum distance $d$ can correct up to $\lfloor \frac{d-1}{2} \rfloor$ bit-flip errors

Every linear code has an $n$x$m$ ***parity-check matrix*** $M$ such that:

- For every codeword $v$, $vM = 0$
- For any ***error-vector*** $e \in \{0,1\}^m$ with weight ≤ $\lfloor \frac{d-1}{2} \rfloor$, $e$ can be uniquely determined by multiplying the disturbed codeword (which is $v$+$e$ ) by $M$

**Error syndrome:** $(v+e)\,M = s_e$ and $e$ is a function of $s_e$ only

**Exercise:** determine the parity check matrix for $C_1$ and for $C_2$

# Encoding

Since , $|C_2| = 8$, it can encode 3 bits

To encode a 3-bit string $b = b_1 b_2 b_3$ in $C_2$, one multiplies $b$ (on the right) by an appropriate 3×7 ***generator matrix***

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Similarly, $C_1$ can encode 4 bits and an appropriate generator matrix for $C_1$ is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

# Orthogonal complement

For a linear code $C$, define its ***orthogonal complement*** as

$$C^{\perp} = \{w \in \{0,1\}^m : \text{for all } v \in C,\ w \cdot v = 0\}$$

(where $w \cdot v = \sum_{j=1}^{m} w_j v_j \bmod 2$ , the "dot product")

Note that, in the previous example, $C_2^{\perp} = C_1$ and $C_1^{\perp} = C_2$

We will use some of these properties in the CSS construction

# CSS construction

Let $C_2 \subset C_1 \subset \{0,1\}^m$ be two classical linear codes such that:

- The minimum distance of $C_1$ is $d$
- $C_2^{\perp} \subseteq C_1$

Let $r = \dim(C_1) - \dim(C_2) = \log(|C_1|/|C_2|)$

Then the resulting CSS code maps each $r$-qubit basis state $|b_1 \ldots b_r\rangle$ to some "coset state" of the form

$$\frac{1}{\sqrt{|C_2|}} \sum_{v \in C_2} |v + w\rangle$$

where $w = w_1 \ldots w_m$ is a linear function of $b_1 \ldots b_r$ chosen so that each value of $w$ occurs in a unique coset in the quotient space $C_1/C_2$

The quantum code can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors

# Example of CSS construction

For $m = 7$, for the $C_1$ and $C_2$ in the previous example we obtain these basis codewords:

$$|0_L\rangle = |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle$$

$$|1_L\rangle = |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle$$

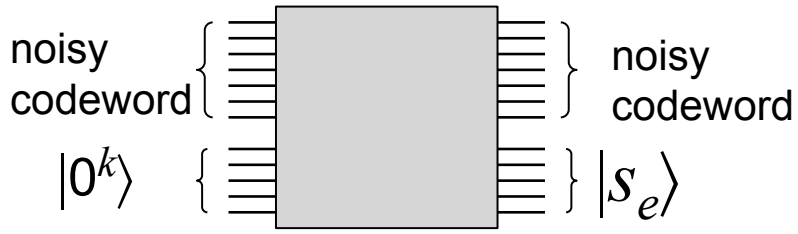
and the linear function mapping $b$ to $w$ can be given as $w = b \cdot G$

$$[w_1\ w_2\ w_3\ w_4\ w_5\ w_6\ w_7] = [b] \underbrace{[1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]}_{G}$$

There is a quantum circuit that transforms between $(\alpha|0\rangle + \beta|1\rangle)|0^{m-1}\rangle$ and $\alpha|0_L\rangle + \beta|1_L\rangle$

# CSS error correction I

Using the error-correcting properties of $C_1$, one can construct a quantum circuit that computes the syndrome $s$ for any combination of up to $d$ $X$-errors in the following sense




Once the syndrome $s_e$, has been computed, the $X$-errors can be determined and undone

What about $Z$-errors?

The above procedure for correcting $X$-errors has no effect on any $Z$-errors that occur

# CSS error correction II

Note that any $Z$-error is an $X$-error in the Hadamard basis

Changing to Hadamard basis is like changing from $C_2$ to $C_1$ since

$$H^{\otimes m}\left(\sum_{v\in C_2}|v\rangle\right)=\sum_{u\in C_2{}^{\perp}}|u\rangle \qquad \text{and} \qquad H^{\otimes m}\left(\sum_{v\in C_2}|v+w\rangle\right)=\sum_{u\in C_2{}^{\perp}}(-1)^{w\cdot u}|u\rangle$$

Applying $H^{\otimes n}$ to a superposition of basis codewords yields

$$H^{\otimes m}\left(\sum_{b\in\{0,1\}^r}\alpha_b\sum_{v\in C_2}|v+b\cdot G\rangle\right)=\sum_{b\in\{0,1\}^r}\alpha_b\sum_{u\in C_2{}^{\perp}}(-1)^{b\cdot G\cdot u}|u\rangle=\sum_{u\in C_2{}^{\perp}}\sum_{b\in\{0,1\}^r}\alpha_b(-1)^{b\cdot G\cdot u}|u\rangle$$

Note that, since $C_2{}^{\perp}\subseteq C_1$, this is a superposition of elements of $C_1$, so we can use the error-correcting properties of $C_1$ to correct

Then, applying Hadamards again, restores the codeword with up to $d$ $Z$-errors corrected

# CSS error correction III

The two procedures together correct up to $d$ errors that can each be either an $X$-error or a $Z$-error — and, since $Y = iXZ$, they can also be $Y$-errors

From this, a standard linearity argument can be applied to show that the code corrects up to $d$ arbitrary errors (that is, the error can be any quantum operation performed on up to $d$ qubits)

Since there exist pretty good classical codes that satisfy the properties needed for the CSS construction, this approach can be used to construct pretty good quantum codes

For any noise rate below some constant, the codes have:

- finite rate (message expansion by a constant factor: $r = n/m$)
- error probability approaching zero as $n \to \infty$