Introduction to Quantum Information Processing
CS 667 / PH 767 / CO 681 / AM 871

Lecture 14 (2011)

Richard Cleve
DC 2117
cleve@cs.uwaterloo.ca

Bloch sphere for qubits

Bloch sphere for qubits (1)

Consider the set of all 2x2 density matrices ρ

They have a nice representation in terms of the ***Pauli matrices***:

$$\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Note that these matrices—combined with I—form a ***basis*** for the vector space of all 2x2 matrices

We will express density matrices ρ in this basis

Note that the coefficient of I is ½, since X, Y, Y are traceless

Bloch sphere for qubits (2)

We will express $\rho = \dfrac{I + c_x X + c_y Y + c_z Z}{2}$

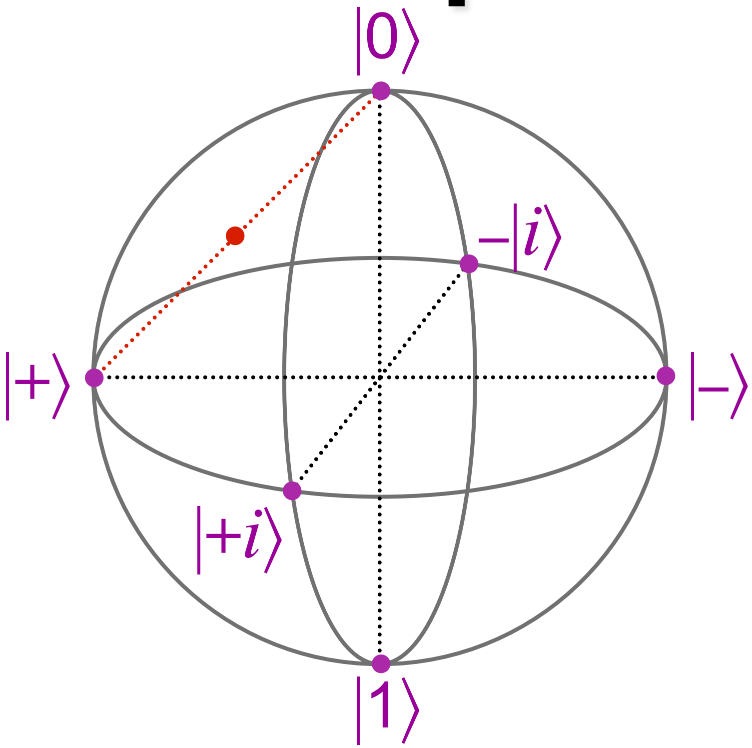
First consider the case of pure states $|\psi\rangle\langle\psi|$, where, without loss of generality, $|\psi\rangle = \cos(\theta)|0\rangle + e^{2i\phi}\sin(\theta)|1\rangle$ $(\theta, \phi \in \mathbf{R})$

$$\rho = \begin{bmatrix} \cos^2\theta & e^{-i2\varphi}\cos\theta\sin\theta \\ e^{i2\varphi}\cos\theta\sin\theta & \sin^2\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\cos(2\theta) & e^{-i2\varphi}\sin(2\theta) \\ e^{i2\varphi}\sin(2\theta) & 1-\cos(2\theta) \end{bmatrix}$$

Therefore $c_z = \cos(2\theta)$, $c_x = \cos(2\phi)\sin(2\theta)$, $c_y = \sin(2\phi)\sin(2\theta)$

These are ***polar coordinates*** of a unit vector $(c_x, c_y, c_z) \in \mathbf{R}^3$

Bloch sphere for qubits (3)



$$|+\rangle = |0\rangle + |1\rangle$$
$$|-\rangle = |0\rangle - |1\rangle$$
$$|+i\rangle = |0\rangle + i|1\rangle$$
$$|-i\rangle = |0\rangle - i|1\rangle$$

Note that ***orthogonal*** corresponds to ***antipodal*** here

Pure states are on the surface, and mixed states are inside (being weighted averages of pure states)

Distinguishing mixed states

Distinguishing mixed states (1)

What's the best distinguishing strategy between these two mixed states?

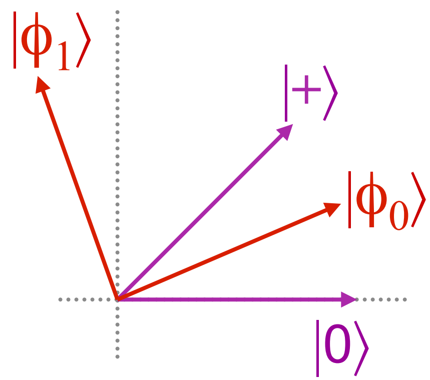
$\begin{cases} |0\rangle & \text{with prob. } 1/2 \\ |0\rangle + |1\rangle & \text{with prob. } 1/2 \end{cases}$

$\begin{cases} |0\rangle & \text{with prob. } 1/2 \\ |1\rangle & \text{with prob. } 1/2 \end{cases}$

$$\rho_1 = \begin{bmatrix} 3/4 & 1/2 \\ 1/2 & 1/4 \end{bmatrix}$$

$$\rho_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ρ_1 also arises from this orthogonal mixture:



… as does ρ_2 from:

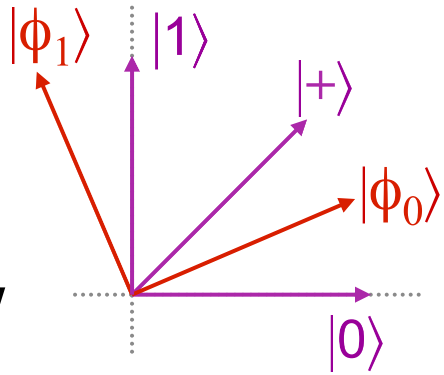
$\begin{cases} |\phi_0\rangle & \text{with prob. } \cos^2(\pi/8) \\ |\phi_1\rangle & \text{with prob. } \sin^2(\pi/8) \end{cases}$

$\begin{cases} |\phi_0\rangle & \text{with prob. } 1/2 \\ |\phi_1\rangle & \text{with prob. } 1/2 \end{cases}$

Distinguishing mixed states (2)

We've effectively found an orthonormal basis $|\phi_0\rangle$, $|\phi_1\rangle$ in which both density matrices are diagonal:

$$\rho_2' = \begin{bmatrix} \cos^2(\pi/8) & 0 \\ 0 & \sin^2(\pi/8) \end{bmatrix} \qquad \rho_1' = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Rotating $|\phi_0\rangle$, $|\phi_1\rangle$ to $|0\rangle$, $|1\rangle$ the scenario can now be examined using classical probability theory:

Distinguish between two ***classical*** coins, whose probabilities of "heads" are $\cos^2(\pi/8)$ and ½ respectively (details: exercise)

Question: what do we do if we aren't so lucky to get two density matrices that are simultaneously diagonalizable?

General quantum operations

General quantum operations (1)

Also known as:
"quantum channels"
"completely positive trace preserving maps",
"admissible operations"

Let $A_1, A_2, \ldots, A_m$ be matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$

Then the mapping $\rho \mapsto \sum_{j=1}^{m} A_j \rho A_j^\dagger$ is a general quantum op

Note: $A_1, A_2, \ldots, A_m$ do not have to be square matrices

Example 1 (unitary op): applying U to ρ yields $U\rho U^\dagger$

General quantum operations (2)

Example 2 (decoherence): let $A_0 = |0\rangle\langle 0|$ and $A_1 = |1\rangle\langle 1|$

This quantum op maps ρ to $|0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$

For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,
$$\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \mapsto \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$$

Corresponds to measuring ρ "without looking at the outcome"

After looking at the outcome, ρ becomes $\begin{cases} |0\rangle\langle 0| & \text{with prob. } |\alpha|^2 \\ |1\rangle\langle 1| & \text{with prob. } |\beta|^2 \end{cases}$

General quantum operations (3)

Example 3 (discarding the second of two qubits):

Let $A_0 = I \otimes \langle 0| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $A_1 = I \otimes \langle 1| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

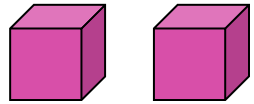
States of the form $\rho \otimes \sigma$ (product states) become ρ

State $\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\langle 00| + \frac{1}{\sqrt{2}}\langle 11|\right)$ becomes $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note 1: it's the same density matrix as for $((½, |0\rangle), (½, |1\rangle))$

Note 2: the operation is called the ***partial trace*** $\mathrm{Tr}_2\, \rho$

More about the partial trace

Two quantum registers  in states σ and μ (resp.) are ***independent*** when the combined system is in state $\rho = \sigma \otimes \mu$

If the 2nd register is discarded, state of the 1st register remains σ

In general, the state of a two-register system may not be of the form $\sigma \otimes \mu$ (it may contain ***entanglement*** or ***correlations***)

The ***partial trace*** Tr_2 gives the effective state of the first register

For d-dimensional registers, Tr_2 is defined with respect to the operators $A_k = I \otimes \langle \phi_k |$, where $|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{d-1}\rangle$ can be any orthonormal basis

The **partial trace** $\mathrm{Tr}_2\, \rho$, can also be characterized as the unique linear operator satisfying the identity $\mathrm{Tr}_2(\sigma \otimes \mu) = \sigma$

Partial trace continued

For 2-qubit systems, the partial trace is explicitly

$$\mathrm{Tr}_2 \begin{bmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{bmatrix} = \begin{bmatrix} \rho_{00,00} + \rho_{01,01} & \rho_{00,10} + \rho_{01,11} \\ \rho_{10,00} + \rho_{11,01} & \rho_{10,10} + \rho_{11,11} \end{bmatrix}$$

and

$$\mathrm{Tr}_1 \begin{bmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{bmatrix} = \begin{bmatrix} \rho_{00,00} + \rho_{10,10} & \rho_{00,01} + \rho_{10,11} \\ \rho_{01,00} + \rho_{11,10} & \rho_{01,01} + \rho_{11,11} \end{bmatrix}$$

General quantum operations (4)

Example 4 (adding an extra qubit):

Just one operator $A_0 = I \otimes |0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

States of the form ρ become $\rho \otimes |0\rangle\langle 0|$

More generally, to add a register in state $|\phi\rangle$, use the operator $A_0 = I \otimes |\phi\rangle$

POVM measurements

(POVM = Positive Operator Valued Measre)

POVM measurements (1)

Let $A_1, A_2, \ldots, A_m$ be matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$

Corresponding ***POVM measurement*** is a stochastic operation on ρ that, with probability $\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)$, produces outcome:

$$\begin{cases} j & \text{(classical information)} \\ \dfrac{A_j \rho A_j^\dagger}{\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)} & \text{(the collapsed quantum state)} \end{cases}$$

Example 1: $A_j = |\phi_j\rangle\langle\phi_j|$ (orthogonal projectors)

This reduces to our previously defined measurements …

POVM measurements (2)

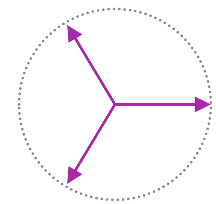
When $A_j = |\phi_j\rangle\langle\phi_j|$ are orthogonal projectors and $\rho = |\psi\rangle\langle\psi|$,

$$\begin{aligned}\mathrm{Tr}\left(A_j \rho A_j^\dagger\right) &= \mathrm{Tr}|\phi_j\rangle\langle\phi_j|\psi\rangle\langle\psi|\phi_j\rangle\langle\phi_j| \\ &= \langle\phi_j|\psi\rangle\langle\psi|\phi_j\rangle\langle\phi_j|\phi_j\rangle \\ &= |\langle\phi_j|\psi\rangle|^2\end{aligned}$$

Moreover, $$\frac{A_j \rho A_j^\dagger}{\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)} = \frac{|\varphi_j\rangle\langle\varphi_j|\psi\rangle\langle\psi|\varphi_j\rangle\langle\varphi_j|}{\left|\langle\varphi_j|\psi\rangle\right|^2} = |\varphi_j\rangle\langle\varphi_j|$$

POVM measurements (3)

Example 3 (trine state "measurent"):



Let $|\varphi_0\rangle = |0\rangle$, $|\varphi_1\rangle = -1/2|0\rangle + \sqrt{3}/2|1\rangle$, $|\varphi_2\rangle = -1/2|0\rangle - \sqrt{3}/2|1\rangle$

Define $A_0 = \sqrt{2/3}|\varphi_0\rangle\langle\varphi_0| = \sqrt{\frac{2}{3}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A_1 = \sqrt{2/3}|\varphi_1\rangle\langle\varphi_1| = \frac{1}{4}\begin{bmatrix} \sqrt{2/3} & +\sqrt{2} \\ +\sqrt{2} & \sqrt{6} \end{bmatrix}$ $A_2 = \sqrt{2/3}|\varphi_2\rangle\langle\varphi_2| = \frac{1}{4}\begin{bmatrix} \sqrt{2/3} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{6} \end{bmatrix}$

Then $A_0^\dagger A_0 + A_1^\dagger A_1 + A_2^\dagger A_2 = I$

If the input itself is an unknown trine state, $|\varphi_k\rangle\langle\varphi_k|$, then the probability that classical outcome is k is 2/3 = 0.6666…

POVM measurements (4)

Often POVMs arise in contexts where we only care about the classical part of the outcome (not the residual quantum state)

The probability of outcome ***j*** is $\mathrm{Tr}\left(A_j \rho A_j^\dagger\right) = \mathrm{Tr}\left(\rho A_j^\dagger A_j\right)$

Simplified definition for POVM measurements:

Let E_1, E_2, …, E_m be positive definite and such that $\sum_{j=1}^{m} E_j = I$

The probability of outcome ***j*** is $\mathrm{Tr}\left(\rho E_j\right)$

This is usually the way POVM measurements are defined

"Mother of all operations"

Let $A_{1,1}, A_{1,2}, \ldots, A_{1,m_1}$
$A_{2,1}, A_{2,2}, \ldots, A_{2,m_2}$
$A_{k,1}, A_{k,2}, \ldots, A_{k,m_k}$
satisfy $\sum_{j=1}^{k} \sum_{i=1}^{m_j} A_{j,i}^{\dagger} A_{j,i} = I$

Then there is a quantum operation that, on input ρ, produces with probability $\sum_{i=1}^{m_j} \mathrm{Tr}\left(A_{j,i} \rho A_{j,i}^{\dagger}\right)$ the state:

j (classical information)

$$\frac{\sum_{i=1}^{m_j} A_{j,i} \rho A_{j,i}^{\dagger}}{\sum_{i=1}^{m_j} \mathrm{Tr}\left(A_{j,i} \rho A_{j,i}^{\dagger}\right)}$$ (the collapsed quantum state)