A Tolerant Independent Set Tester
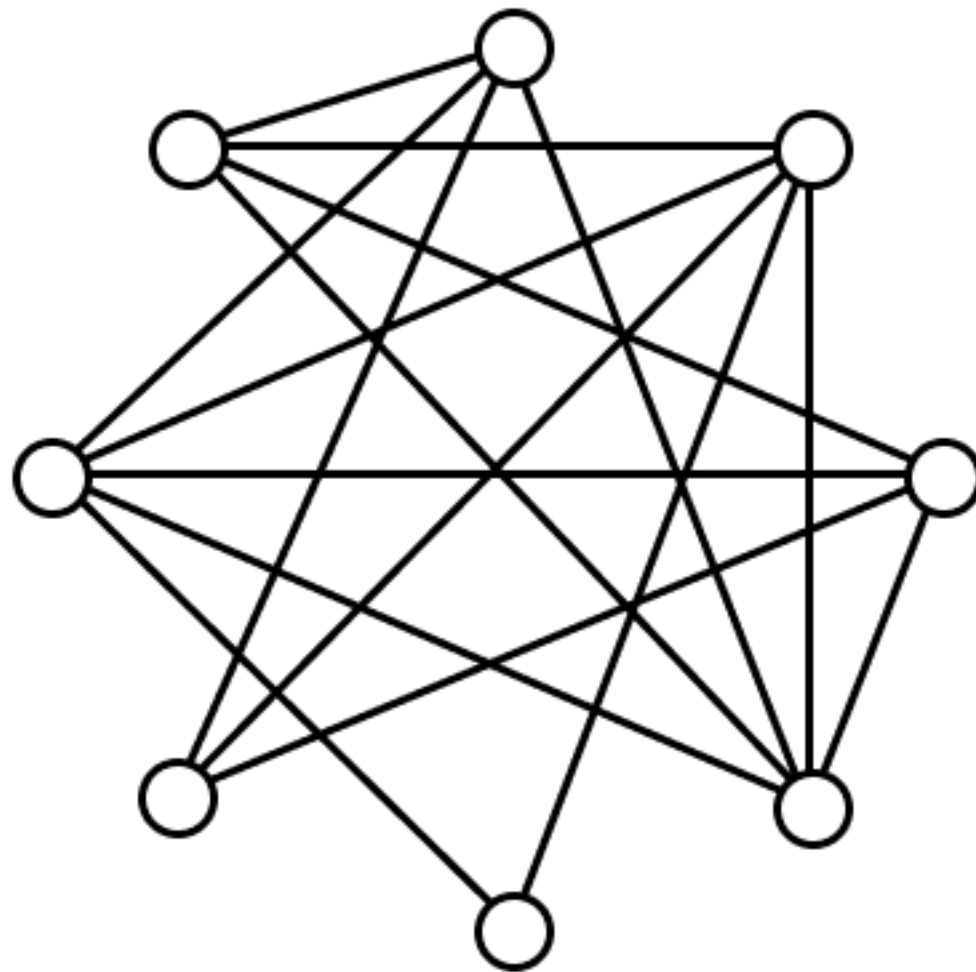
Cameron Seth

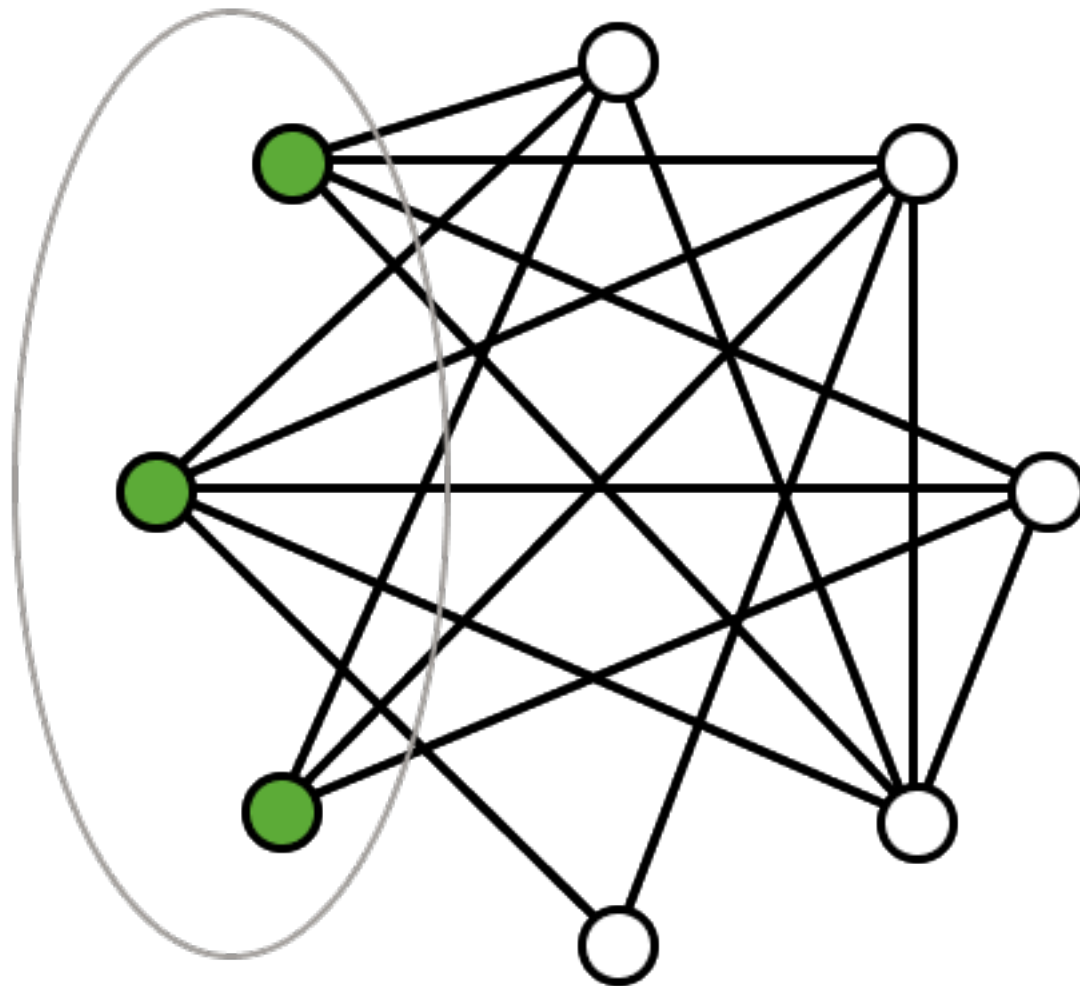**March 12 2025**

UNIVERSITY OF
WATERLOO

Independent Set Problem

Given a graph G on n vertices, does it have an independent set of size ρn?
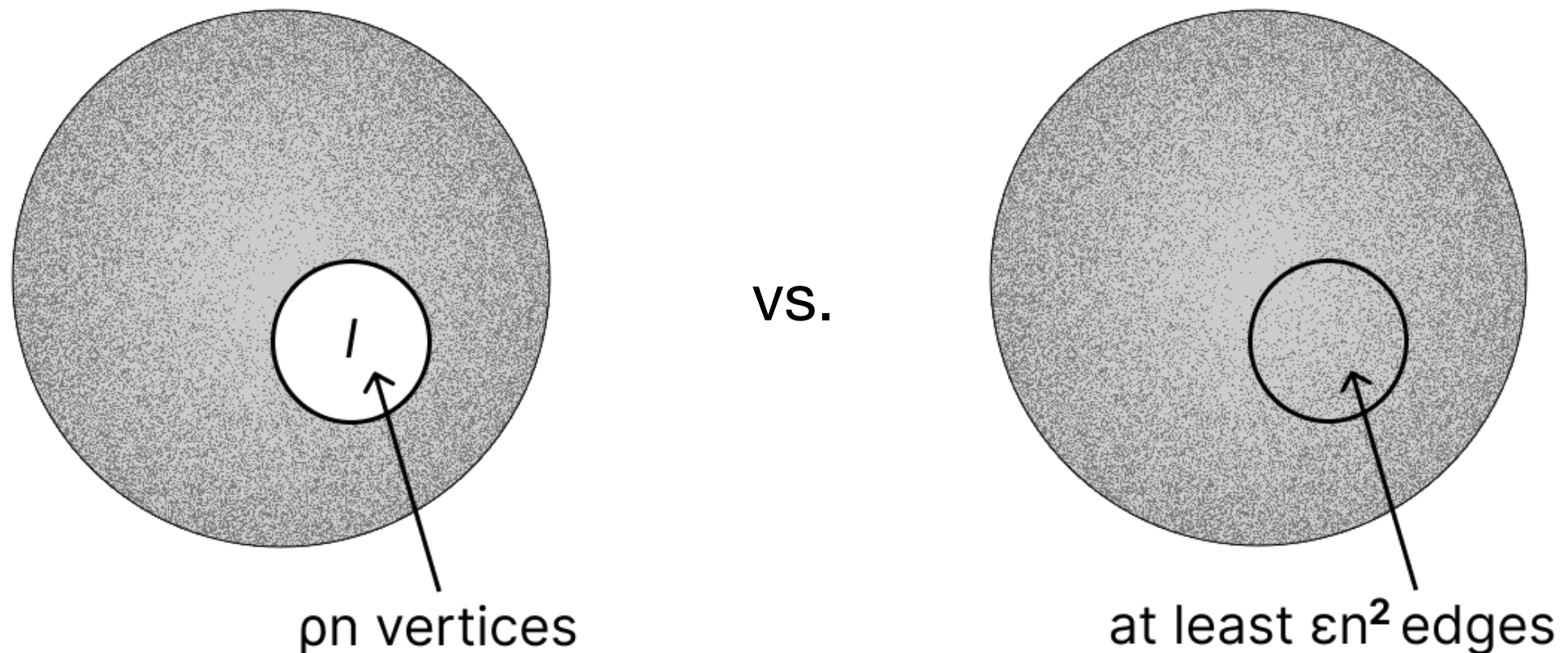
Independent Set Problem

Given a graph G on n vertices, does it have an independent set of size ρn?

Testing Independent Sets

Problem: Distinguish between the cases:

(i) G has a ρn independent set, and

(ii) every induced subgraph of size ρn has at least ϵn^2 edges

vs.

ρn vertices

at least ϵn^2 edges

Theorem: Inspecting a random subgraph on $\tilde{O}(\rho/\epsilon^4)$ vertices is sufficient for distinguishing between (i) and (ii) (whp).

[Goldreich, Goldwasser, Ron '98]

Definitions

An ϵ-tester for the ρn-independent set property is an algorithm that samples a set S of s random vertices, examines the induced subgraph $G[S]$, and distinguishes between the cases (with high probability):

(i) G has a ρn independent set, and

(ii) every induced subgraph of size ρn has at least ϵn^2 edges ($\epsilon-$far)
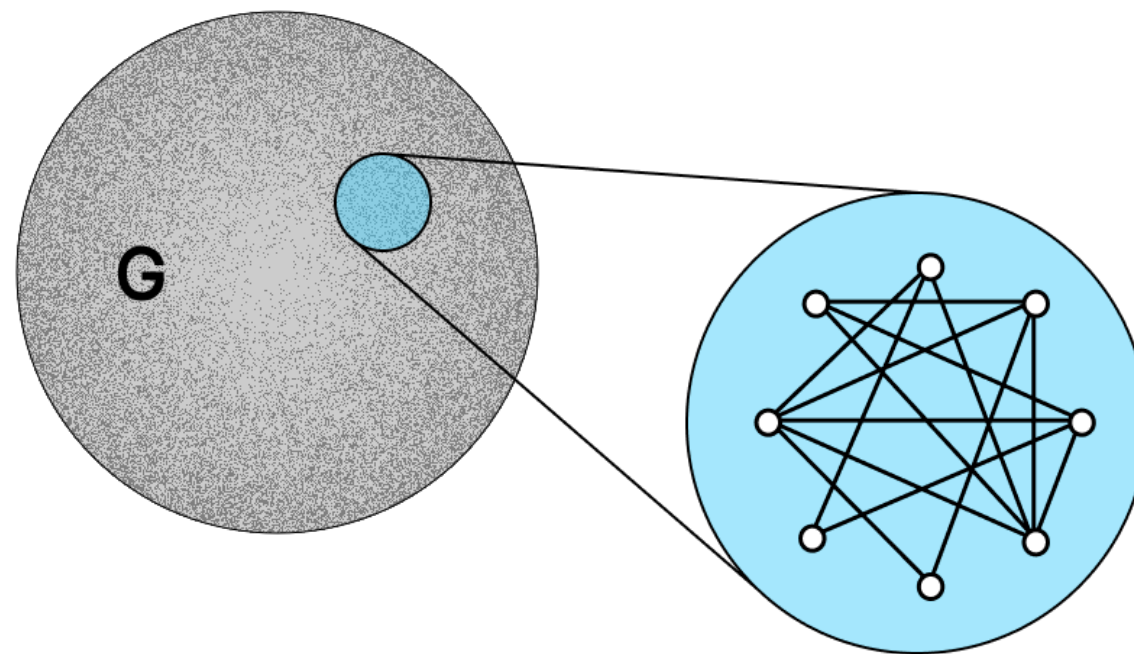
s is the sample complexity of the tester.

Testing Independent Sets

Theorem: There exists an ϵ—tester for the ρn independent set property with sample complexity $\tilde{O}(\rho/\epsilon^4)$.

[Goldreich, Goldwasser, Ron '98]

Testing Independent Sets

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Theorem: There exists an $\epsilon-$tester for the ρn independent set property with sample complexity $\tilde{O}(\rho^3/\epsilon^2)$, [Blais, Seth '23]

and any such tester has sample complexity $\Omega(\rho^3/\epsilon^2)$.

[Feige, Langberg, Schechtman '04]

Weakness with Standard Testing
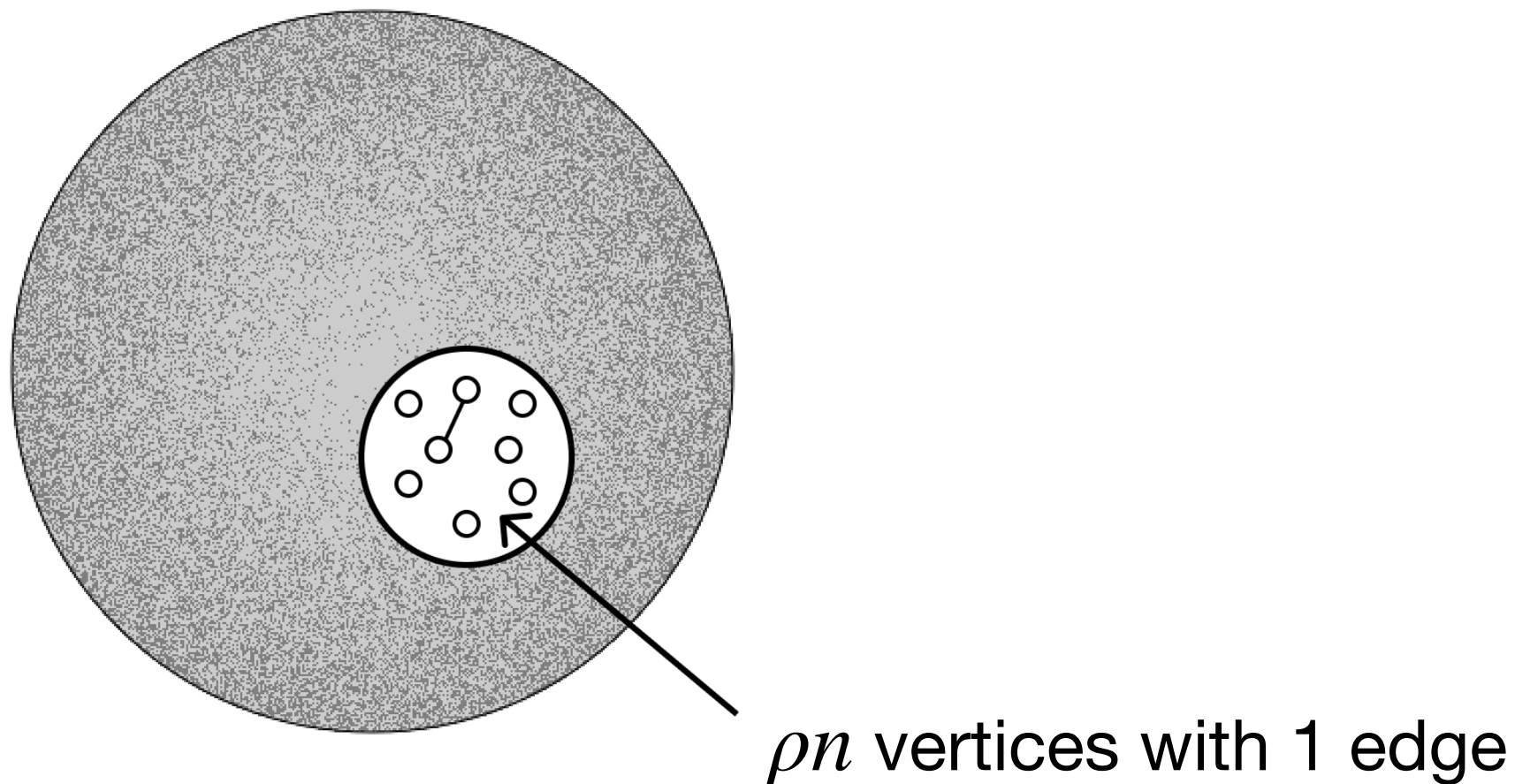
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Question: What if input graph has a set $U \subset V$ with $|U| = \rho n$ such that $G[U]$ has exactly one edge?
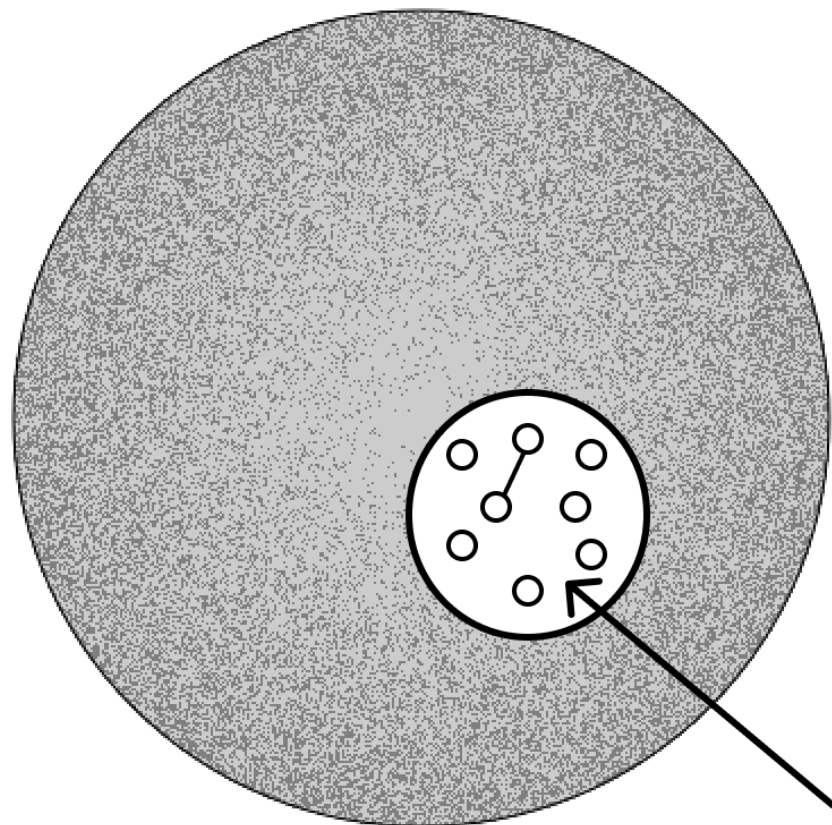

ρn vertices with 1 edge

Weakness with Standard Testing

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Question: What if input graph has a set $U \subset V$ with $|U| = \rho n$ such that $G[U]$ has exactly one edge?

ρn vertices with 1 edge

Answer: Testing algorithms have no guarantee on this type of input!!
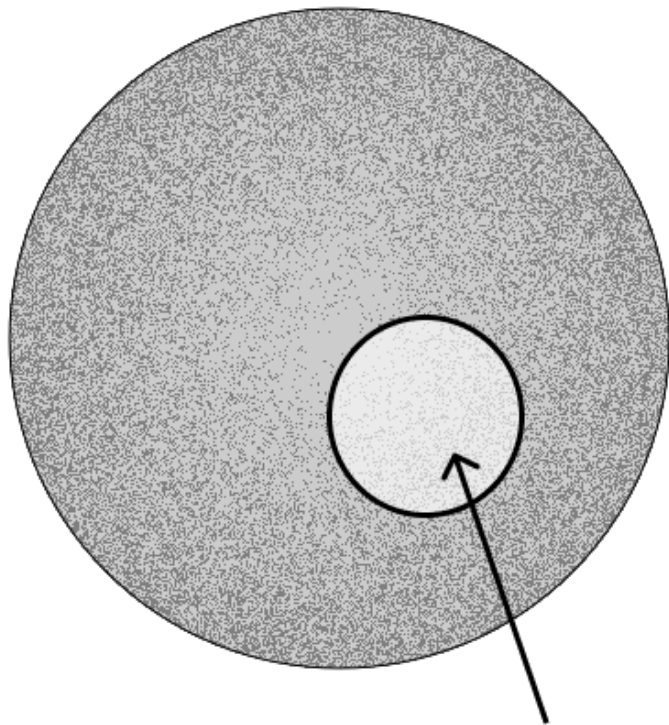
Ideally we would like the algorithm to accept this type of graph.

Tolerant Testing Independent Sets

Problem: For $\epsilon_1 < \epsilon_2$, distinguish between the cases:

(i) G has an induced subgraph of size ρn with less than $\epsilon_1 n^2$ edges (ϵ_1-close)

(ii) Every induced subgraph of size ρn has at least $\epsilon_2 n^2$ edges (ϵ_2-far)

vs.

ρn vertices with at most $\epsilon_1 n^2$ edges

at least $\epsilon_2 n^2$ edges

An algorithm that, with high probability, distinguishes between
(i) and (ii) is called an $(\epsilon_1, \epsilon_2)-$tester.

[Parnas, Ron, Rubinfeld '06]
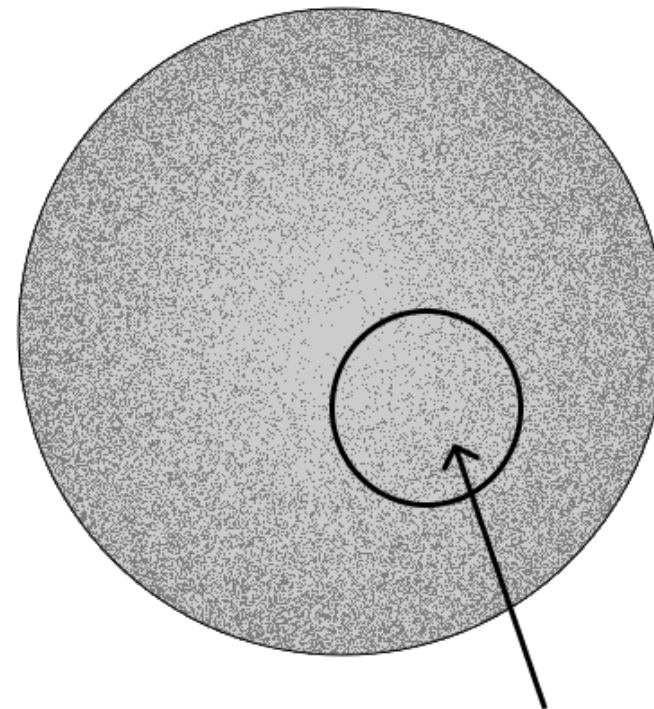
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An algorithm that, with high probability, distinguishes between (i) and (ii) is called an $(\epsilon_1, \epsilon_2)-$tester.

Remarks:

- Generalizes the standard testing problem ($\epsilon_1 = 0$)

- In general ϵ_1 may be a function of ϵ_2

- In some other settings (bounded degree model, boolean strings), there is exponential gap between the query complexity of $\epsilon-$testing and $(\tilde{\Theta}(\epsilon), \epsilon)-$tolerant testing.
  [Fischer, Fortnow '05]
  [Goldreich, Wigderson '22]

Main Result

Theorem: There is a $\left(\frac{\epsilon}{\text{polylog}(1/\epsilon)}, \epsilon \right)-$ tolerant tester for the ρn

independent set property with sample complexity $\tilde{O}(\rho^3/\epsilon^2)$.

[This Work]

Main Result

Theorem: There is a $\left(\frac{\epsilon}{\text{polylog}(1/\epsilon)}, \epsilon \right)$ — tolerant tester for the ρn independent set property with sample complexity $\tilde{O}(\rho^3/\epsilon^2)$.

[This Work]

Remarks:

- Matches the (optimal) sample complexity bound for ϵ —testing

- Generalizes container method approach of Blais, Seth '23

- Best prior result is from a general result for all graph partition properties, which gives sample complexity of roughly $(1/\epsilon)^{12}$
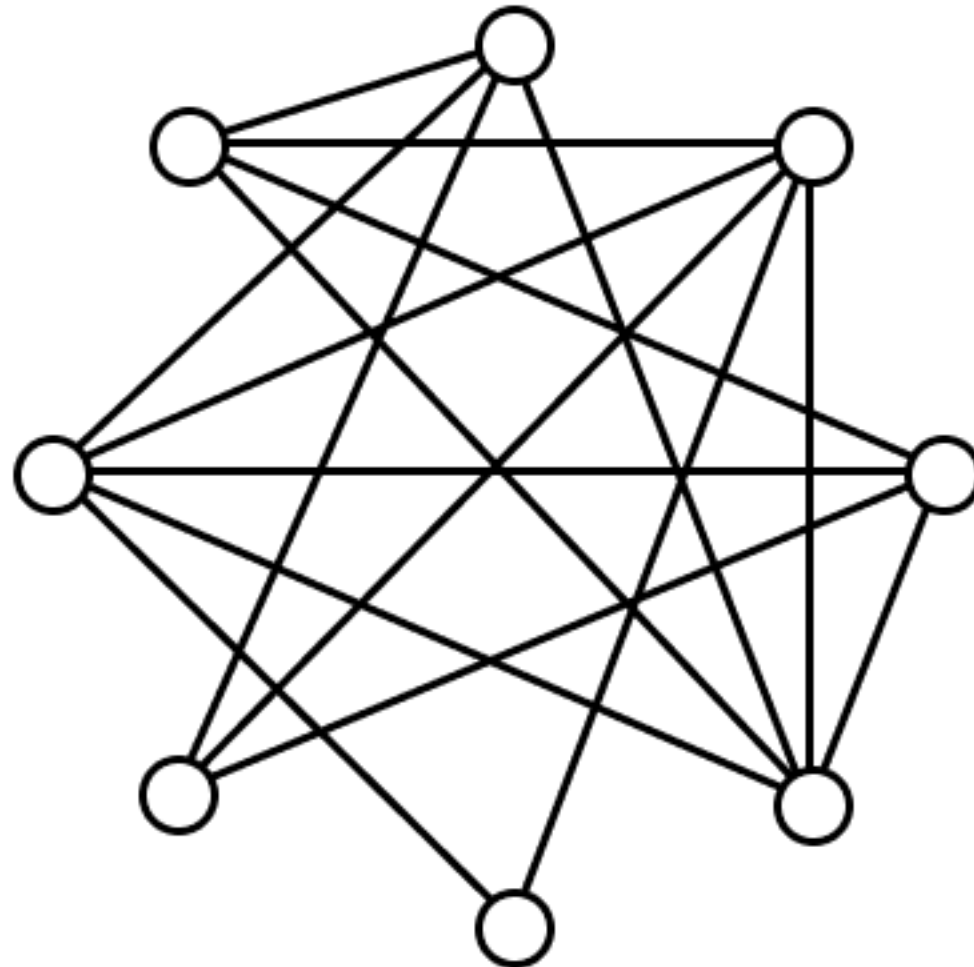
[Fiat, Ron '21]

[Goldreich, Goldwasser, Ron '98]

Outline of Talk

- Main technique to prove theorem: graph container method

  - What is the container method?

  - How to use the container method to prove testing results

    [Blais, Seth '23]

- A new graph container lemma for sparse subgraphs

  [This Work]

  - Proof ideas of new container lemma

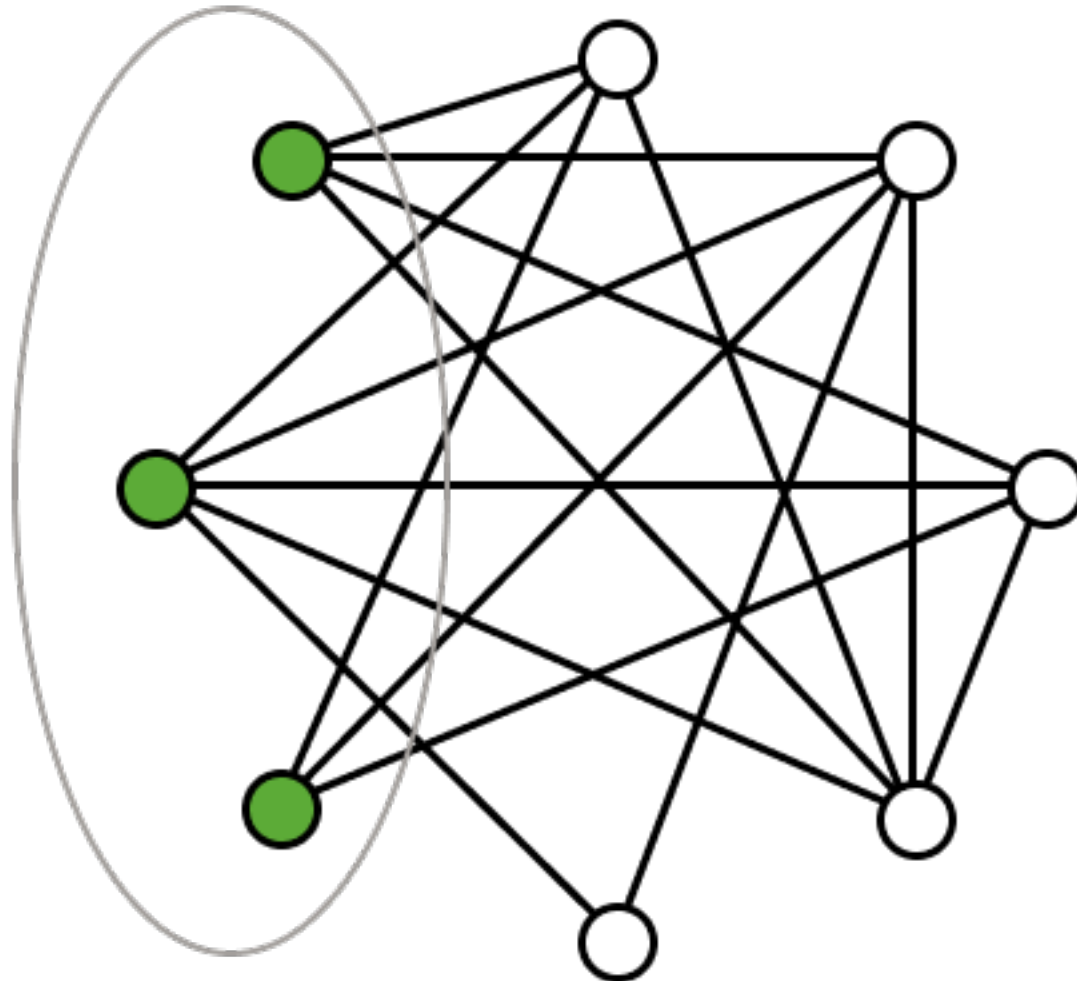 - Another application of new container lemma

What is the Container Method?

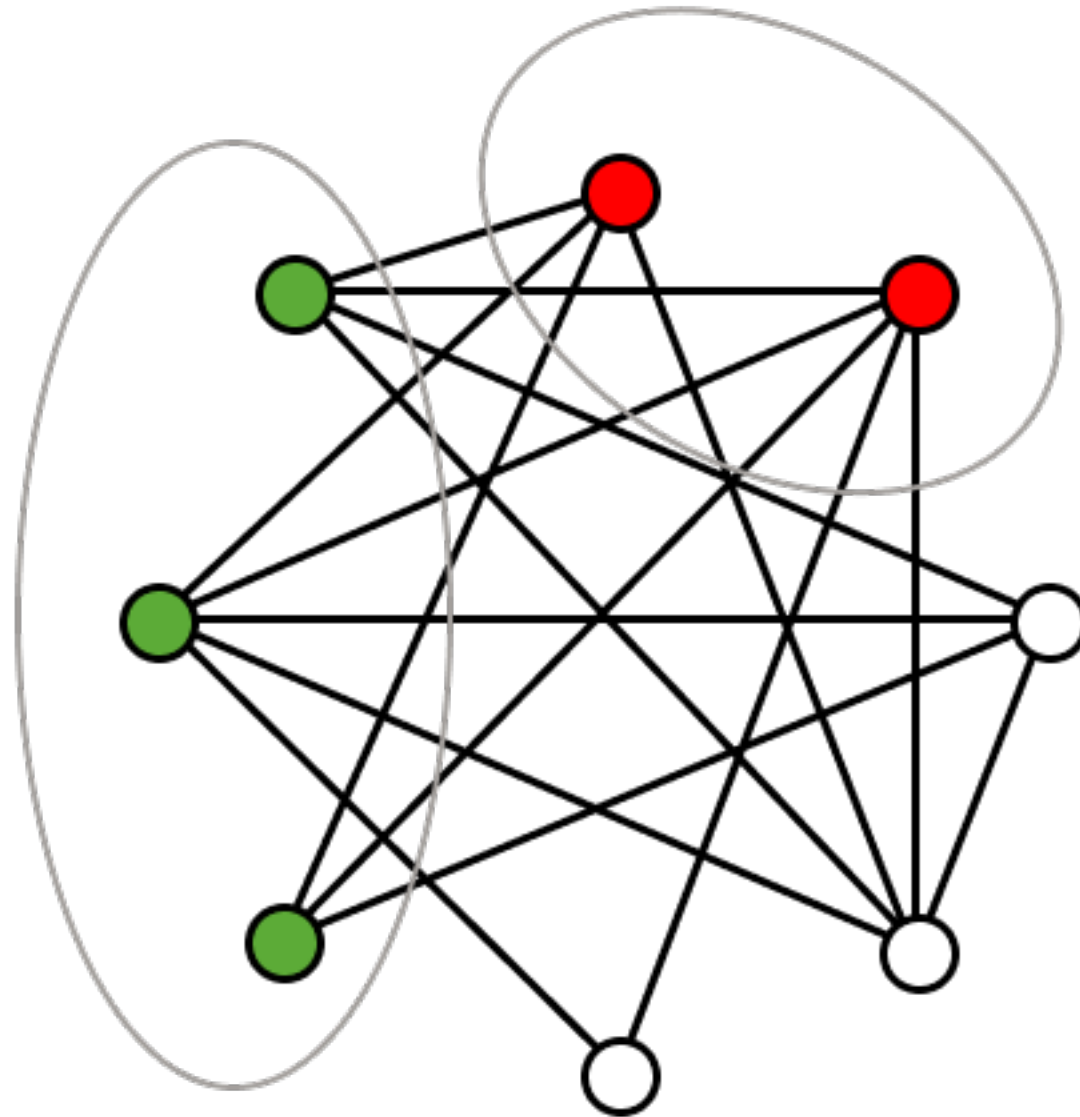**Answer**: A tool for characterizing independent sets in some graphs.

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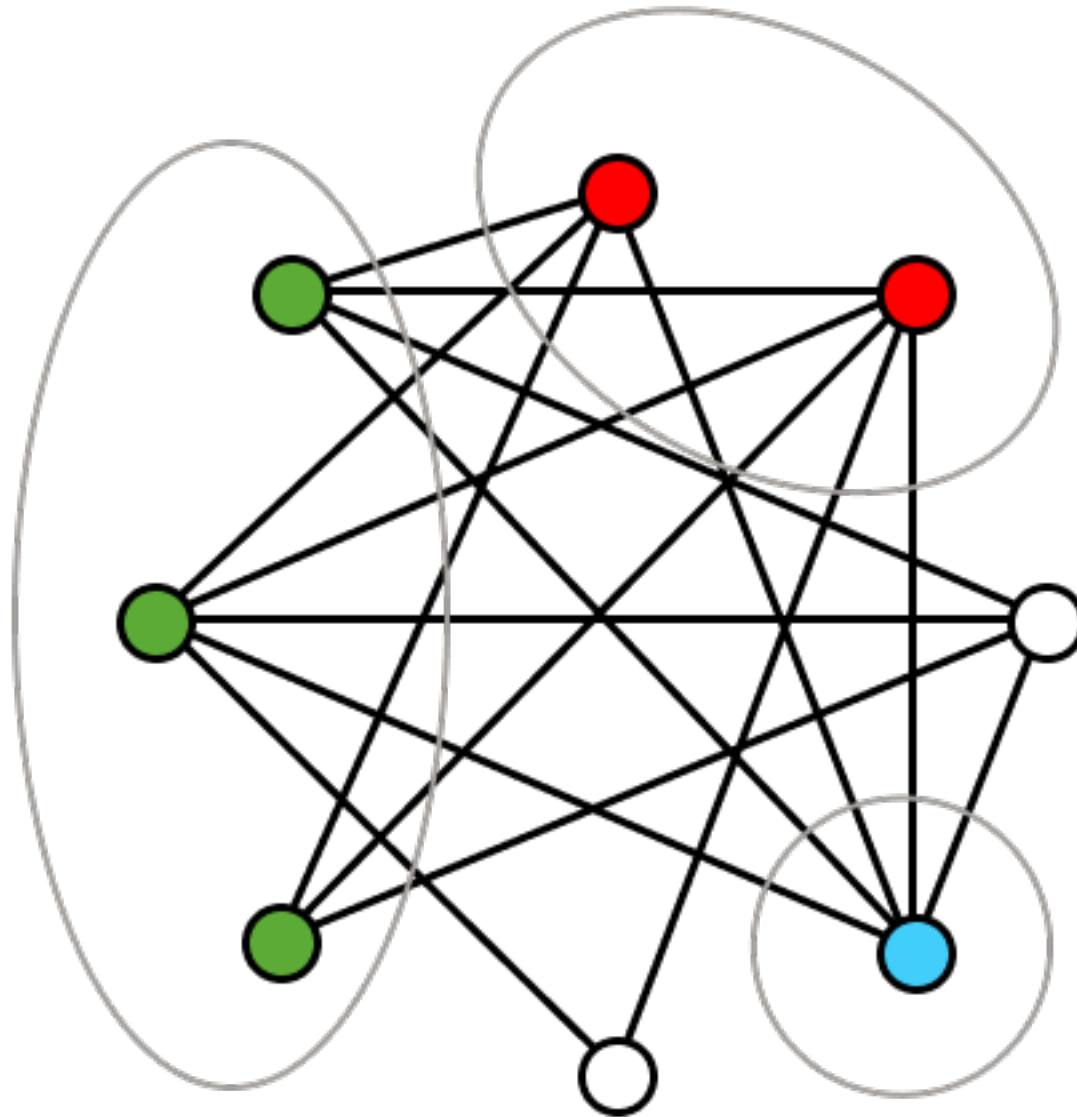**Answer**: A tool for characterizing independent sets in some graphs.

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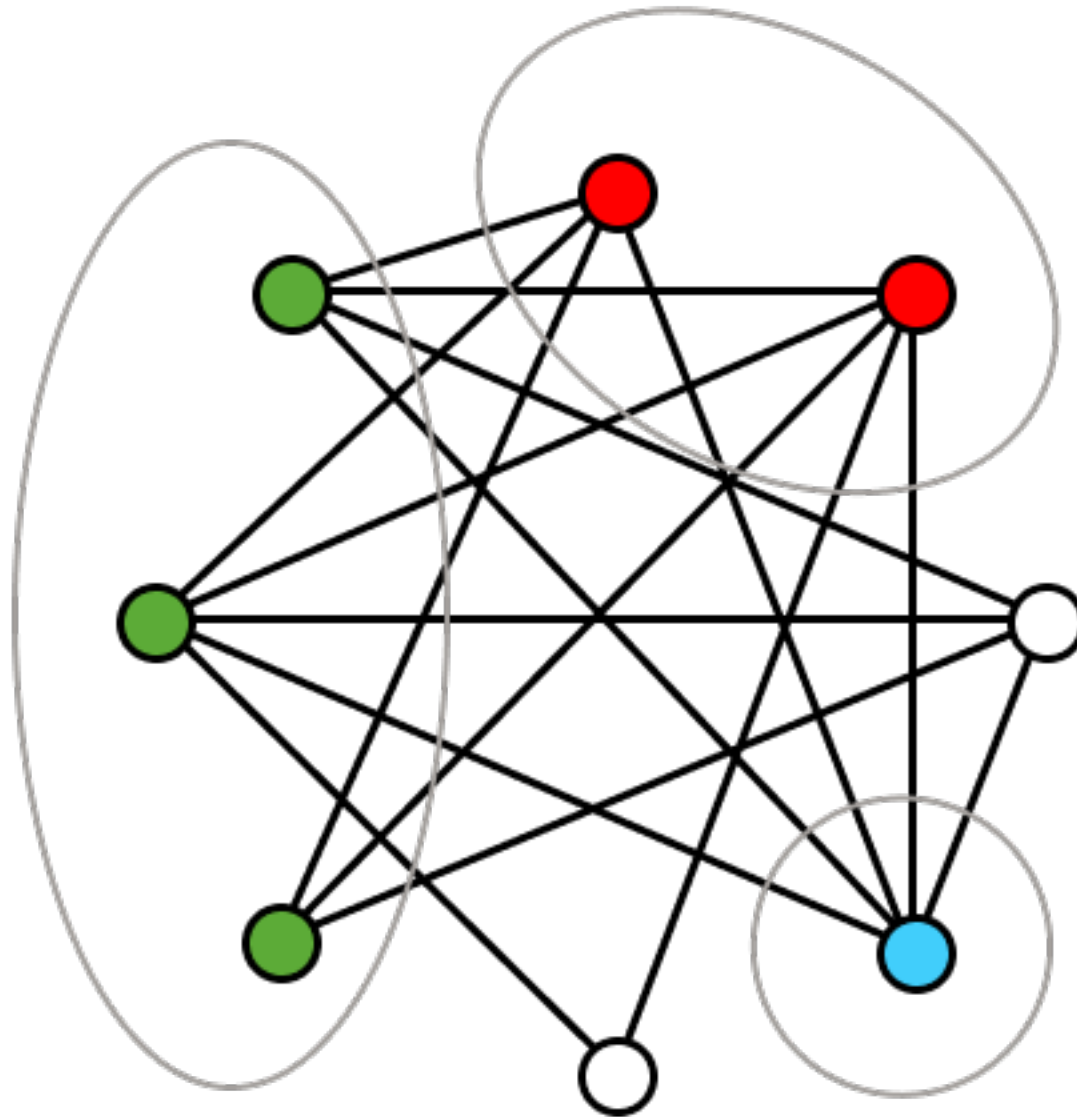**Answer**: A tool for characterizing independent sets in some graphs.

Informal Idea: For any graph satisfying some "nice" conditions, all independent sets in the graph are clustered together into a small number of containers (each container is a subset of vertices).
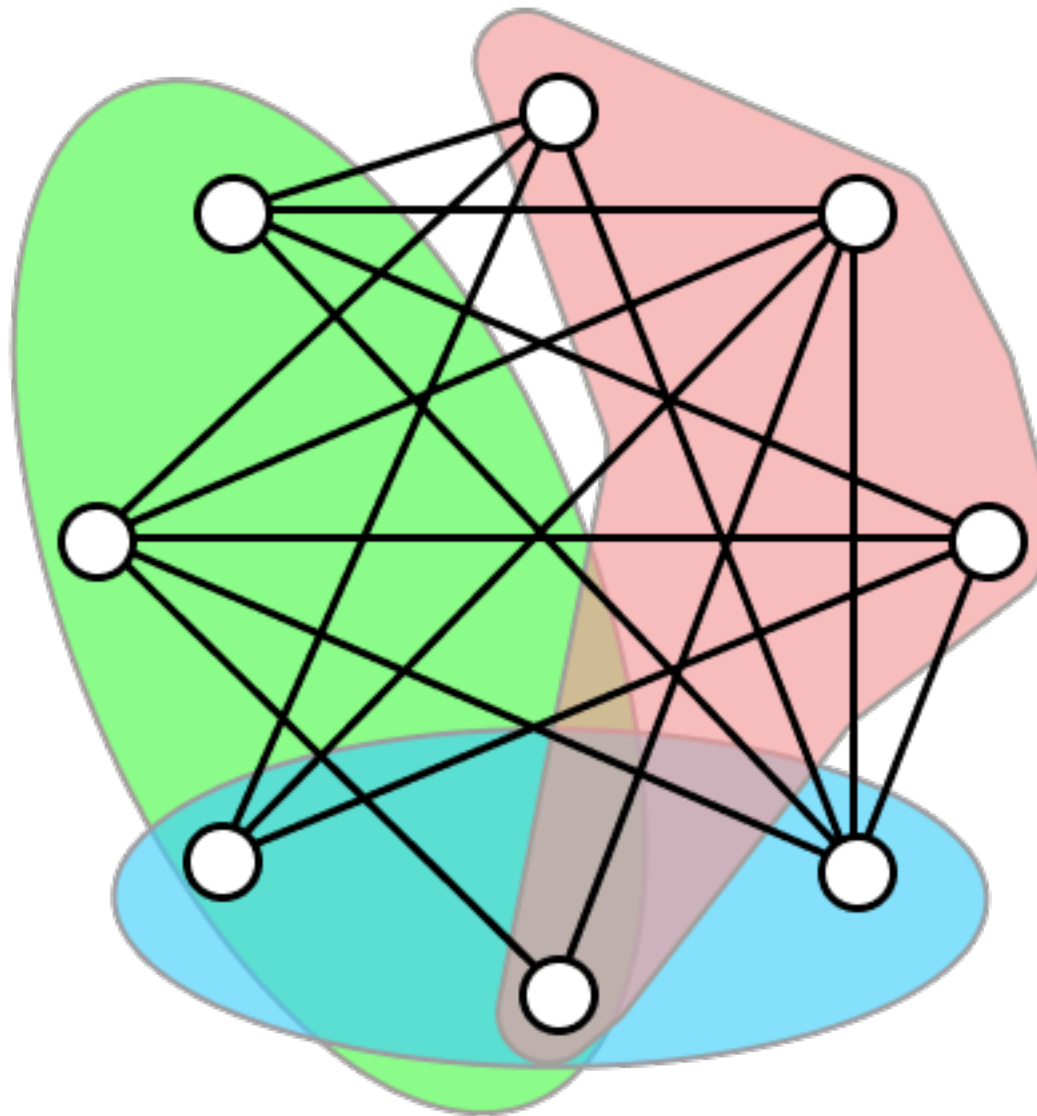
What is the Container Method?

Answer: A tool for characterizing independent sets in some graphs.

Informal Idea: For any graph satisfying some "nice" conditions, all independent sets in the graph are clustered together into a small number of containers (each container is a subset of vertices).

An Initial Graph Container Lemma

Lemma: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,

2. for every $C \in \mathscr{C}$, $|C| < \rho n$.

3. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$.

[Kleitman, Winston '82]

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Note: for survey of combinatorial applications see "Counting Indepedent Sets in Graphs" by Samotij or "The method of hypergraph containers" by Balogh, Morris, and Samotij.

Application to Non-Tolerant Testing

ρn vertices

vs.
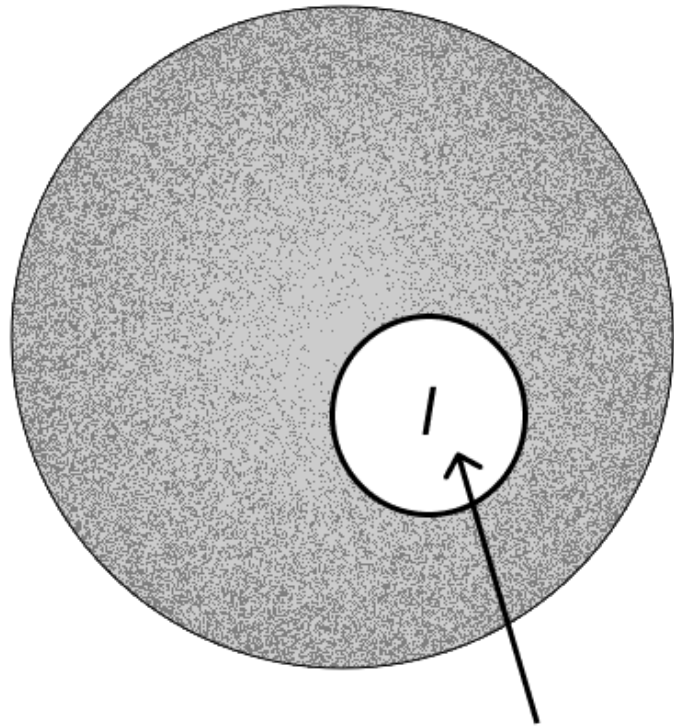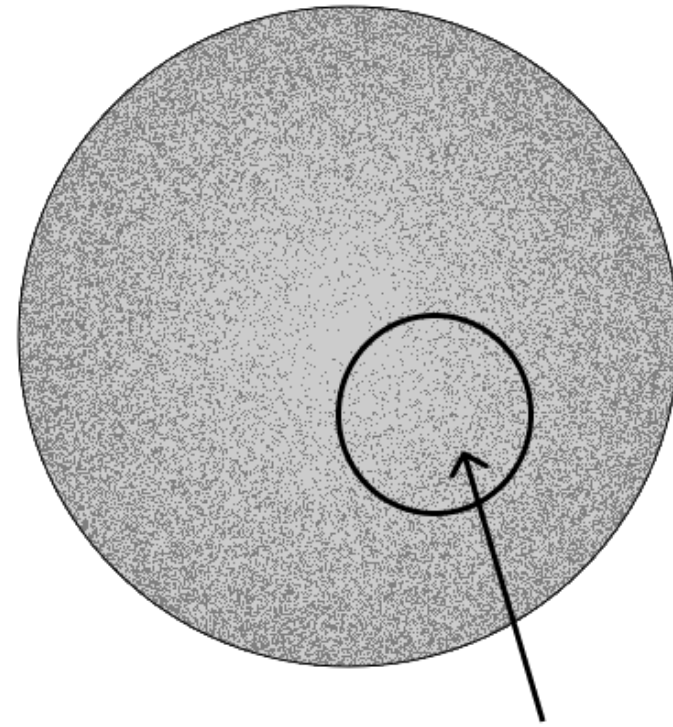
at least εn^2 edges

Application to Non-Tolerant Testing

vs.

ρn vertices   at least εn² edges

Testing Algorithm: Take a random sample S of $s \sim \rho^3/\epsilon^2$ vertices, check if the induced subgraph $G[S]$ has a ρs independent set.

[Blais, Seth '23]

Application to Non-Tolerant Testing

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Application to Non-Tolerant Testing

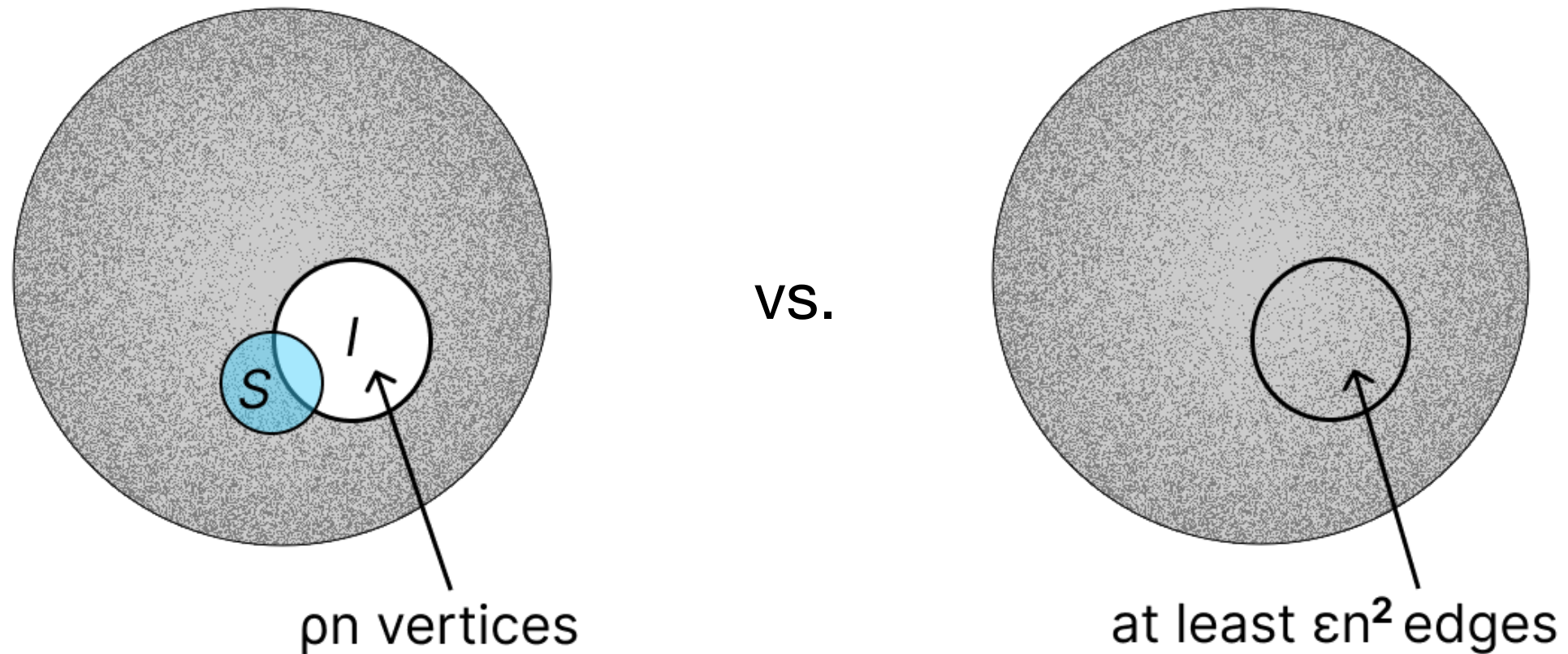ρn vertices   vs.   at least εn² edges

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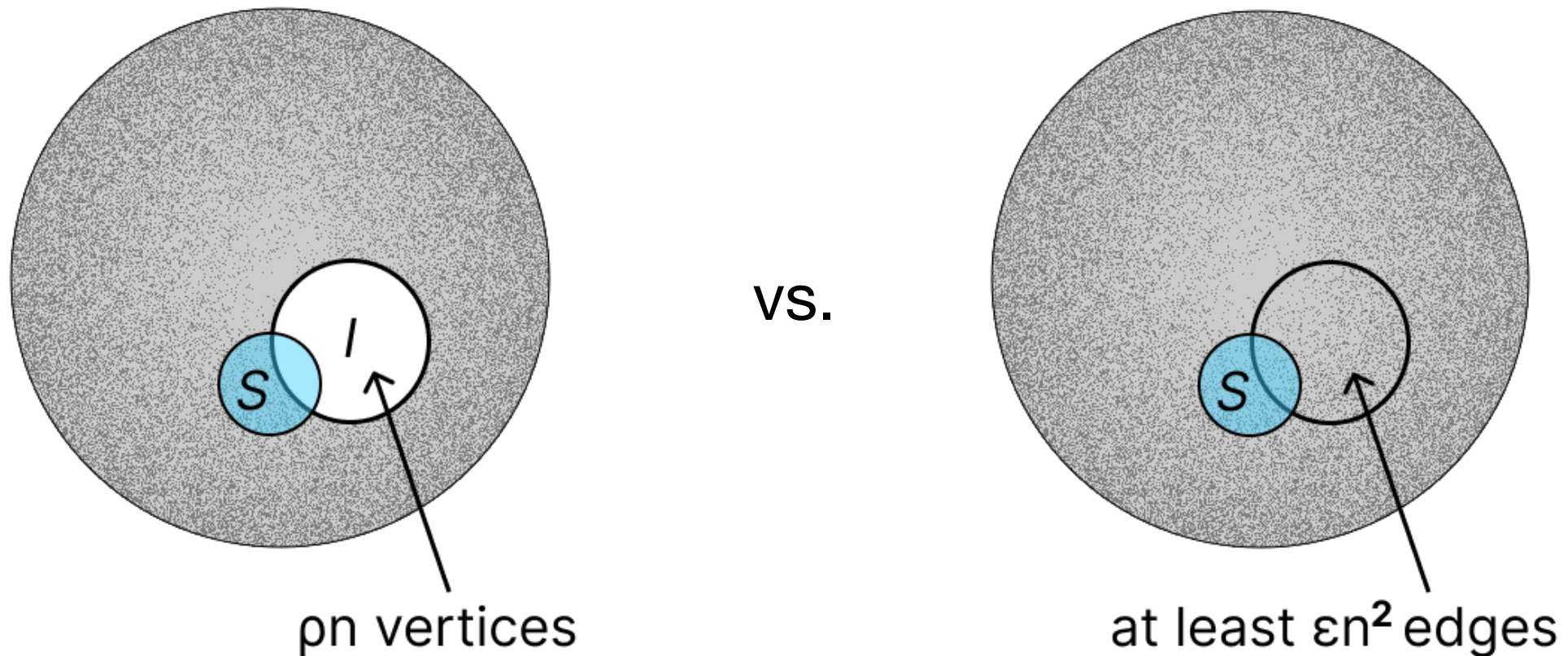
[Blais, Seth '23]

Key Challenge: If G is ϵ-far from having a ρn independent set, show that S has a ρs independent set with only small probability.

Application of Container Lemma

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Naive Approach:

- Union bound over all possible independent sets in the graph.

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Using Container Lemma*:

- Any independent set is contained in a container!

- The probability that S has ρs vertices from any container is small because each container is significantly smaller than ρn.

- Union bound over relatively small number of containers.

Towards a Tolerant Tester

vs.

ρn vertices with at most $\dfrac{\epsilon}{\text{polylog}(1/\epsilon)}\, n^2$ edges

at least ϵn^2 edges

Towards a Tolerant Tester

ρn vertices with at most $\dfrac{\epsilon}{\text{polylog}(1/\epsilon)} \, n^2$ edges

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Towards a Tolerant Tester

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vs.

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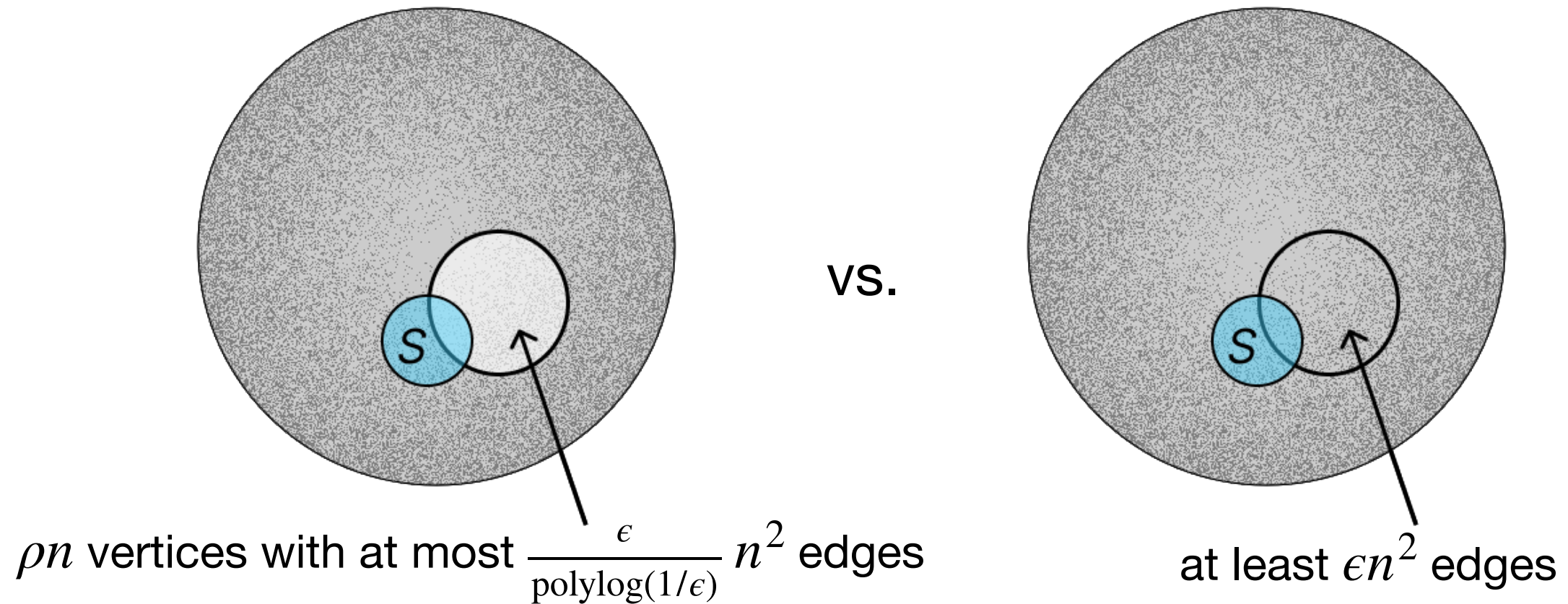
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Key Challenge: If G is ϵ-far from having a ρn independent set, show that S has a ρs sparse set with only small probability.

Towards a Tolerant Tester

vs.

ρn vertices with at most $\dfrac{\epsilon}{\text{polylog}(1/\epsilon)} n^2$ edges

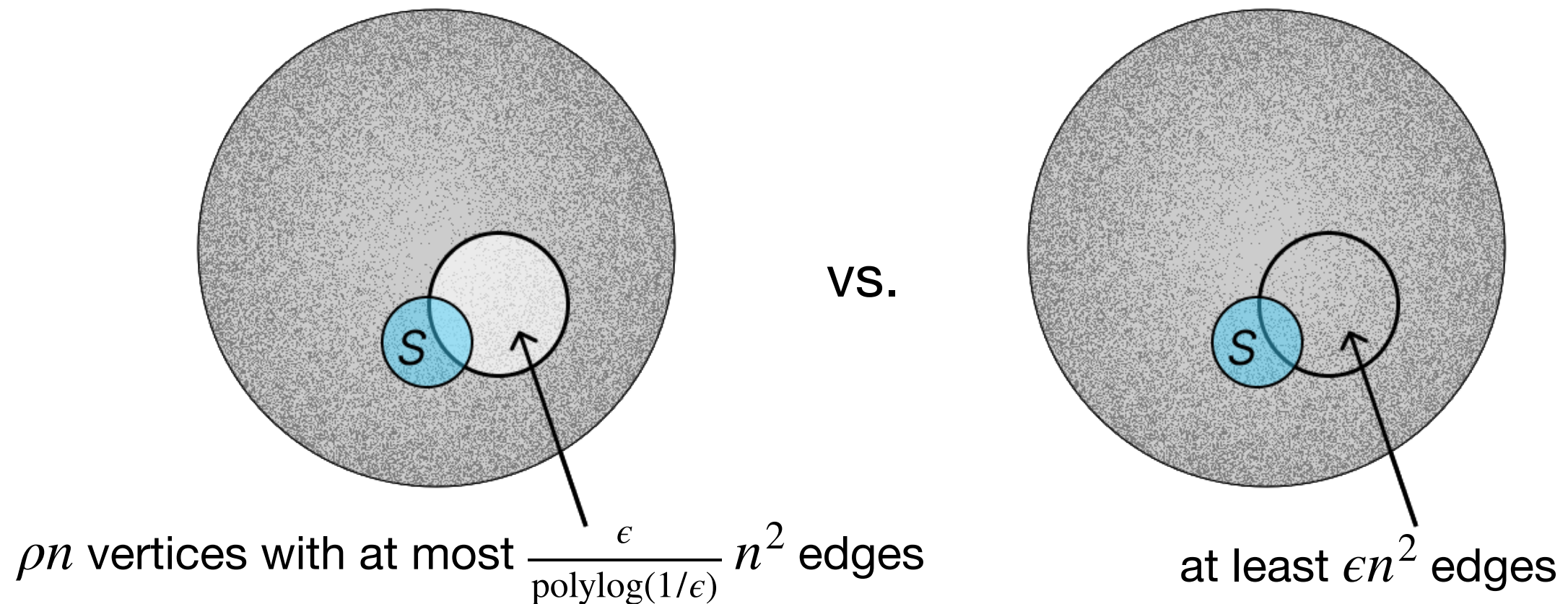at least ϵn^2 edges

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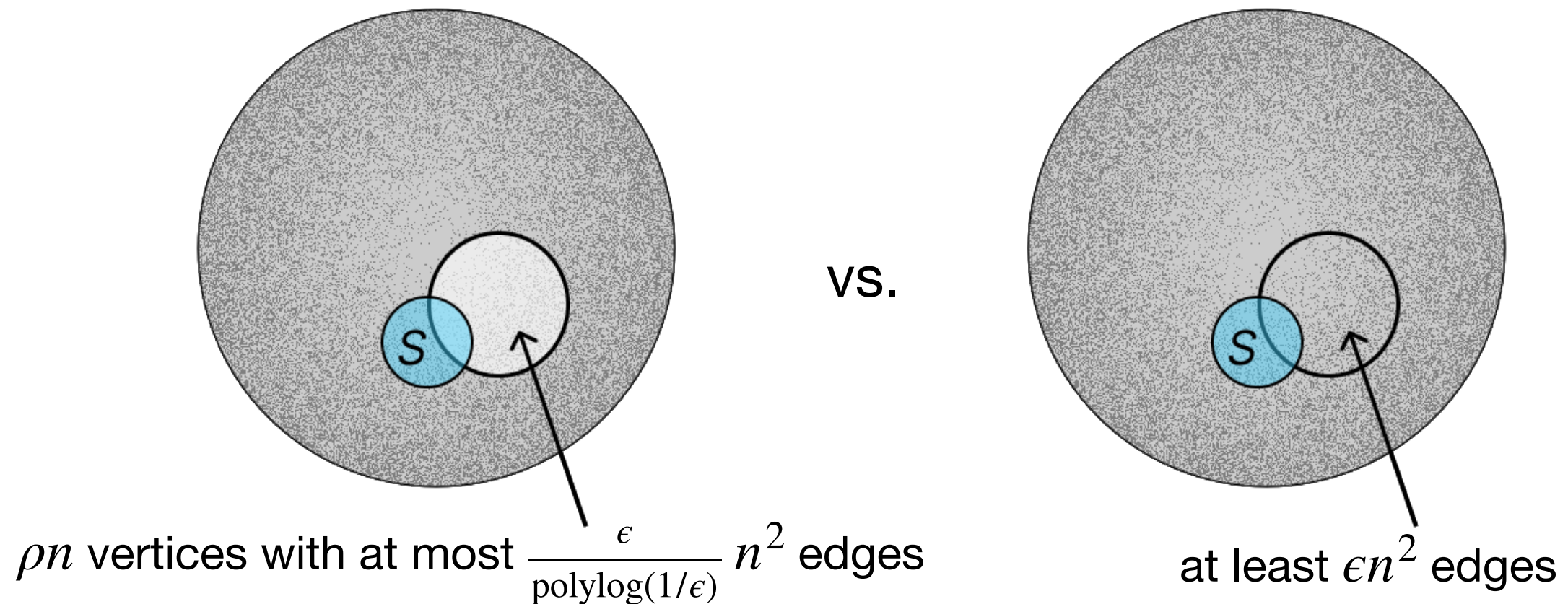
Key Challenge: If G is ϵ-far from having a ρn independent set, show that S has a ρs sparse set with only small probability.

Induced subgraph has less than $\dfrac{\epsilon}{\text{polylog}(1/\epsilon)} s^2$ edges

A Container Lemma For Sparse Sets

Desired Lemma?: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,

2. for every $C \in \mathscr{C}$, $|C| < \rho n$.

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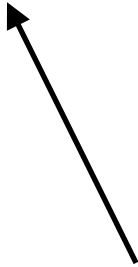
This is not possible
(more details in a few slides)

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No prior results of this form. Only similar results on "sparse subgraphs" apply to subgraphs with smaller density or bounded max degree.

This is not possible (more details in a few slides)

[Nenadov '24] [Saxton, Thomason '15]

A Container Lemma For Sparse Sets

Lemma*: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

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$$|C \cap J| \geq \left(1 - \frac{\alpha}{2}\right) |J|.$$

* Omitting some details

A Container Lemma For Sparse Sets

Lemma*: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

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Informally: each sparse set is "mostly" contained in a container

A Container Lemma For Sparse Sets

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1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,

Using this new container lemma instead of standard container lemma used by Blais, Seth ('23) we can the prove theorem.

2. for every $C \in \mathscr{C}$, $|C| < \rho n$.

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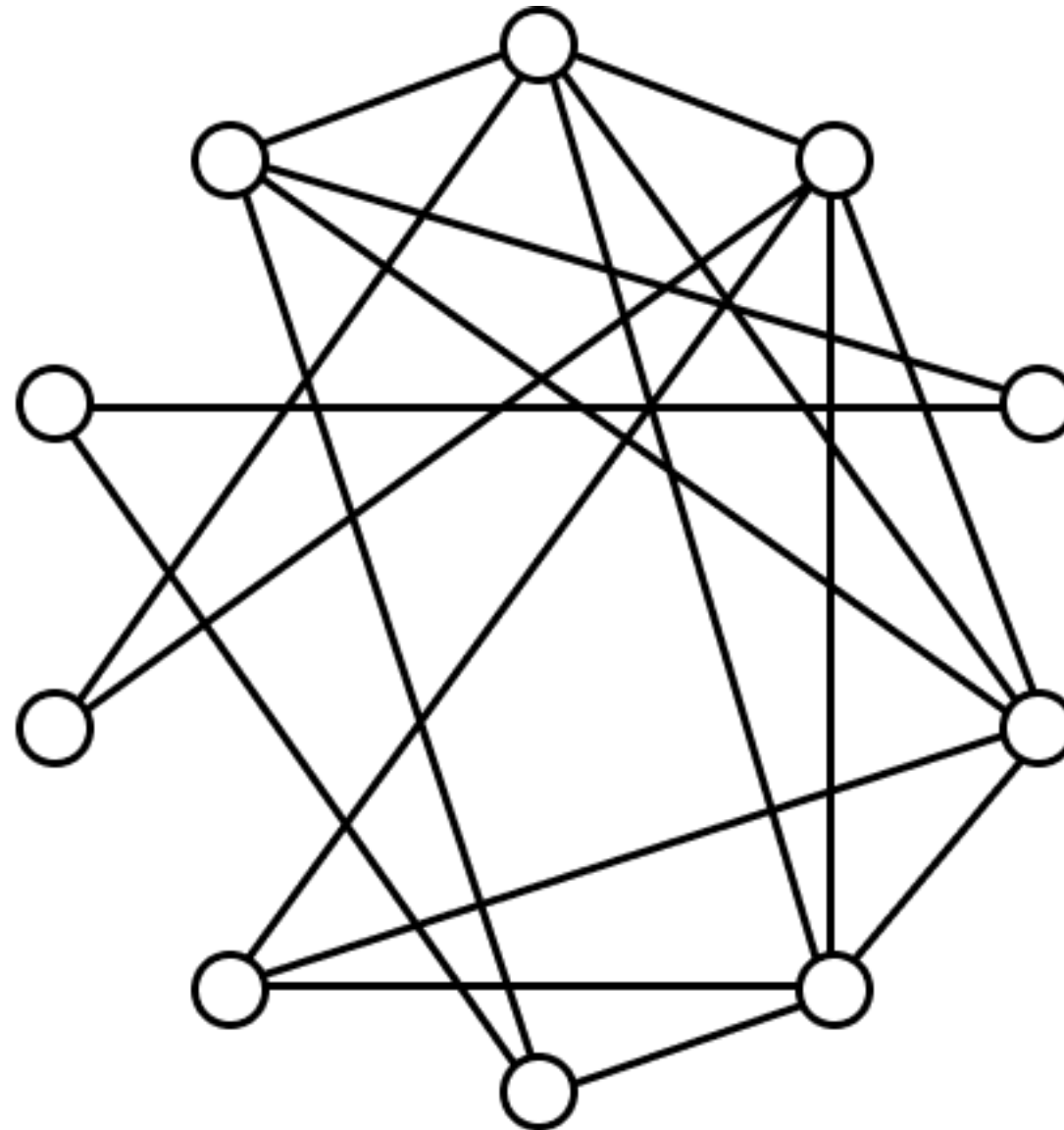
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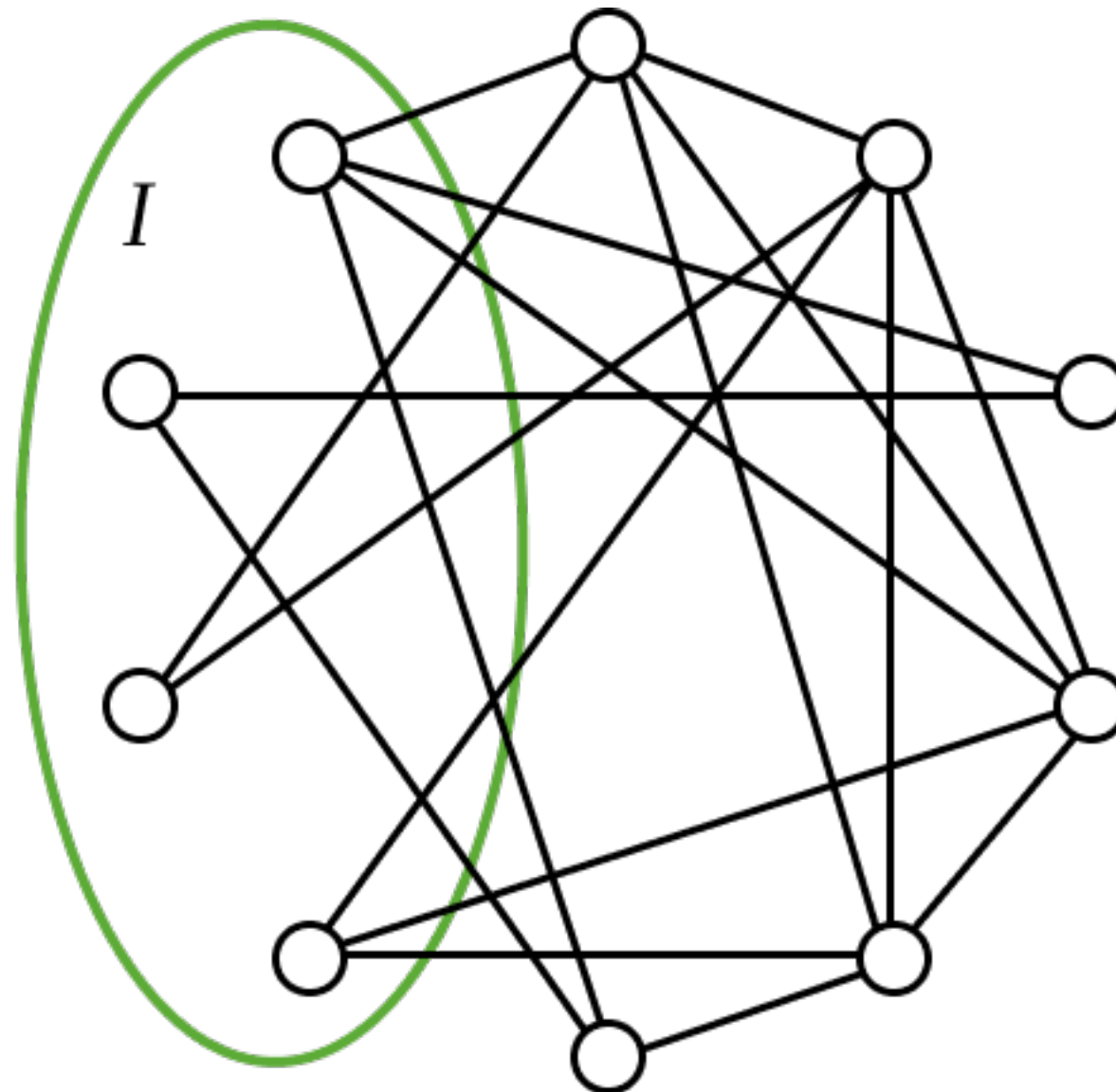Informally: each sparse set is "mostly" contained in a container

Warmup: How to prove Container Lemma for Independent Sets - An Encoding Argument

Encoding Independent Sets

Encoding Independent Sets

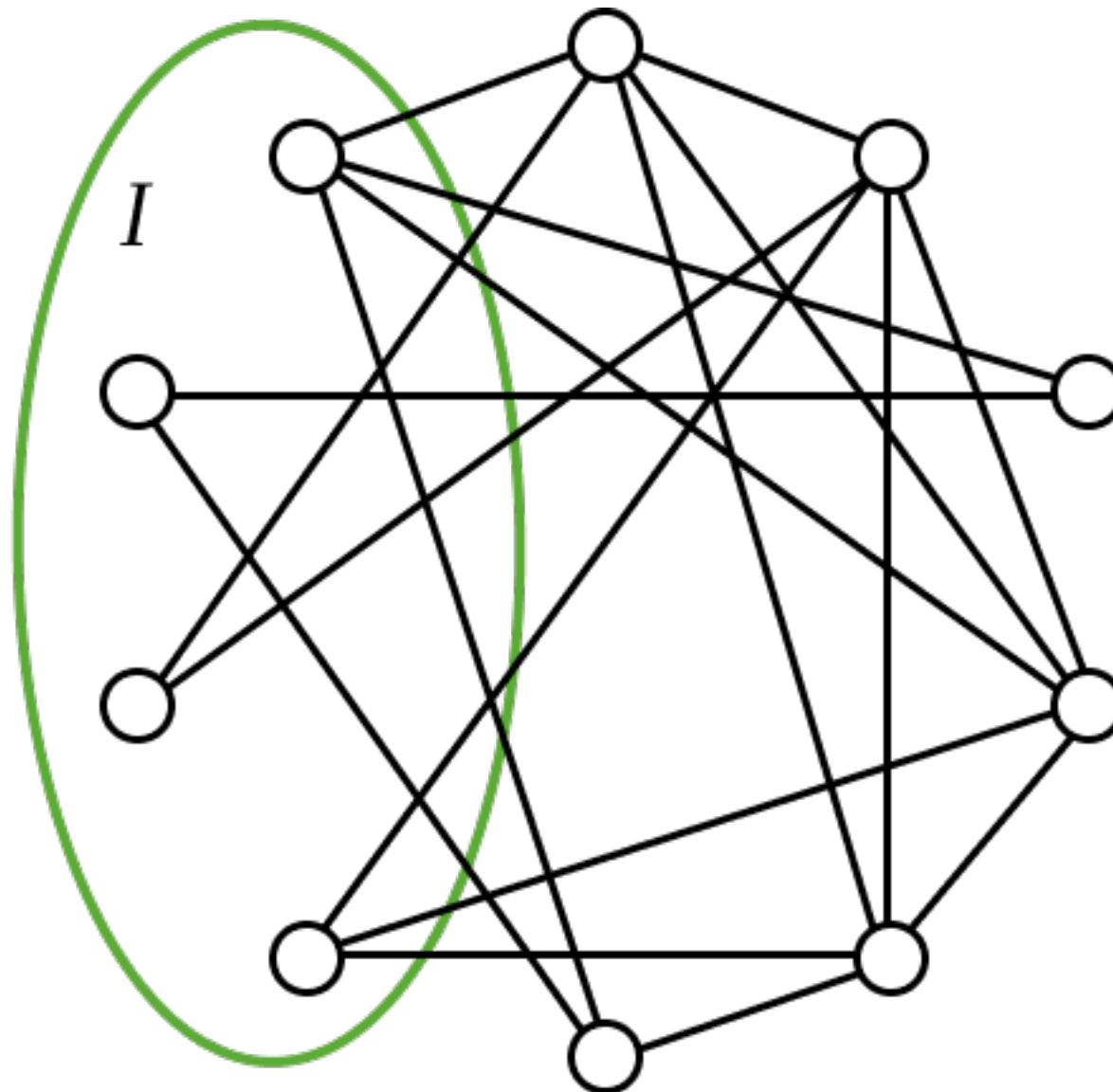# Encoding Independent Sets

Encoding Independent Sets

How can I give you the most information about an independent set I by just telling you about one of the vertices in I?

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Answer: Send the vertex $v \in I$ with highest degree.

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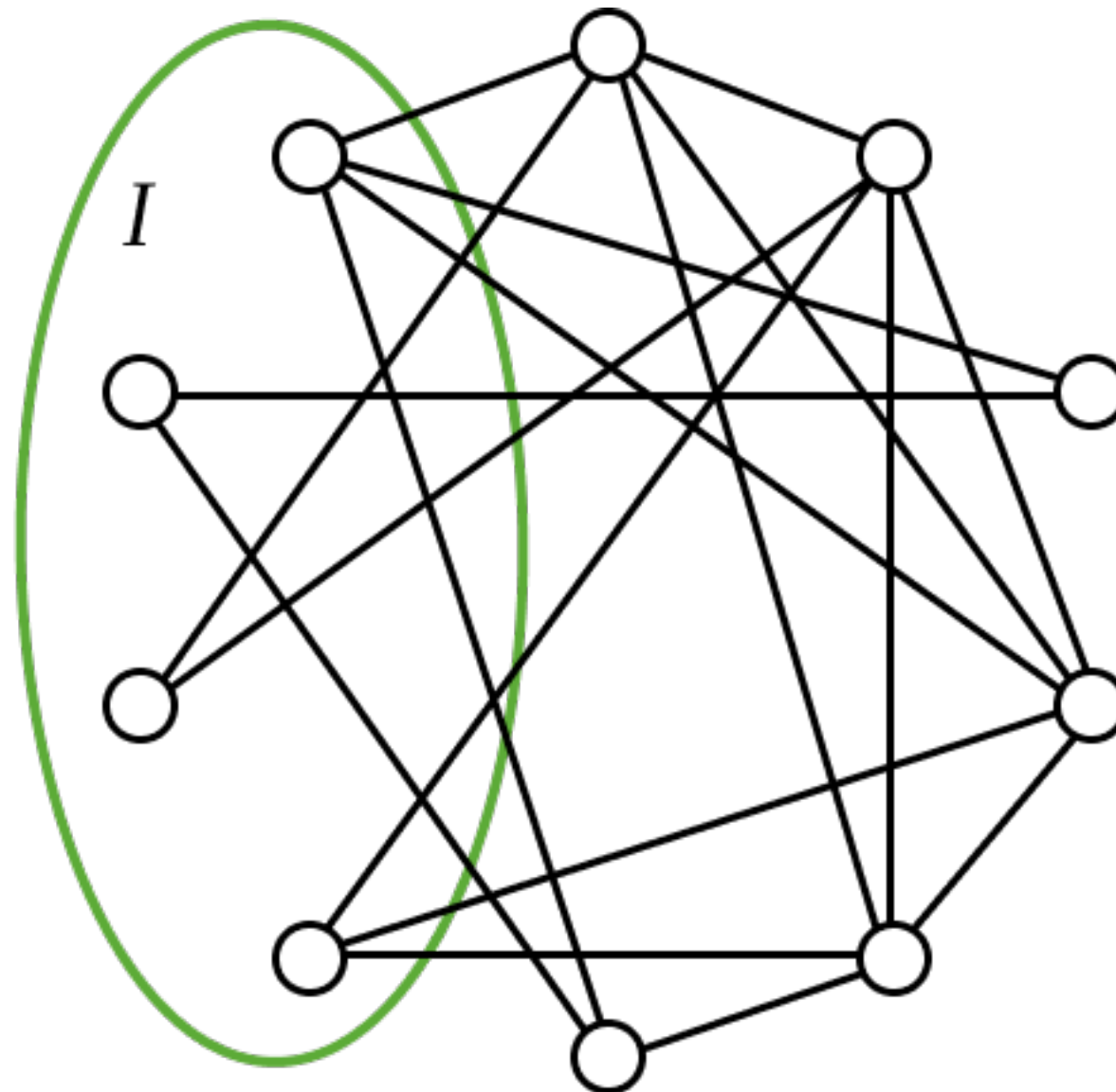
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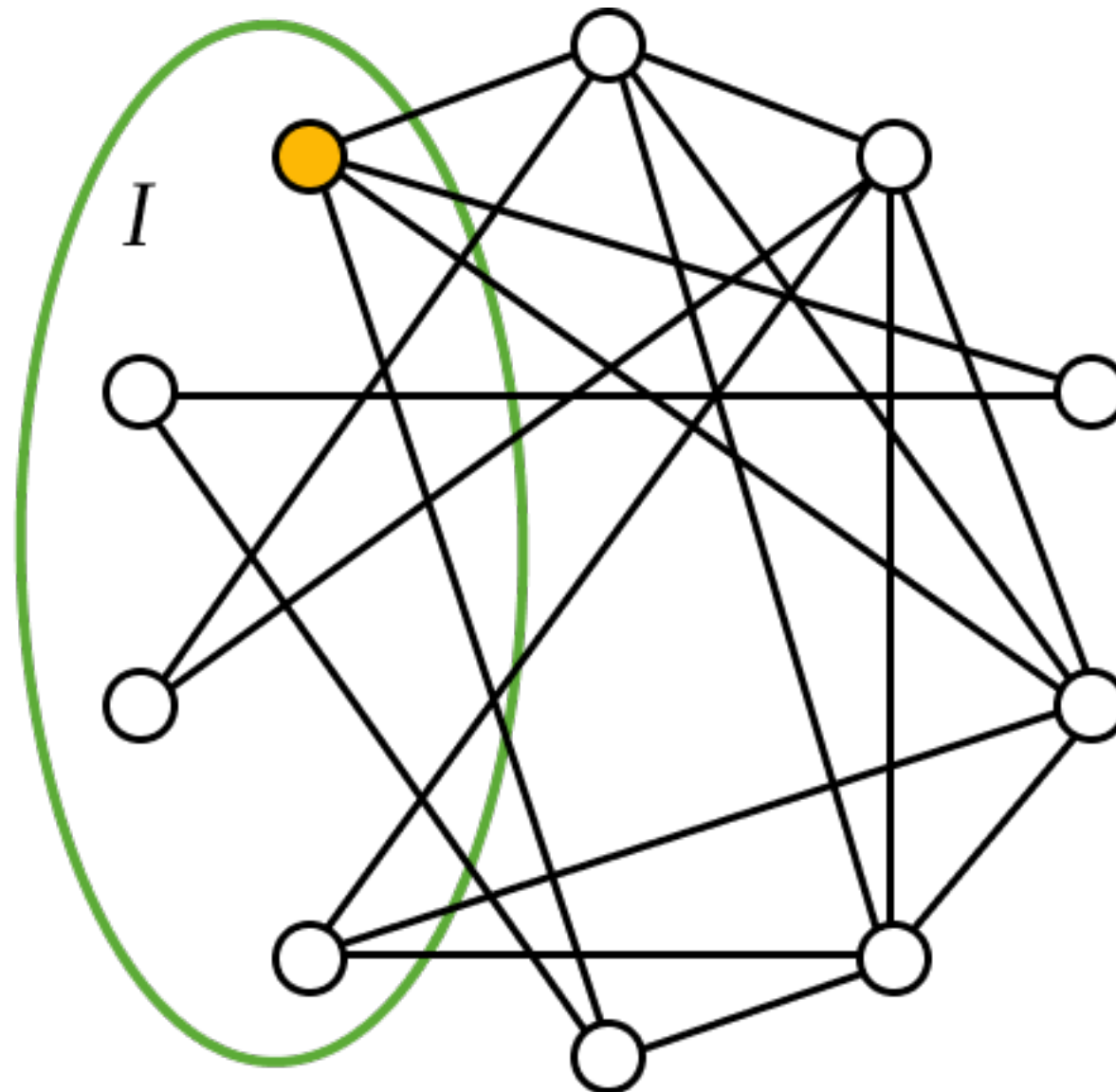**Answer**: Send the vertex $v \in I$ with highest degree.

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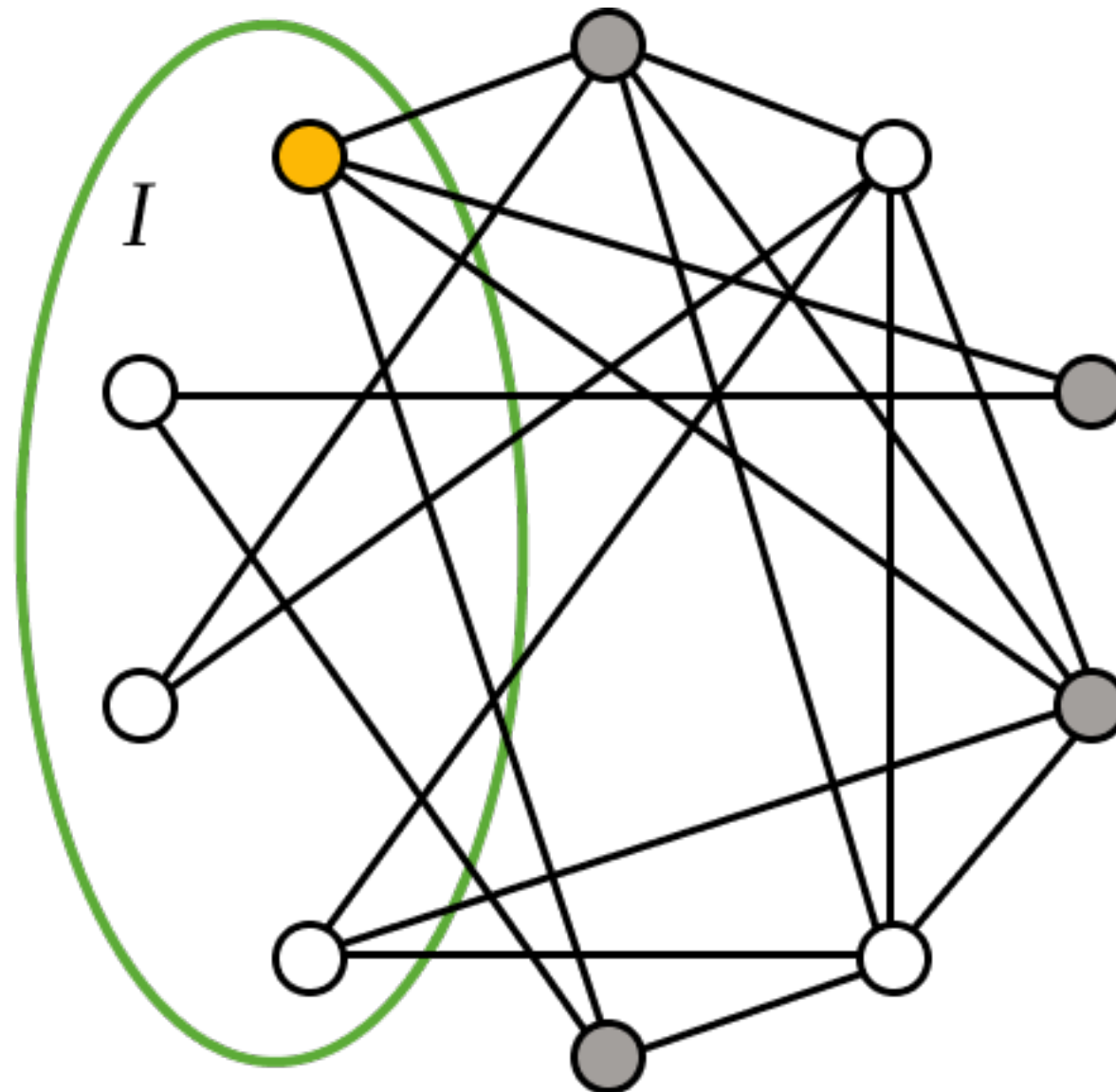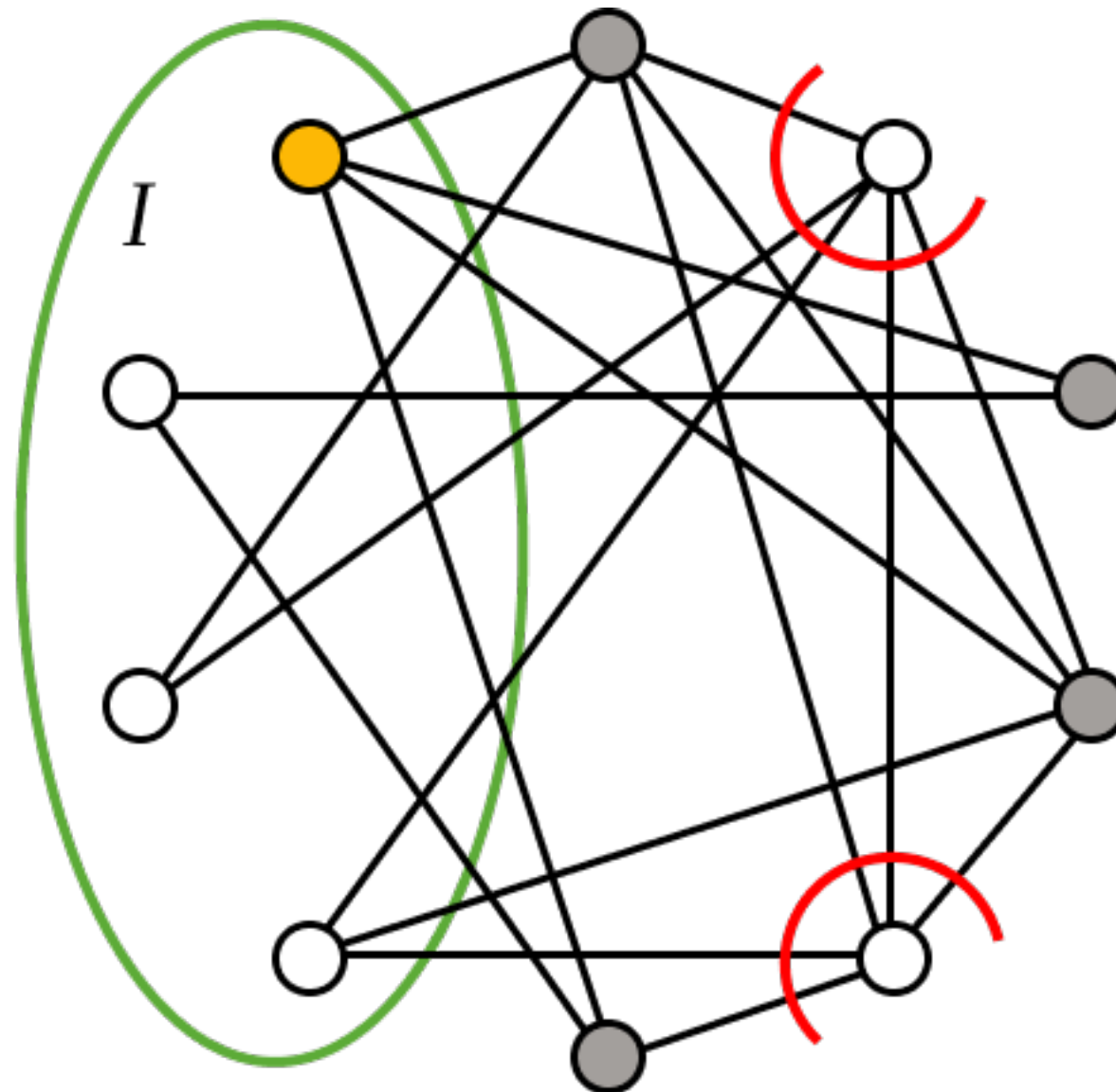**Answer**: Send the vertex $v \in I$ with highest degree.

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Container Generator Algorithm

Input: Graph G and an independent set I

- Initialize fingerprint $F = \varnothing$ and container $C = V$

- Repeatedly select a vertex $v \in I$ with highest degree in $G[C]$ and add it to F. Remove $N(v)$ and all vertices with higher degree than v from C.

Container Generator Algorithm

Input: Graph G and an independent set I

- Initialize fingerprint $F = \emptyset$ and container $C = V$

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1st Iteration

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1st Iteration

C

I

F

Container Generator Algorithm

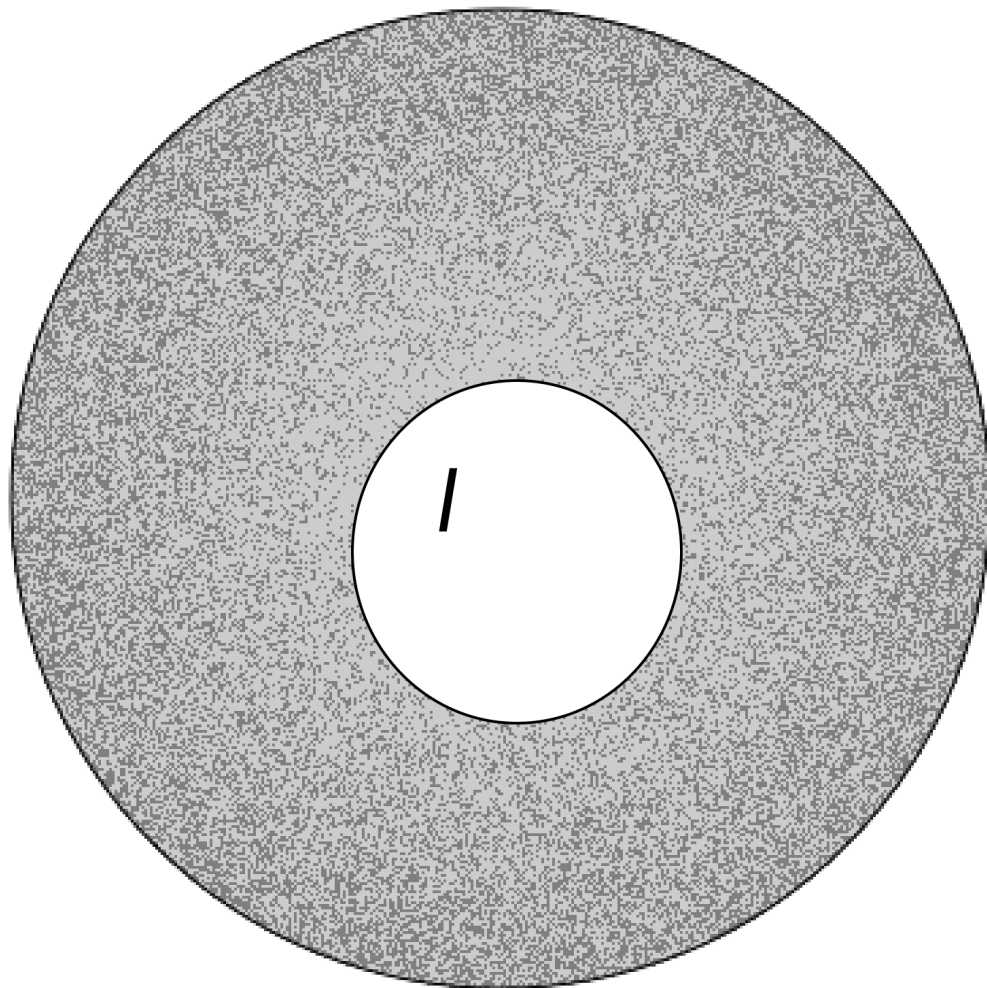**Input**: Graph G and an independent set I
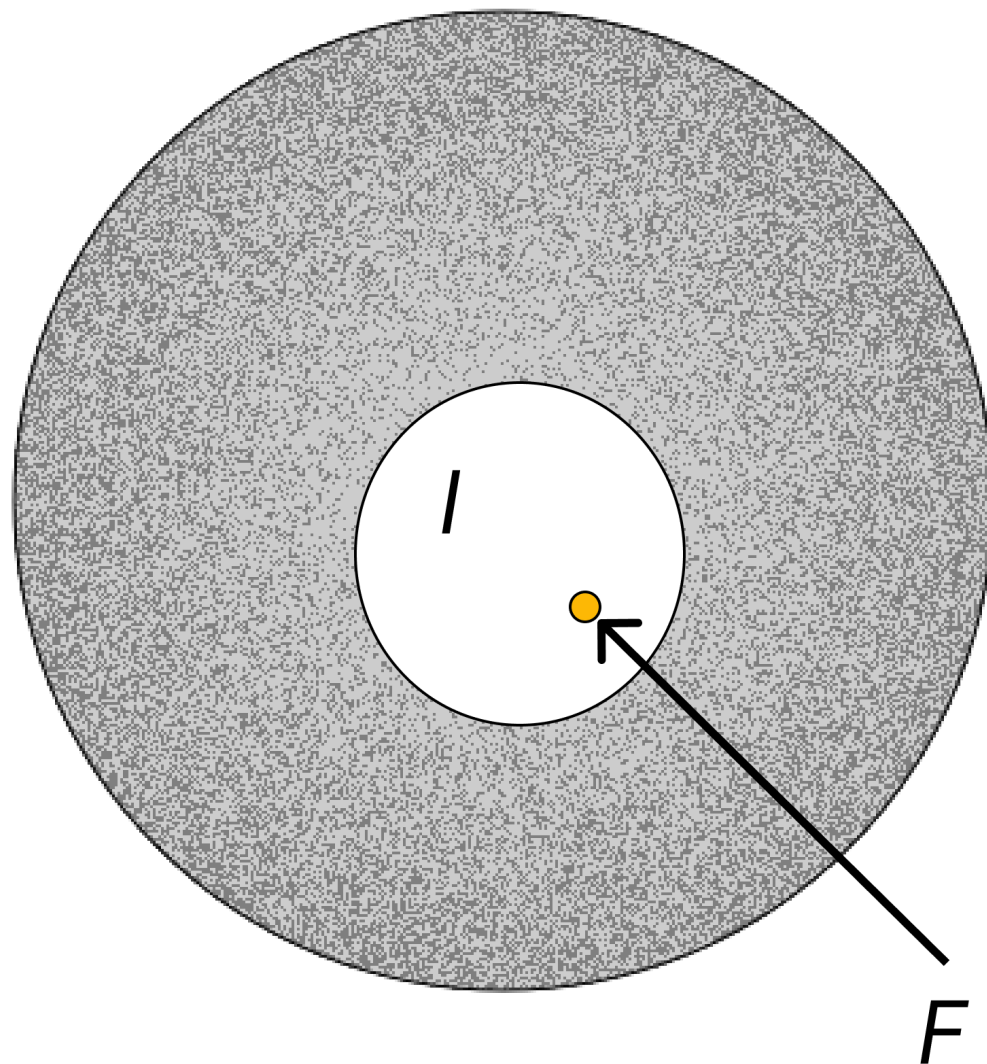
- Initialize fingerprint $F = \emptyset$ and container $C = V$

- Repeatedly select a vertex $v \in I$ with highest degree in $G[C]$ and add it to F. Remove $N(v)$ and all vertices with higher degree than v from C.

2nd Iteration

C

I

F

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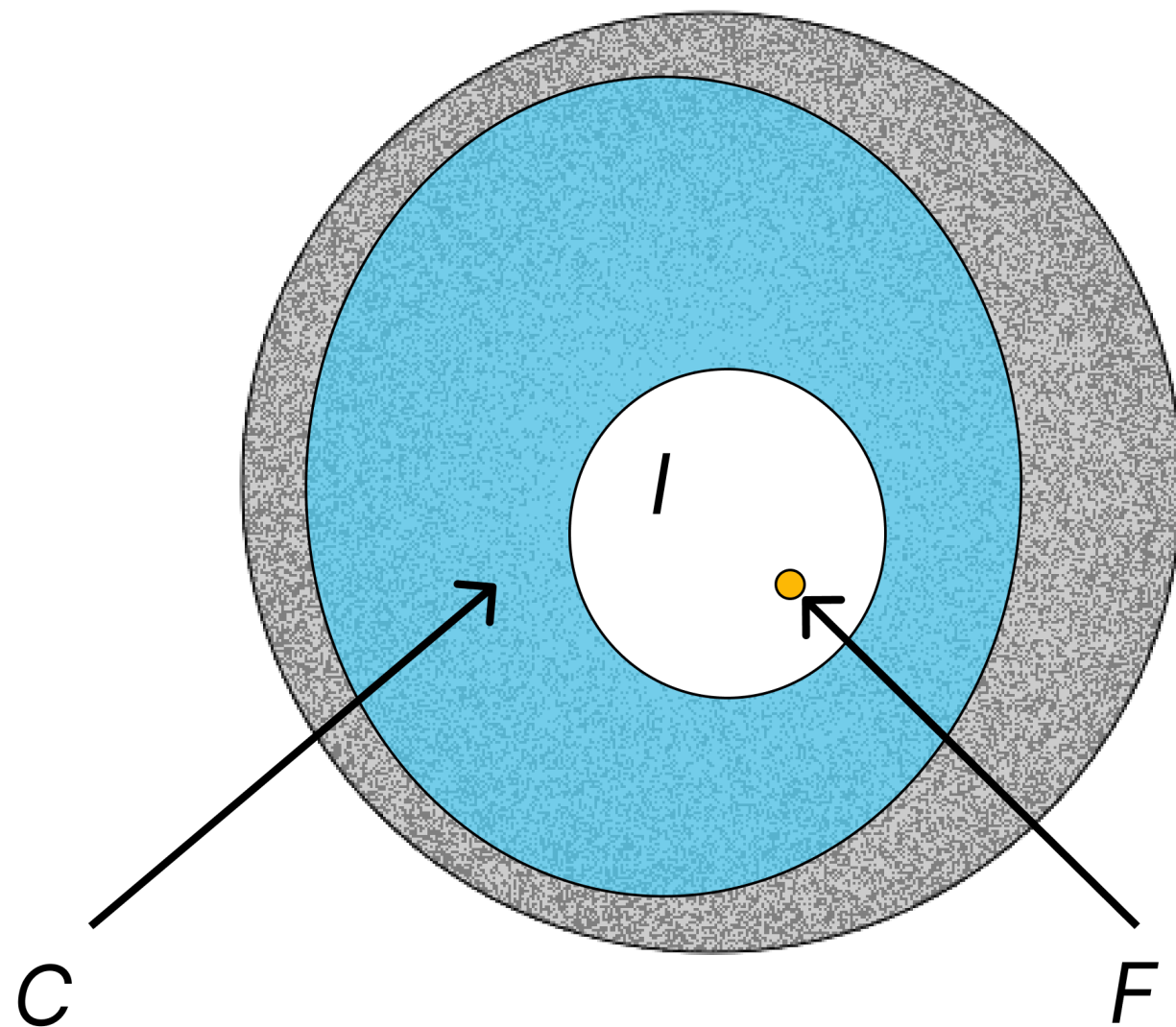

Final Iteration

I

$C(I)$

$F(I)$

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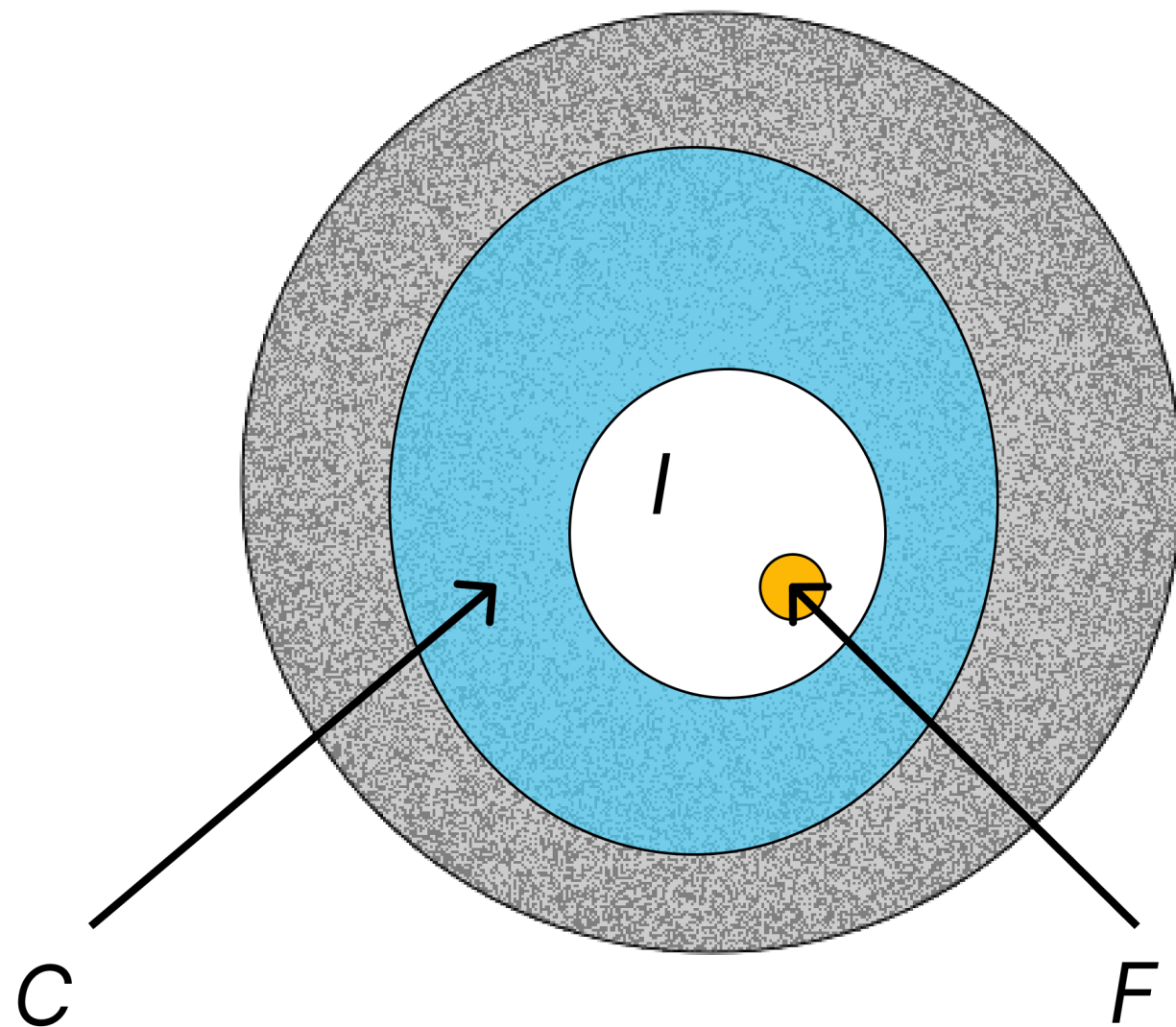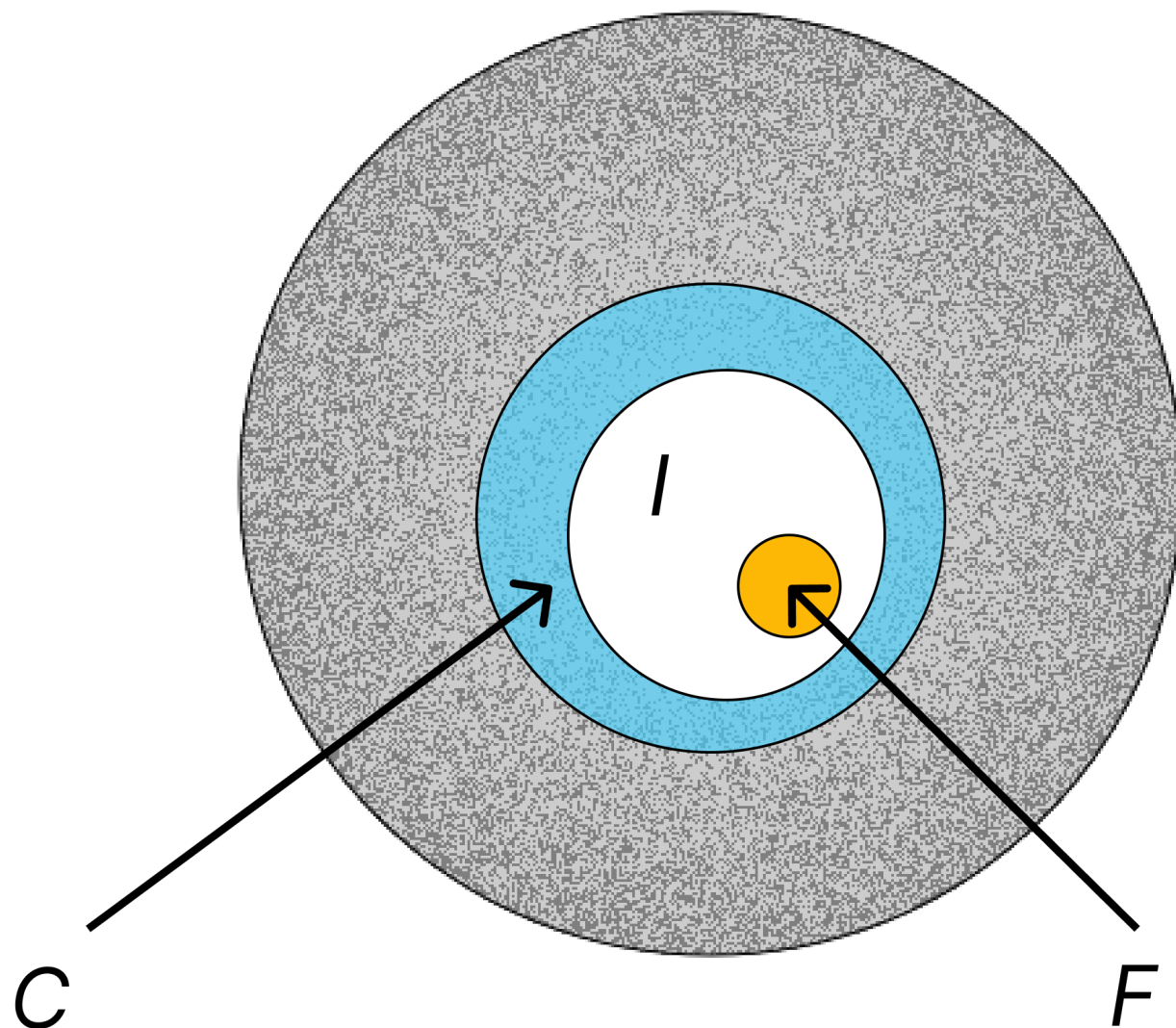

C(I)

F(I)

Final Iteration

Key Observations:

- $I \subseteq C(I)$

- $C(I) = C(F(I))$

Container Lemma for Independent Sets - Restated

Lemma: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,

2. for every $C \in \mathscr{C}$, $|C| < \rho n$.   [Kleitman, Winston '82]

3. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$.

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$$= \{C(F) : F = F(I) \text{ for some independent set } I \text{ in } G\}$$

How to prove new Container Lemma for Sparse Sets - An Encoding Argument

A Container Lemma For Sparse Sets (restated)

Lemma*: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,

2. for every $C \in \mathscr{C}$, $|C| < \rho n$.

3. For every set $J \subseteq V$ such that $G[J]$ has less than $\frac{\epsilon}{\text{polylog}(1/\epsilon)} |J|^2$ edges, there exists $C \in \mathscr{C}$ and α such that $|C| \leq (1 - \alpha)\rho n$ and

$$|C \cap J| \geq \left(1 - \frac{\alpha}{2}\right) |J|.$$

* Omitting some details

Encoding Sparse Sets
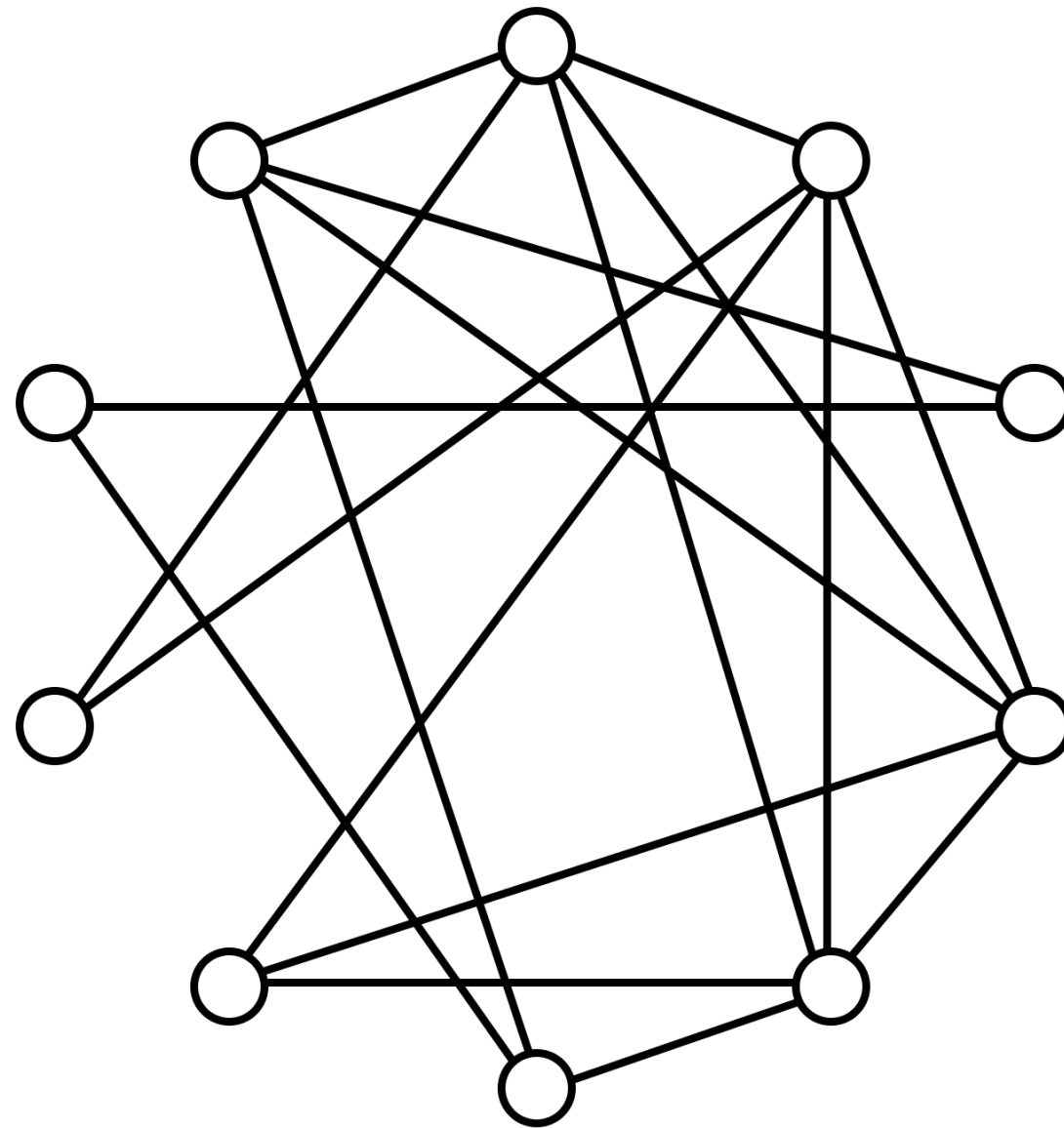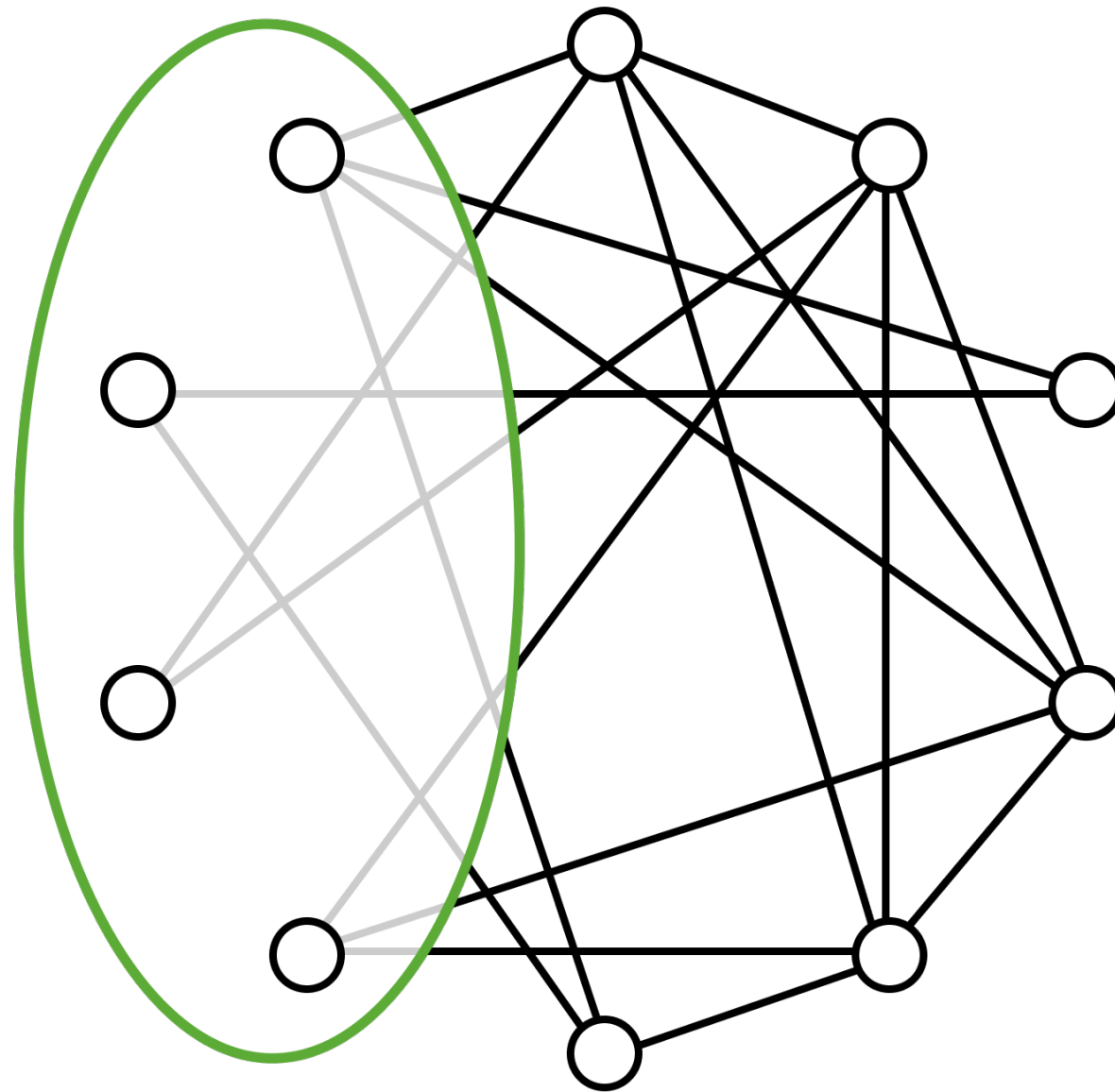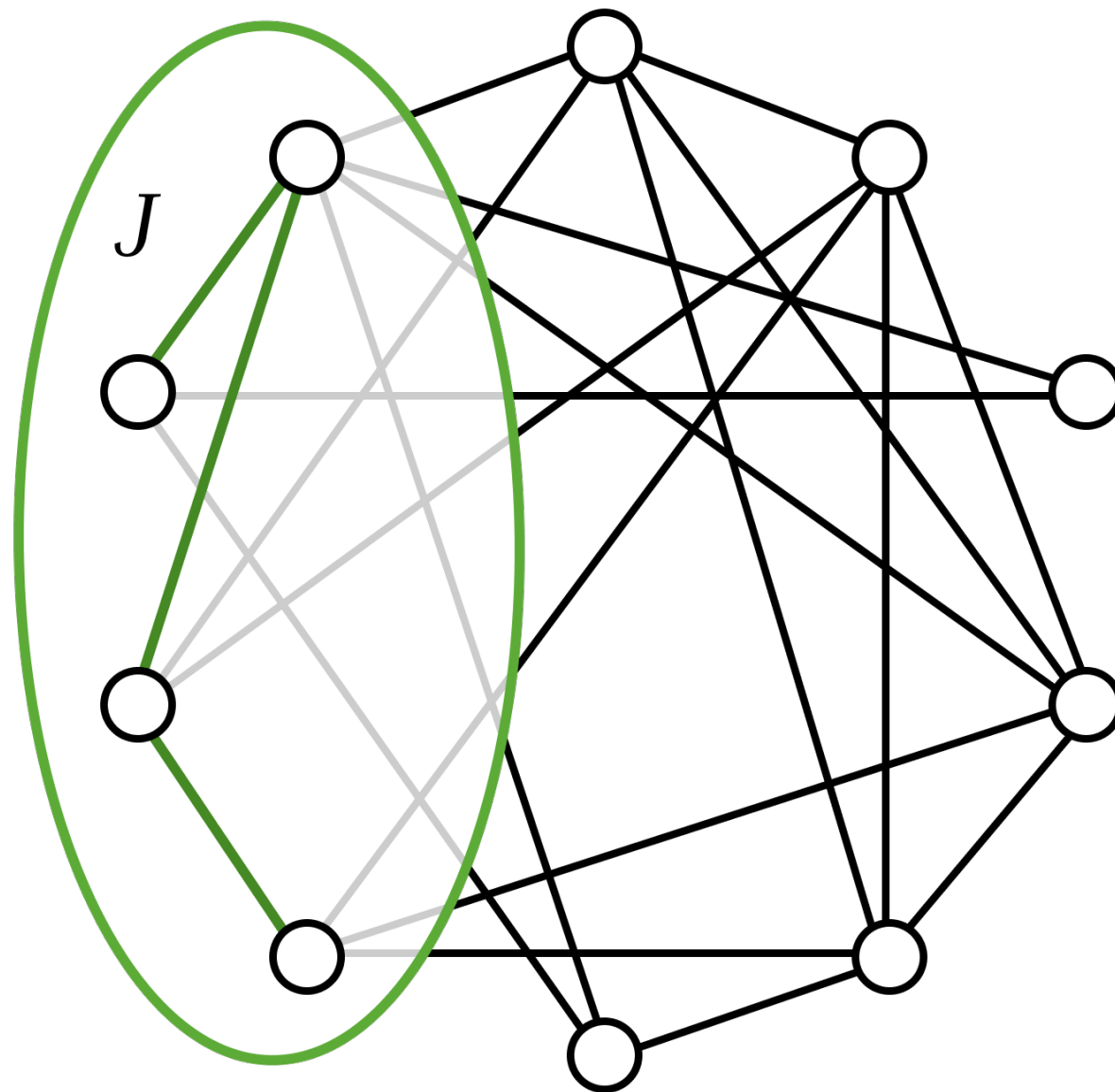
Encoding Sparse Sets

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How can I give you the most information about a sparse set J by just telling you about one (or two) of the vertices in J?

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Answer: Send ● and remove higher degree vertices

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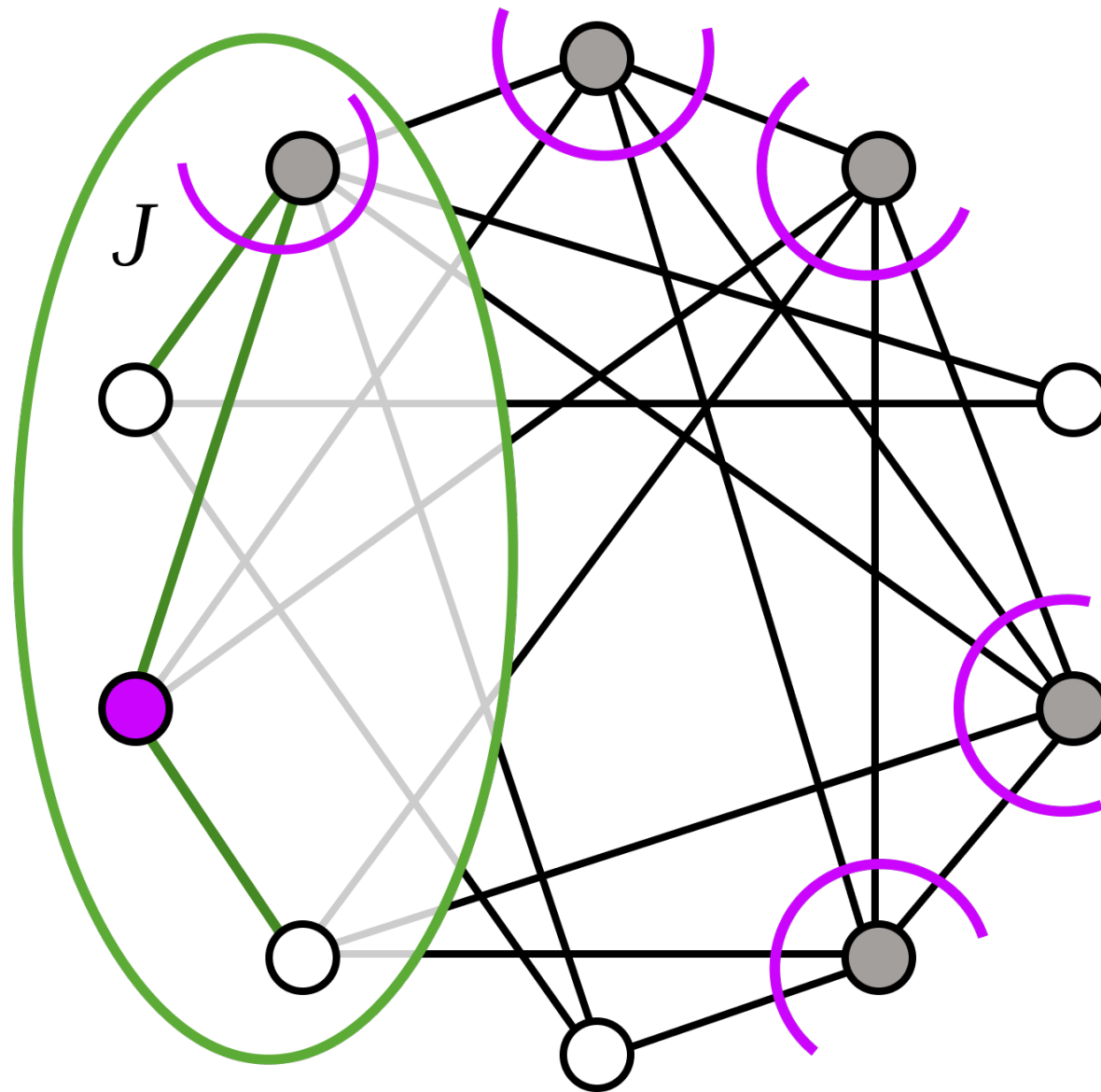
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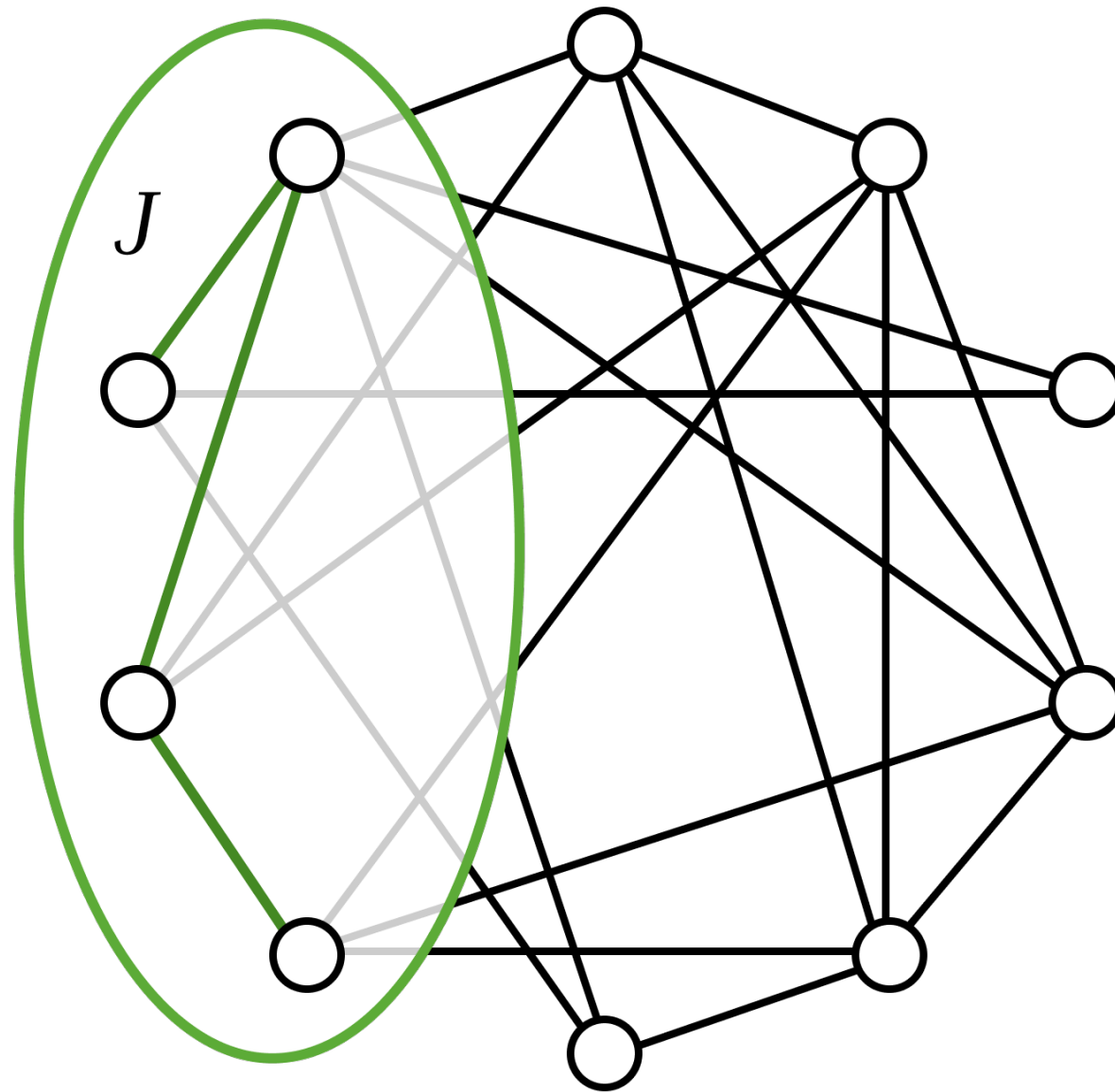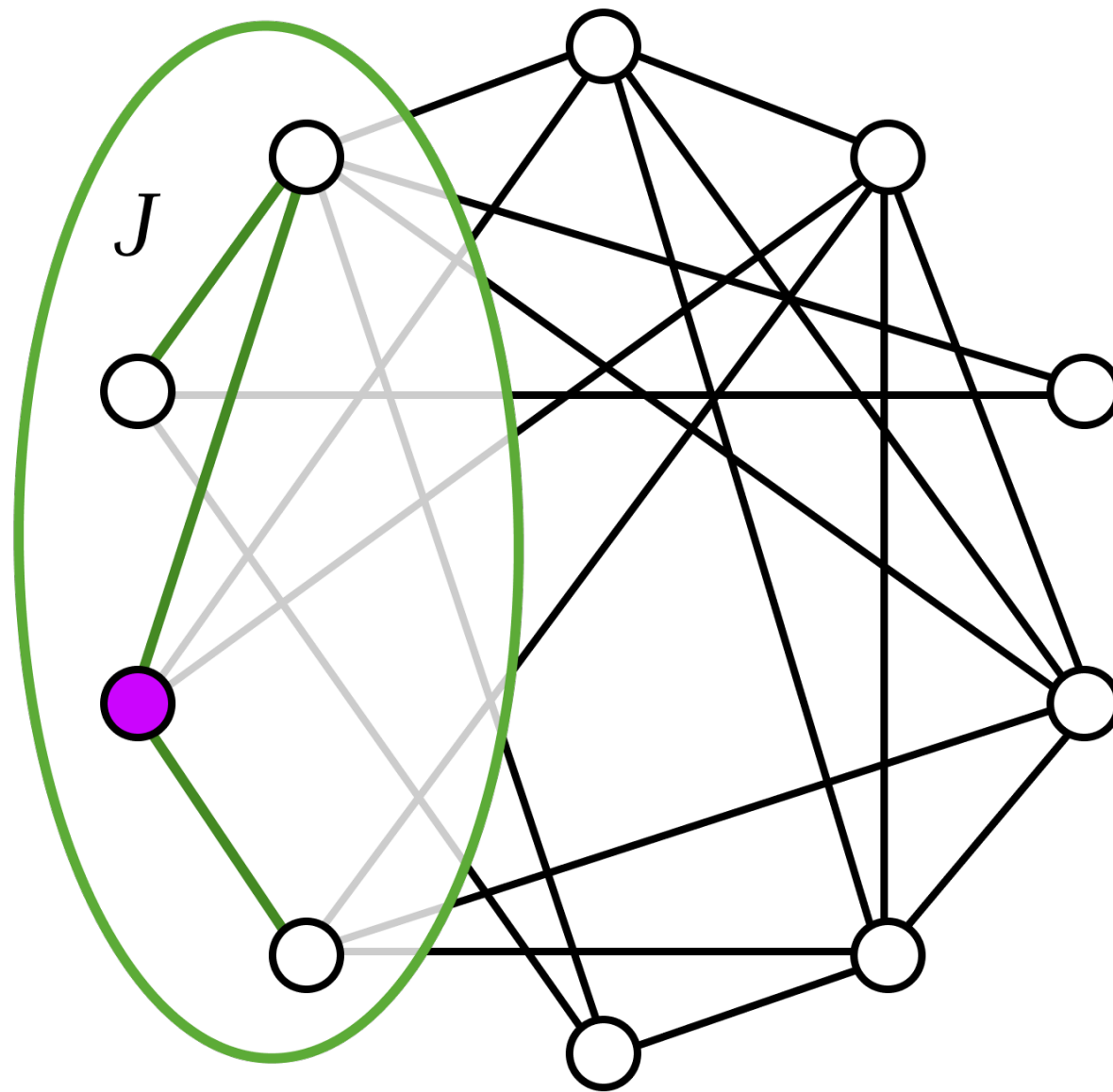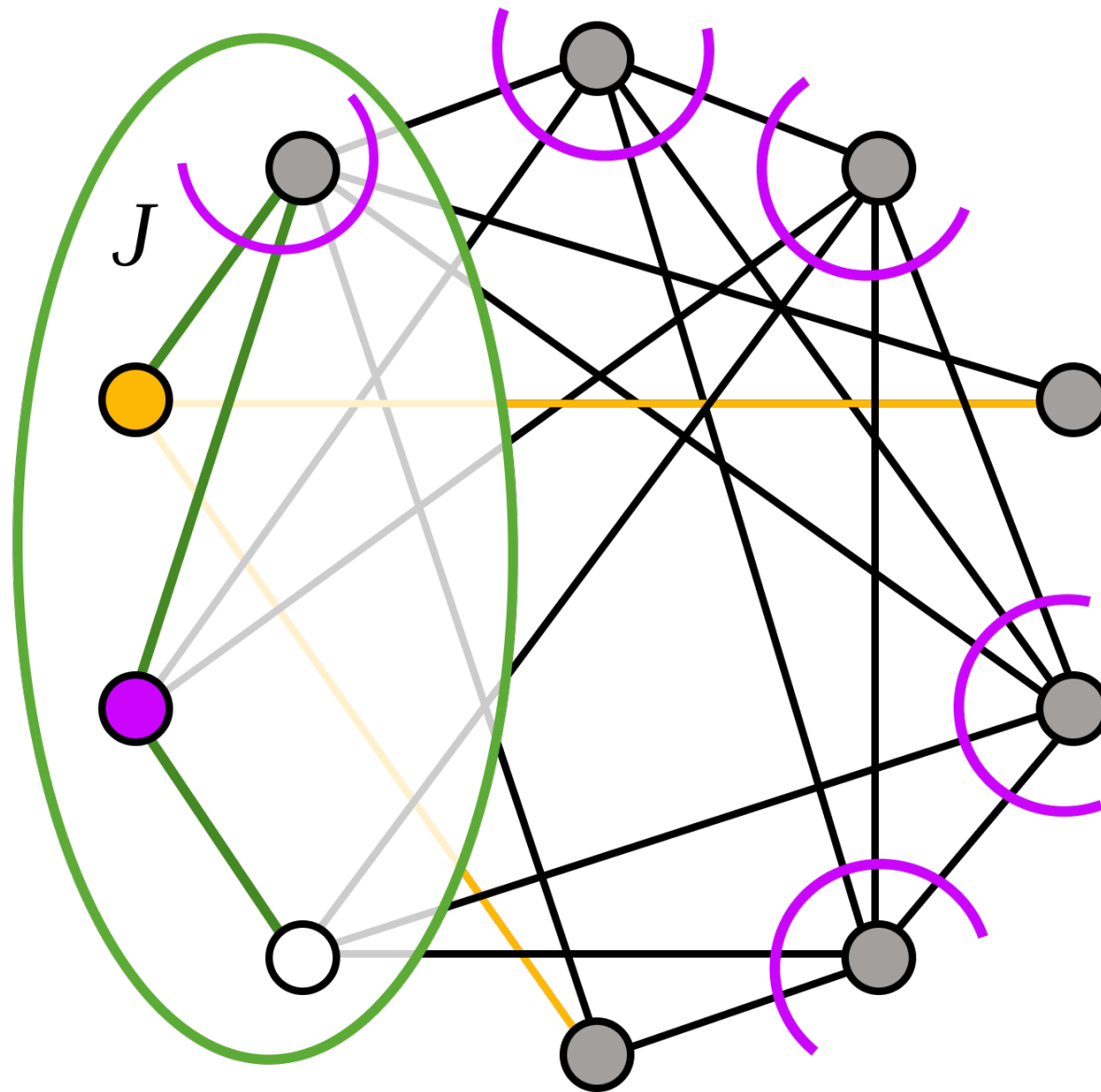

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C(J)

F(J)

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$C(J)$ $F(J)$

Observations:

- $C(J)$ can be reconstructed from $F(J)$

- J is NOT fully contained in $C(J)$

Weak Containment Lemma

Lemma: Let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Let $J \subseteq V$ have less than $\frac{\epsilon}{\log^2(1/\epsilon)} |J|^2$ edges. Then the final container $C(J)$ contains at least $\left(1 - \frac{1}{\log(1/\epsilon)}\right)$ fraction of J.

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Proof Idea:

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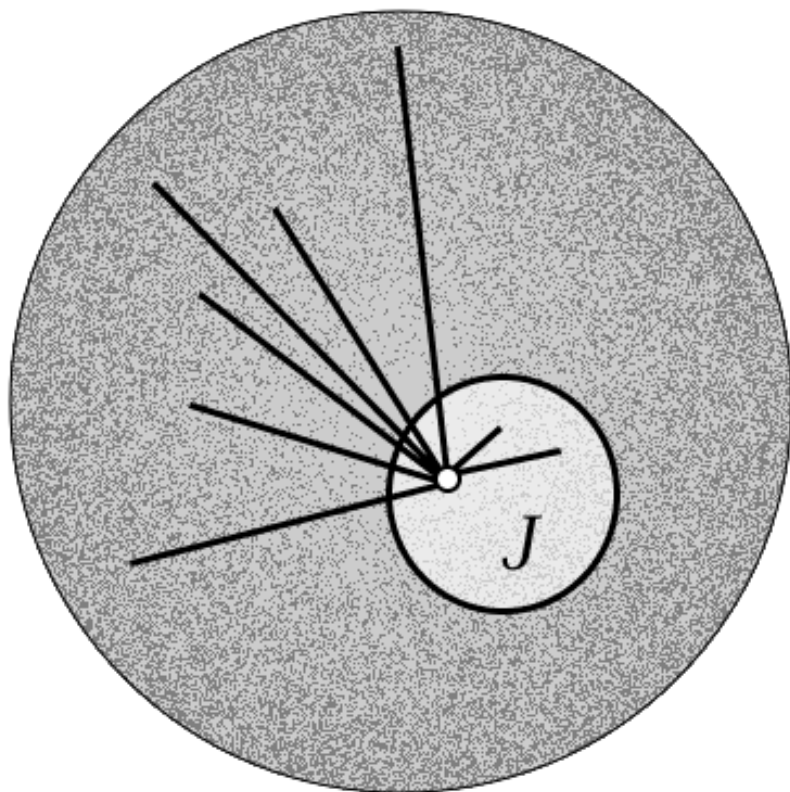
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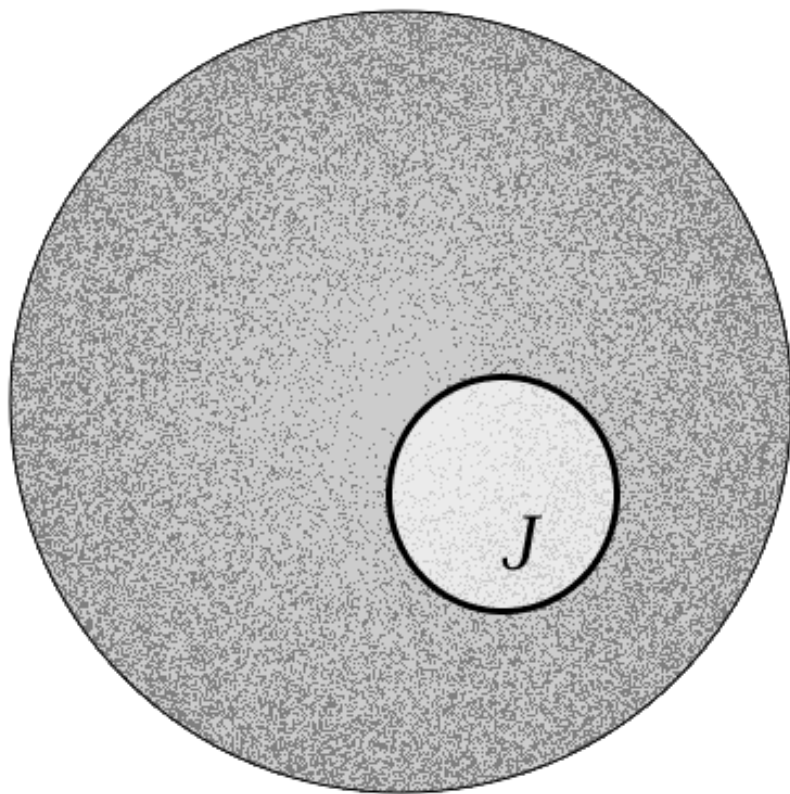

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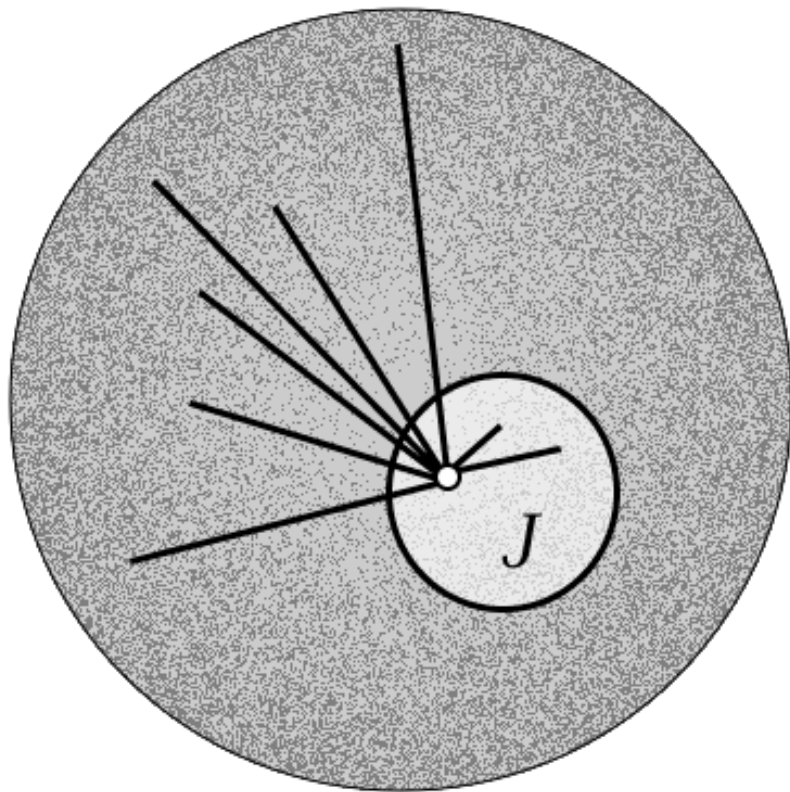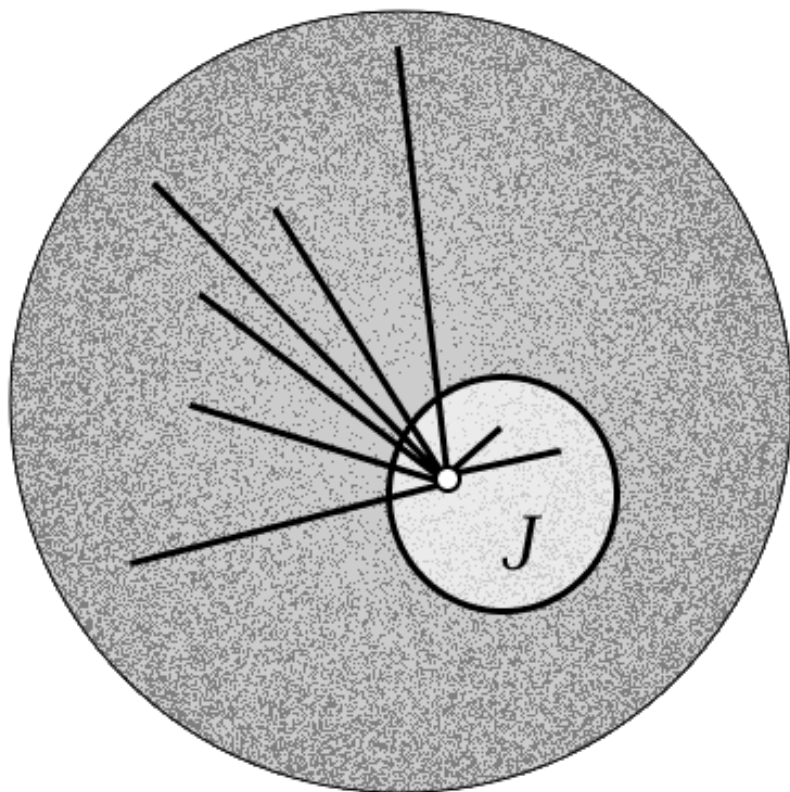

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- Using "remove neighbours" operation, can remove ϵn vertices from C and less than $\frac{\epsilon}{\log^2(1/\epsilon)}|J|$ vertices of J from C.

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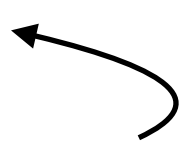ϵn vertices adjacent to everything

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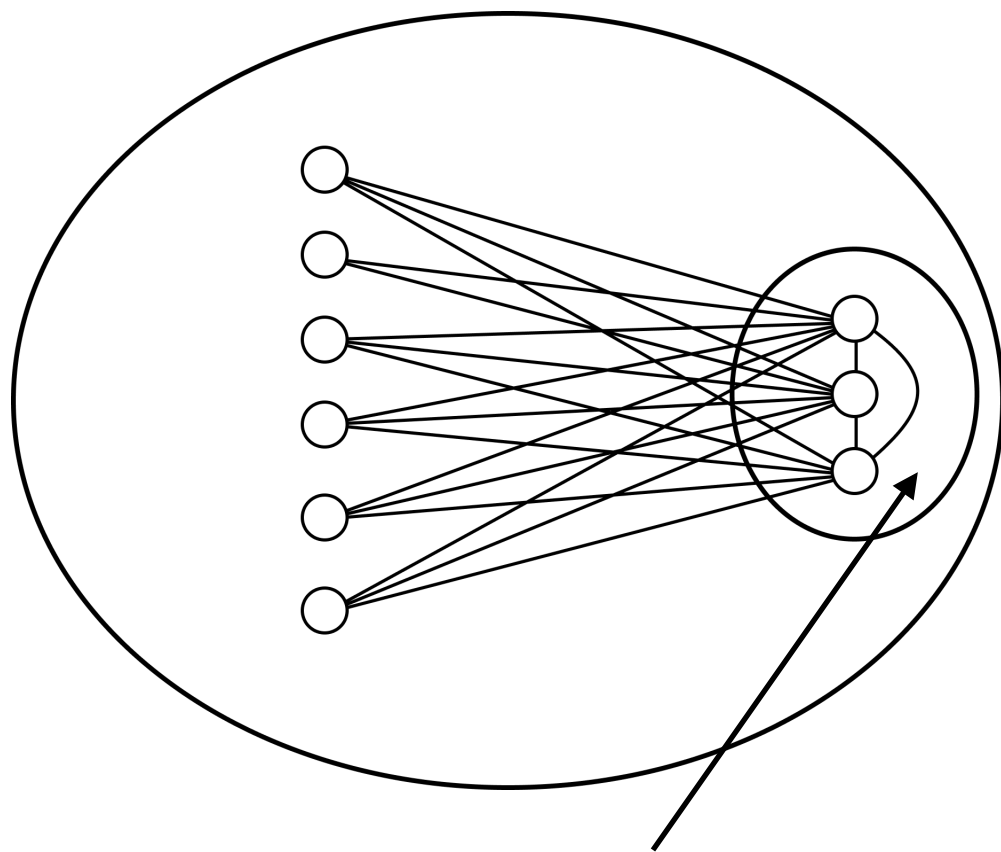
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Proof: Use the following lemma to generalize 2 cases.

Lemma: If $G[C]$ has more than ϵn^2 edges, then there exists $U \subseteq V$ such that every vertex in U has degree at least (roughly) $\dfrac{\epsilon n^2}{|U|}$ in $G[C]$.

$|U|$ vertices each with degree $\gtrsim \dfrac{\epsilon n^2}{|U|}$

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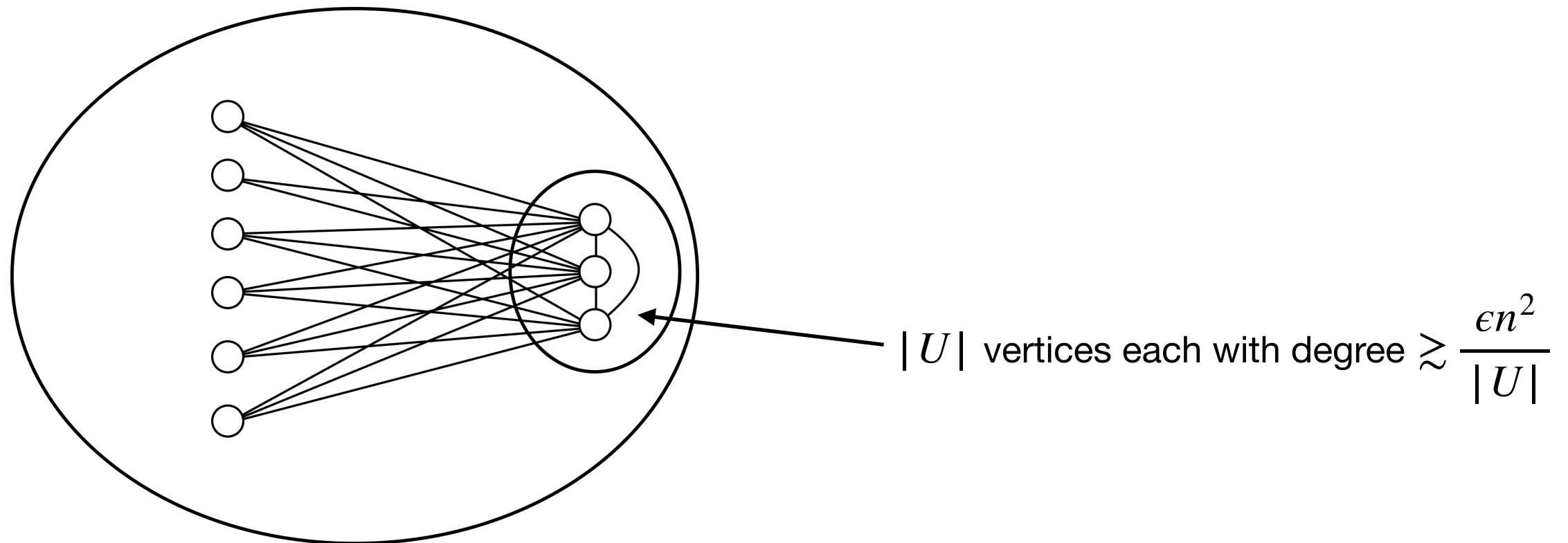
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Weak Containment Lemma Proof

Lemma: Let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Let $J \subseteq V$ have less than $\frac{\epsilon}{\log^2(1/\epsilon)} |J|^2$ edges. Then the final container $C(J)$ contains at least $\left(1 - \frac{1}{\log(1/\epsilon)}\right)$ fraction of J.

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Then, by the Claim above, at least $\log(1/\epsilon) \, \beta_1 n, \ldots, \log(1/\epsilon) \, \beta_k n$ vertices are removed from the container at each step.

Weak Containment Lemma Proof

Lemma: Let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Let $J \subseteq V$ have less than $\frac{\epsilon}{\log^2(1/\epsilon)} |J|^2$ edges. Then the final container $C(J)$ contains at least $\left(1 - \frac{1}{\log(1/\epsilon)}\right)$ fraction of J.

Proof:

Claim: In every step of procedure where $G[C]$ has more than ϵn^2 edges:

$$\frac{\text{\# vertices removed from } C \text{ by operation}}{\text{\# vertices in } J \text{ removed from } C \text{ by operation}} \gtrsim \log(1/\epsilon) \cdot \frac{n}{|J|}$$

Let $\beta_1 |J|, \ldots, \beta_k |J|$ be the number of vertices of J removed from the container at each step.

Then, by the Claim above, at least $\log(1/\epsilon) \beta_1 n, \ldots, \log(1/\epsilon) \beta_k n$ vertices are removed from the container at each step.

Since the initial container is of size n, then $\sum_i \beta_i \leq \frac{1}{\log(1/\epsilon)}$, otherwise the final container would have less than 0 vertices. $\qquad \square$

Towards Proving Stronger Lemma

Stronger Lemma (Restated): For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq P(V)$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,

2. for every $C \in \mathscr{C}$, $|C| < \rho n$.

3. For every set $J \subseteq V$ such that $G[J]$ has less than $\dfrac{\epsilon}{\text{polylog}(1/\epsilon)} |J|^2$ edges, there exists $C \in \mathscr{C}$ and α such that $|C| \leq (1 - \alpha)\rho n$ and

$$|C \cap J| \geq \left(1 - \frac{\alpha}{2}\right) |J|.$$

- Weak containment lemma shows that $|C \cap J| \geq \left(1 - \dfrac{1}{\log(1/\epsilon)}\right) |J|$

- Full proof involves showing that when container is large (close to ρn) the container procedure makes faster progress OR we can shrink the container at the end of the process (full details in paper)

Another Application of new Container Lemma

Counting Independent Sets in Regular Graphs

Theorem: Let G be a $d-$regular graph. Then the number of independent sets in G is at most $2^{\frac{n}{2}\left(1 + \frac{1}{d}\right)}$, and there exists a $d-$regular graph achieving this maximum ($\frac{n}{2d}$ copies of $K_{d,d}$).

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One of the classic applications of container method

Question: What about sparse sets in a $d-$regular graph? Can we use the new container lemma in place of the standard container lemma used by Sapozhenko?

Counting Sparse Sets in Regular Graphs

Theorem: Let G be a $d-$regular graph. Let $k \geq \text{polylog}(n)$. Then the number of induced subgraphs in G with edge density less $\frac{1}{k}\frac{d}{n}$ is at most

$$2^{\frac{n}{2}\left(1 + O\left(\frac{\text{polylog}(n)}{d}\right) + O\left(\frac{\text{polylog}(n)}{k^{1/3}}\right)\right)}.$$ [This Work]

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Remarks:

- Observe that the edge density of G is roughly $\frac{d}{n}$ and there are at least $\frac{1}{2}2^n$ induced subgraphs with edge density at least $\frac{4d}{n}$

- [Zhao '10] says that there are at most $2^{\frac{n}{2}\left(1 + \frac{1}{d}\right)}$ induced subgraphs with edge density 0.

- Our theorem shows that even for edge density up to $\frac{1}{\text{polylog}(n)}\frac{d}{n}$ there are still only roughly $2^{\frac{n}{2}}$ induced subgraphs.

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- $\frac{n}{2d}$ copies of $K_{d,d}$ gives lower bound of $2^{\frac{n}{2}\left(1 + \frac{1}{d} + \frac{1}{k}\right)}$

Open Questions

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Thank you!