Introduction to Quantum Information Processing

CS 667 / PH 767 / CO 681 / AM 871

Lecture 20 (2009)

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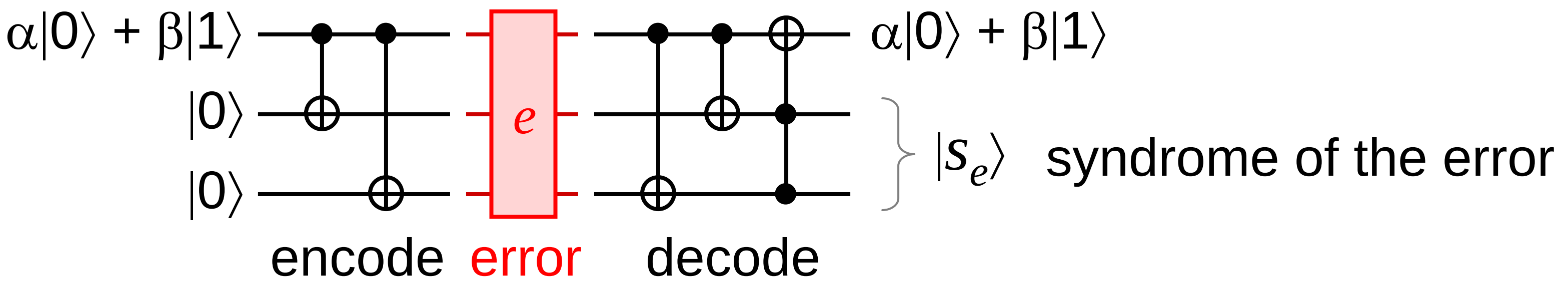
Shor’s 9-qubit code

3-qubit code for one X-error

The following 3-qubit quantum code protects against up to one error, if the error can only be a quantum bit-flip (an X operation)



Error can be any one of: $I\otimes I\otimes I$ $\quad X\otimes I\otimes I$ $\quad I\otimes X\otimes I$ $\quad I\otimes I\otimes X$

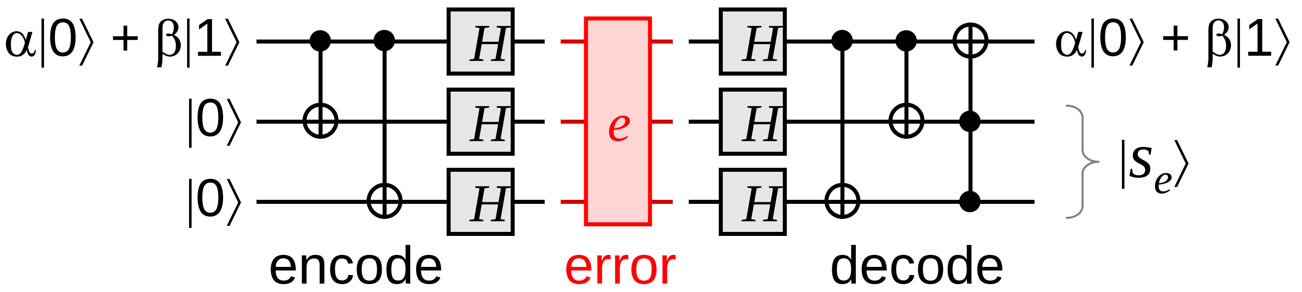
Corresponding syndrome: $|00\rangle$ $\quad |11\rangle$ $\quad |10\rangle$ $\quad |01\rangle$

The essential property is that, in each case, the data $\alpha|0\rangle + \beta|1\rangle$ is shielded from (i.e., unaffected by) the error

What about Z errors? This code leaves them intact …

3-qubit code for one Z-error

Using the fact that $HZH = X$, one can adapt the previous code to protect against Z-errors instead of X-errors

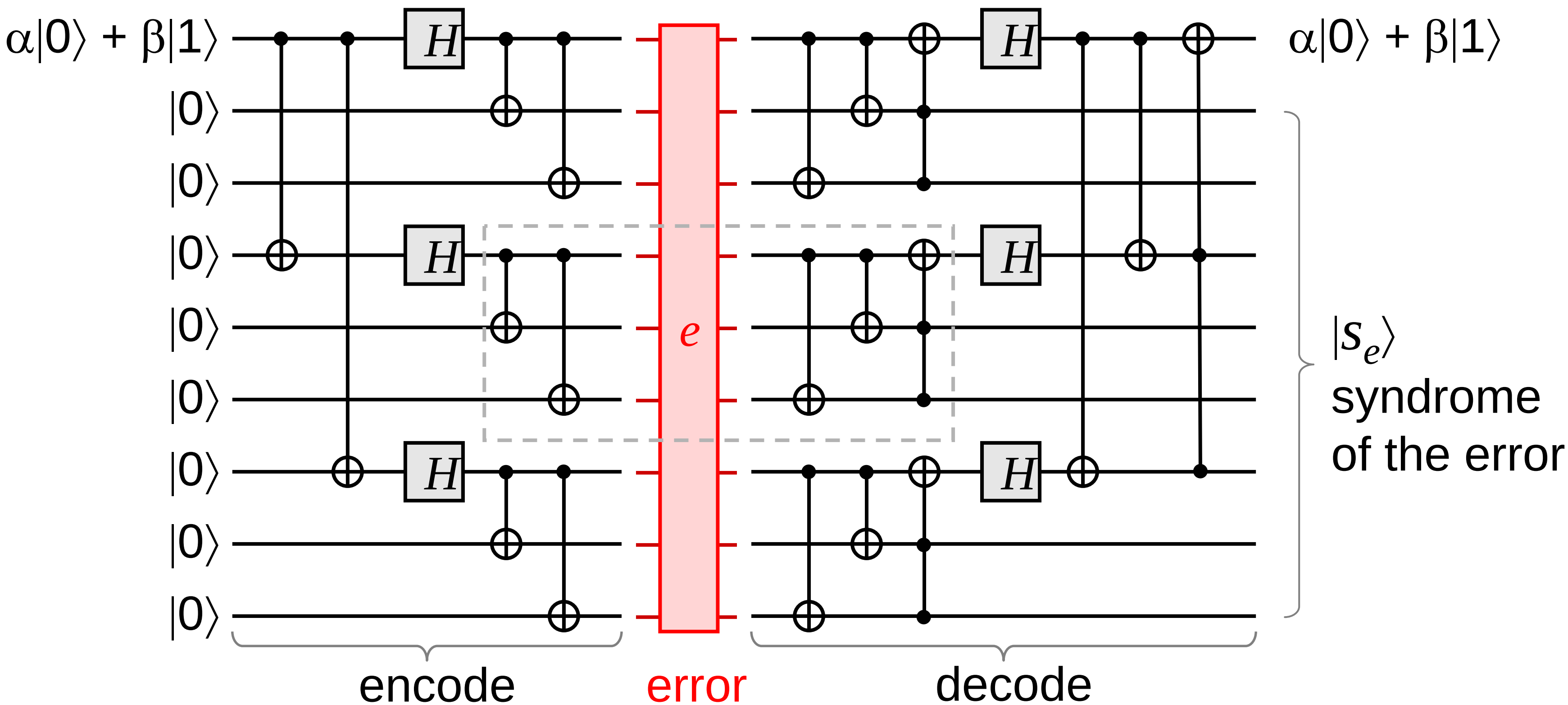


Error can be any one of: $I\otimes I\otimes I$ $Z\otimes I\otimes I$ $I\otimes Z\otimes I$ $I\otimes I\otimes Z$

This code leaves X-errors intact

Is there a code that protects against errors that are arbitrary one-qubit unitaries?

Shor' 9-qubit quantum code



The "inner" part corrects any single-qubit X-error

The "outer" part corrects any single-qubit Z-error

Since $Y = iXZ$, single-qubit Y-errors are also corrected

Arbitrary one-qubit errors

Suppose that the error is some arbitrary one-qubit unitary operation U

Since there exist scalars $\lambda_1, \lambda_2, \lambda_3$ and λ_4, such that

$$U = \lambda_1 I + \lambda_2 X + \lambda_3 Y + \lambda_4 Z$$

a straightforward calculation shows that, when a U-error occurs on the k^{th} qubit, the output of the decoding circuit is

$$(\alpha|0\rangle + \beta|1\rangle)(\lambda_1 |s_{e_1}\rangle + \lambda_2 |s_{e_2}\rangle + \lambda_3 |s_{e_3}\rangle + \lambda_4 |s_{e_4}\rangle)$$

where s_{e_1}, s_{e_2}, s_{e_3} and s_{e_4} are the syndromes associated with the four errors (I, X, Y and Z) on the k^{th} qubit

Hence the code actually protects against ***any*** unitary one-qubit error (in fact the error can be any one-qubit quantum operation)

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CSS Codes

Introduction to CSS codes

CSS codes (named after Calderbank, Shor, and Steane) are quantum error correcting codes that are constructed from classical error-correcting codes with certain properties

A classical ***linear*** code is one whose codewords (a subset of $\{0,1\}^n$) constitute a vector space

In other words, they are closed under linear combinations (here the underlying field is $\{0,1\}$ so the arithmetic is mod 2)

Examples of linear codes

For $n = 7$, consider these codes (which are linear):

basis for space

C_2 = {0000000, 1010101, 0110011, 1100110,
0001111, 1011010, 0111100 , 1101001}

C_1 = {0000000, 1010101, 0110011, 1100110,
0001111, 1011010, 0111100, 1101001,
1111111, 0101010, 1001100, 0011001,
1110000, 0100101, 1000011 , 0010110}

Note that the minimum Hamming distance between any pair of codewords is: 4 for C_2 and 3 for C_1

The minimum distances imply each code can correct one error

Parity check matrix

Linear codes with maximum distance d can correct up to $\lfloor (d-1)/2 \rfloor$ bit-flip errors

Moreover, for any ***error-vector*** $e \in \{0,1\}^n$ with weight $\leq \lfloor (d-1)/2 \rfloor$, e can be uniquely determined by multiplying the disturbed codeword $v+e$ by a ***parity-check matrix*** M

More precisely, $(v+e)\,M = s_e$ (called the ***error syndrome***), and e is a function of s_e only

Exercise: determine the parity check matrix for C_1 and for C_2

Encoding

Since , $|C_2| = 8$, it can encode 3 bits

To encode a 3-bit string $b = b_1 b_2 b_3$ in C_2, one multiplies b (on the right) by an appropriate 3×7 ***generator matrix***

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Similarly, C_1 can encode 4 bits and an appropriate generator matrix for C_1 is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Orthogonal complement

For a linear code C, define its ***orthogonal complement*** as

$$C^{\perp} = \{w \in \{0,1\}^n : \text{for all } v \in C,\ w \cdot v = 0\}$$

(where $w \cdot v = \sum_{j=1}^{n} w_j v_j \bmod 2$, the "dot product")

Note that, in the previous example, $C_2^{\perp} = C_1$ and $C_1^{\perp} = C_2$

We will use some of these properties in the CSS construction

CSS construction

Let $C_2 \subset C_1 \subset \{0,1\}^n$ be two classical linear codes such that:

- The minimum distance of C_1 is d
- $C_2^{\perp} \subseteq C_1$

Let $r = \dim(C_1) - \dim(C_2) = \log(|C_1|/|C_2|)$

Then the resulting CSS code maps each r-qubit basis state $|b_1 \ldots b_r\rangle$ to some "coset state" of the form

$$\frac{1}{\sqrt{|C_2|}} \sum_{v \in C_2} |v + w\rangle$$

where $w = w_1 \ldots w_n$ is a linear function of $b_1 \ldots b_r$ chosen so that each value of w occurs in a unique coset in the quotient space C_1/C_2

The quantum code can correct $\lfloor (d-1)/2 \rfloor$ errors

Example of CSS construction

For $n = 7$, for the C_1 and C_2 in the previous example we obtain these basis codewords:

$$|0_L\rangle = |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$
$$+ |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle$$

$$|1_L\rangle = |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$$
$$+ |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle$$

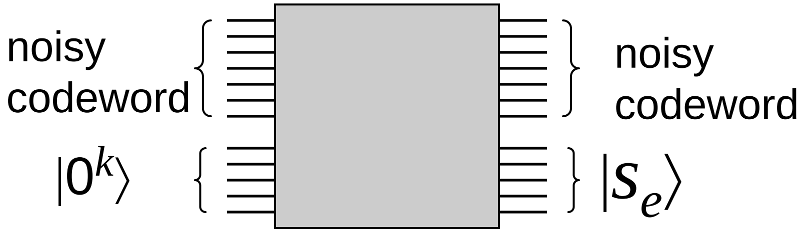
and the linear function maping b to w can be given as $w = b \cdot G$

$$\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{G}$$

There is a quantum circuit that transforms between $(\alpha|0\rangle + \beta|1\rangle)|0^{n-1}\rangle$ and $\alpha|0_L\rangle + \beta|1_L\rangle$

CSS error correction I

Using the error-correcting properties of C_1, one can construct a quantum circuit that computes the syndrome s for any combination of up to d X-errors in the following sense



Once the syndrome s_e, has been computed, the X-errors can be determined and undone

What about Z-errors?

The above procedure for correcting X-errors has no effect on any Z-errors that occur

CSS error correction II

Note that any Z-error is an X-error in the Hadamard basis

The codewords in the Hadamard basis can be computed using

$$H^{\otimes n}\left(\sum_{v\in C_2}|v\rangle\right)=\sum_{u\in C_2^{\perp}}|u\rangle \qquad \text{and} \qquad H^{\otimes n}\left(\sum_{v\in C_2}|v+w\rangle\right)=\sum_{u\in C_2^{\perp}}(-1)^{w\cdot u}|u\rangle$$

Applying $H^{\otimes n}$ to a superposition of basis codewords yields

$$H^{\otimes n}\left(\sum_{b\in\{0,1\}^r}\alpha_b\sum_{v\in C_2}|v+b\cdot G\rangle\right)=\sum_{b\in\{0,1\}^r}\alpha_b\sum_{u\in C_2^{\perp}}(-1)^{b\cdot G\cdot u}|u\rangle=\sum_{u\in C_2^{\perp}}\sum_{b\in\{0,1\}^r}\alpha_b(-1)^{b\cdot G\cdot u}|u\rangle$$

Note that, since $C_2^{\perp}\subseteq C_1$, this is a superposition of elements of C_1, so we can use the error-correcting properties of C_1 to correct

Then, applying Hadamards again, restores the codeword with up to d Z-errors corrected

CSS error correction III

The two procedures together correct up to d errors that can each be either an X-error or a Z-error — and, since $Y = iXZ$, they can also be Y-errors

From this, a standard linearity argument can be applied to show that the code corrects up to d arbitrary errors (that is, the error can be any quantum operation performed on up to d qubits)

Since there exist pretty good classical codes that satisfy the properties needed for the CSS construction, this approach can be used to construct pretty good quantum codes

For any noise rate below some constant, the codes have:

- finite rate (message expansion by a constant factor, r to cr)
- error probability approaching zero as $r \to \infty$