CS667/CO681/PH767 Quantum Information Processing (Fall 07)

Assignment 4

Due date: November 15, 2007

1. **Amplitude amplification.** This is a generalization of Grover's algorithm. We are given a unitary U_f that computes $f : \{0,1\}^n \to \{0,1\}$ in the usual way and our goal is to find $x_0 \in \{0,1\}^n$ such that $f(x_0) = 1$ (for simplicity, assume that this x_0 is unique). In addition to U_f, we are given a unitary operation A (acting on n qubits) such that $A|00\dots0\rangle$ overlaps with $|x_0\rangle$ in the sense that $|\langle x_0|A|00\dots0\rangle|^2 = p$ for some p that is small, but larger than $1/2^n$. Assume we have black boxes for both A and $A^\dagger$.

   The obvious classically-minded approach is to repeatedly compute $A|00\dots0\rangle$, measure in the computational basis, and then plug the resulting state $|x\rangle$ into U_f to check if $f(x) = 1$. Each such trial succeeds with probability p so, on average, order $1/p$ trials are needed until x_0 is found. The resulting algorithm makes $O(1/p)$ queries to U_f and to A.

   We will show that there is a quantum algorithm making $O(\sqrt{1/p})$ queries to U_f, A, and $A^\dagger$ that succeeds with a positive constant probability.

   (a) Consider the subspace spanned by $|x_0\rangle$ and $A|00\dots0\rangle$. Show that $-AU_0A^\dagger U_f$ is a rotation in this space by 2θ, where θ is an angle such that $\sin\theta = |\langle x_0|A|00\dots0\rangle|$. (Recall that U_0 is the unitary such that $U_0|x\rangle = (-1)^{[x=00\dots0]}|x\rangle$, for all $x \in \{0,1\}^n$.)

   (b) Show that, by setting k to some value that is order $O(\sqrt{1/p})$, $(-AU_0A^\dagger U_f)^k A|00\dots0\rangle$ is close to $|x_0\rangle$ in the sense that the absolute value of their inner product is at least 3/4. (Note that this implies an $O(\sqrt{1/p})$ query algorithm.)

2. **Basic properties of density matrices.**

   (a) A density matrix ρ corresponds to a *pure* state if and only if $\rho = |\psi\rangle\langle\psi|$. Show that ρ corresponds to a pure state if and only if $\mathrm{Tr}(\rho^2) = 1$.

   (b) Show that for any operator ρ that is Hermitian, positive semidefinite, and has trace 1, there is a probabilistic mixture of pure states whose density matrix is ρ.

   (c) Show that every 2×2 density matrix ρ can be expressed as an *equally weighted mixture* of pure states. That is

   $$\rho = \tfrac{1}{2}|\psi_1\rangle\langle\psi_1| + \tfrac{1}{2}|\psi_2\rangle\langle\psi_2|$$

   for states $|\psi_1\rangle$ and $|\psi_2\rangle$ (note that, in general, the two states will not be orthogonal).

3. **Separable versus entangled mixed states.** A pure bipartite state shared by A(lice) and B(ob) is entangled iff it is not of the form $\rho_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$ (the subscripts are to help keep track of who has what). In the case of mixed states, it's more complicated because a bipartite state can include classical correlations. For example, the state $\frac{1}{2}|00\rangle\langle 00|_{AB} + \frac{1}{2}|11\rangle\langle 11|_{AB}$ corresponds to A and B sharing $|00\rangle_{AB}$ with probability $\frac{1}{2}$ and $|00\rangle_{AB}$ with

probability $\frac{1}{2}$. Such a state is classically correlated but has no entanglement—it could never be used for, say, teleportation or superdense coding.

A bipartite state ρ_{AB} is *separable* iff it can be written as a mixture of product states:

$$\rho_{AB} = \sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j|_A \otimes |\phi_j\rangle\langle\phi_j|_B \quad \text{(where, for all } j,\ p_j \geq 0\text{)}.$$

(Note that $|\psi_j\rangle\langle\psi_j|_A \otimes |\phi_j\rangle\langle\phi_j|_B = (|\psi_j\rangle_A \otimes |\phi_j\rangle_B)(\langle\psi_j|_A \otimes \langle\phi_j|_B)$.) A mixed state is deemed *entangled* if it is not separable.

(a) Show that the mixture of two Bell states $\frac{1}{2}|\Phi^+\rangle\langle\Phi^+|_{AB} + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-|_{AB}$ is separable by giving another expression for its density matrix that's a mixture of product states.

(b) Is $\frac{1}{2}|\Phi^+\rangle\langle\Phi^+|_{AB} + \frac{1}{2}|00\rangle\langle 00|_{AB}$ entangled or separable? Justify your answer.

4. **Constructing a specific Krauss operation.**

(a) Give a general quantum operation that maps each two-qubit basis state to another basis state as follows. For all $a, b \in \{0, 1\}$, $|a\rangle|b\rangle \mapsto |a\rangle|a \wedge b\rangle$. (Note that this mapping cannot be implemented by a unitary operation, since it is not reversible.)

(b) What quantum state does the operation in part (a) map $\frac{1}{\sqrt{2}}|1\rangle(|0\rangle + |1\rangle)$ to? (You can describe the state in terms of its density matrix.)

5. **Is the transpose a valid quantum operation?** Here we consider an operation on qubits that we denote by Λ, defined as $\Lambda(\rho) = \rho^T$ for each density matrix ρ (where ρ^T is the transpose of T).

(a) Give an example of a one-qubit pure state $|\psi\rangle$ such that $\Lambda(|\psi\rangle\langle\psi|)$ is a pure state orthogonal to $|\psi\rangle$.

(b) Show that Λ is not a valid quantum operation in that there is no general quantum operation, of the form $\rho \mapsto \sum_{k=1}^{m} A_k \rho A_k^\dagger$, where $\sum_{k=1}^{m} A_k^\dagger A_k = I$ that implements Λ. For this question, you may use without proof the fact that any general quantum operation can be extended so that it acts on part of a larger system. More precisely, you may assume that if Λ is a valid one-qubit quantum operation then there is a valid two-qubit quantum operation that, for all pairs of density matrices σ and ρ, maps $\sigma \otimes \rho$ to $\sigma \otimes \Lambda(\rho)$.

(Hint: consider how this extended operation would act on the state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.)