A Tolerant Independent Set Tester

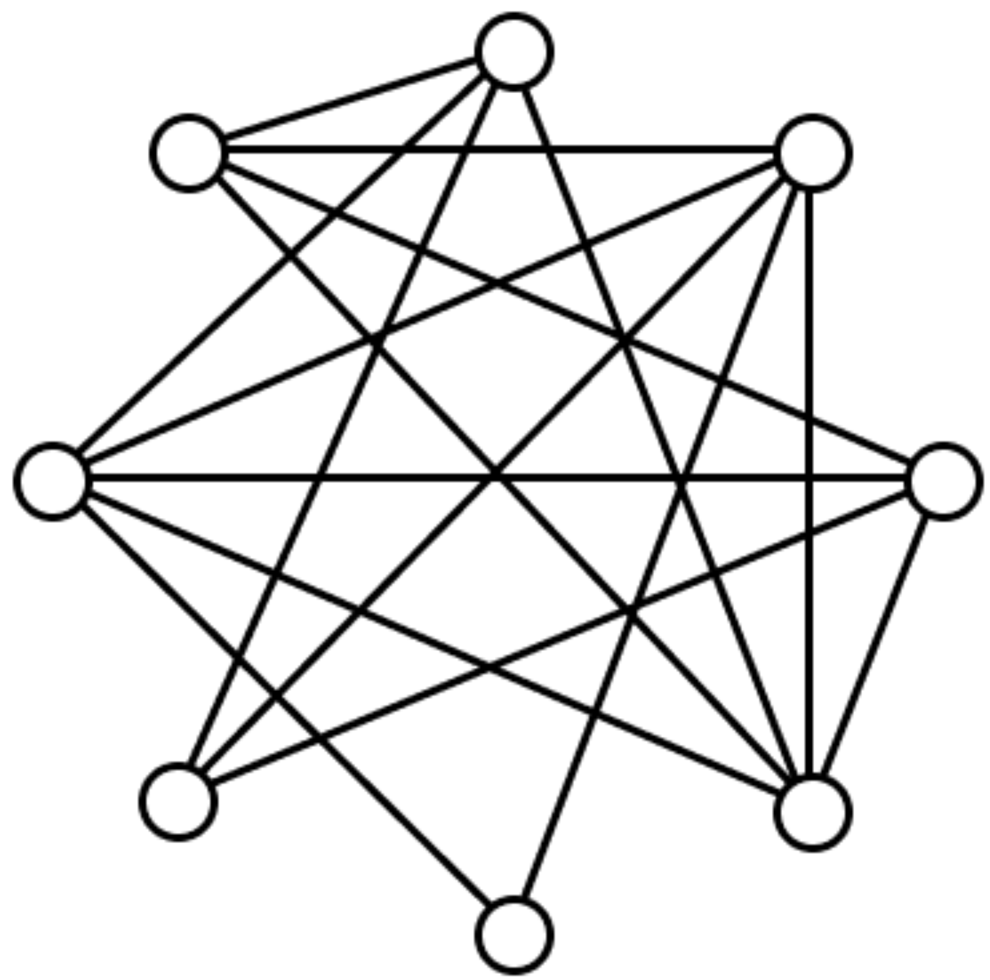
Cameron Seth

STOC 2025

UNIVERSITY OF WATERLOO

Independent Set Problem

Given a graph G on n vertices, does it have an independent set of size ρn?



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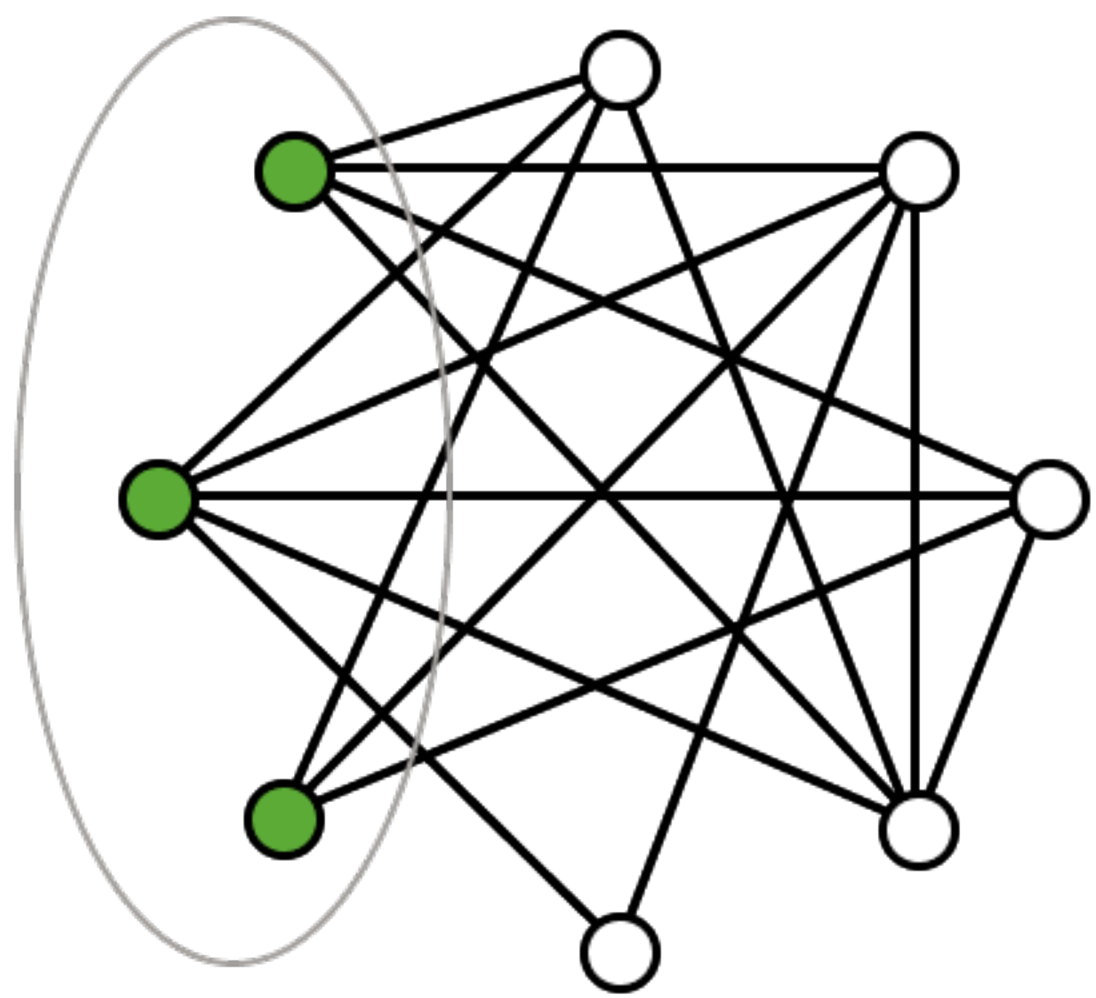
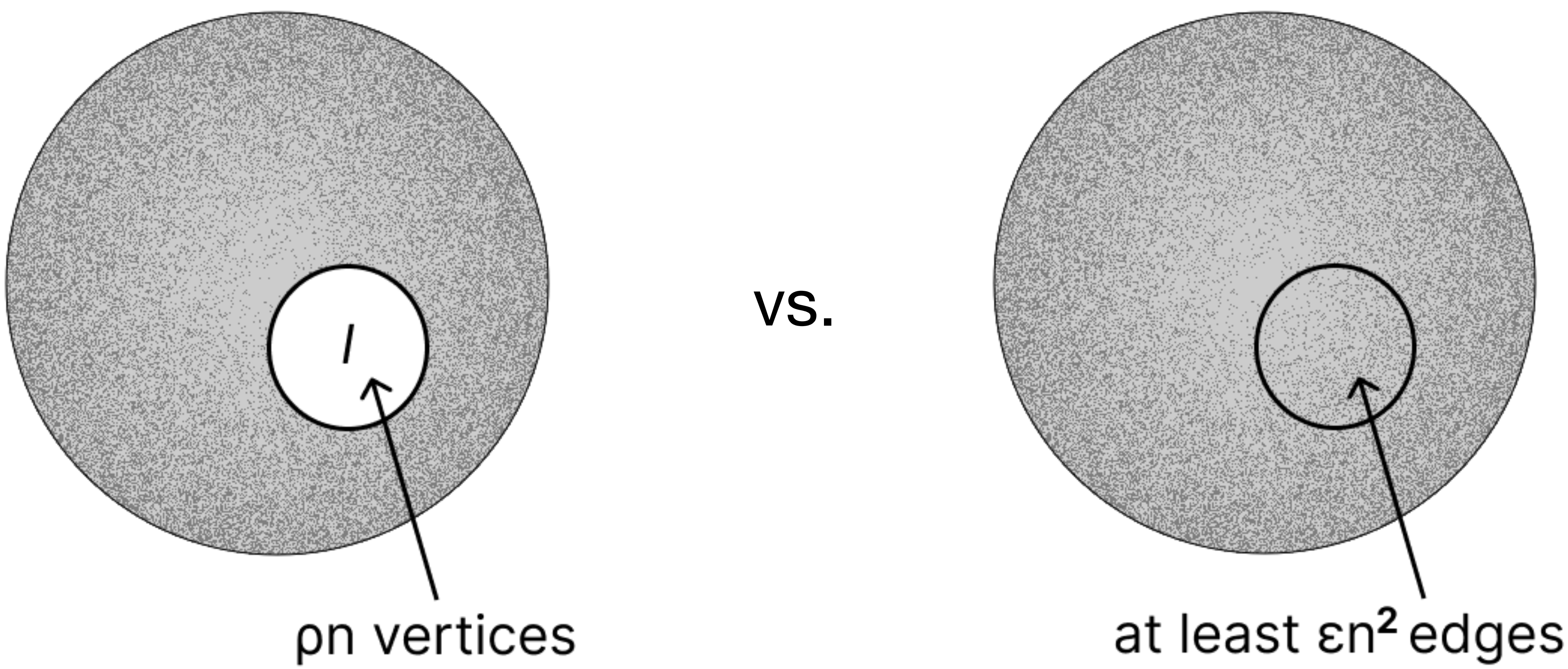


Testing Independent Sets

Problem: Distinguish between the cases:

(i) G has a ρn independent set, and

(ii) every induced subgraph of size ρn has at least ϵn^2 edges



Theorem: Inspecting a random subgraph on $\tilde{O}(\rho/\epsilon^4)$ vertices is sufficient for distinguishing between (i) and (ii) (whp).

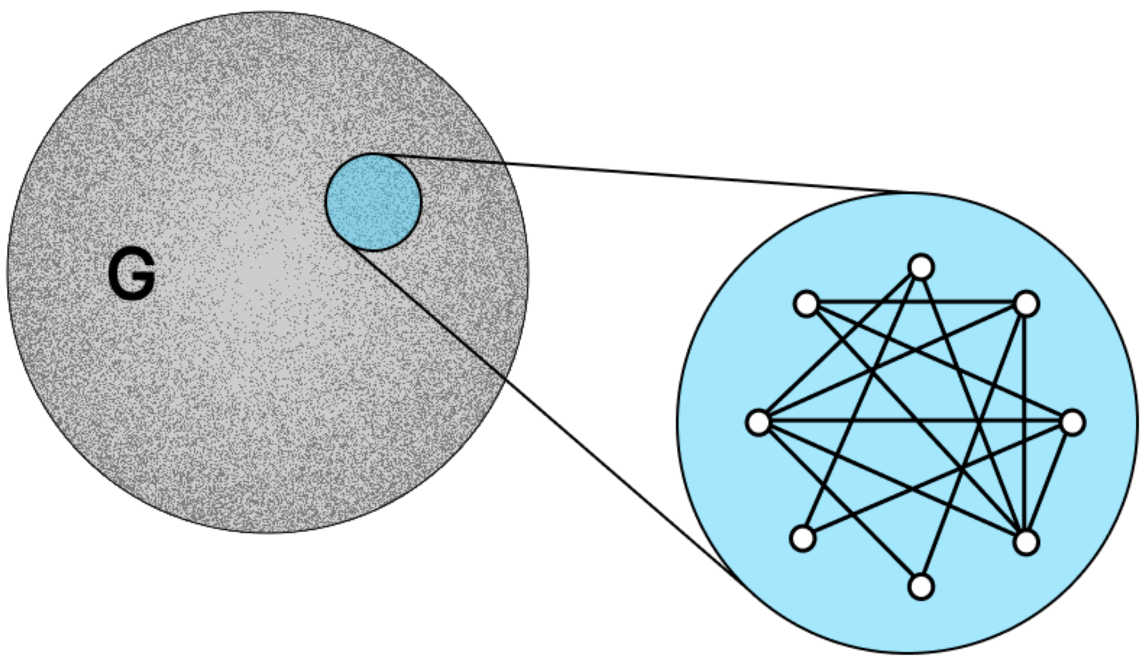
[Goldreich, Goldwasser, Ron '98]

Definitions

An ϵ-tester for the ρn-independent set property is an algorithm that samples a set S of s random vertices, examines the induced subgraph $G[S]$, and distinguishes between the cases (with high probability):

(i) G has a ρn independent set, and

(ii) every induced subgraph of size ρn has at least ϵn^2 edges (ϵ—far)



s is the sample complexity of the tester.

Testing Independent Sets

Theorem: There exists an $\epsilon-$tester for the ρn independent set property with sample complexity $\tilde{O}(\rho/\epsilon^4)$.

[Goldreich, Goldwasser, Ron ’98]

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Theorem: There exists an $\epsilon-$tester for the ρn independent set property with sample complexity $\tilde{O}(\rho^3/\epsilon^2)$, [Blais, Seth '23]

and any such tester has sample complexity $\Omega(\rho^3/\epsilon^2)$.

[Feige, Langberg, Schechtman '04]

Weakness with Standard Testing

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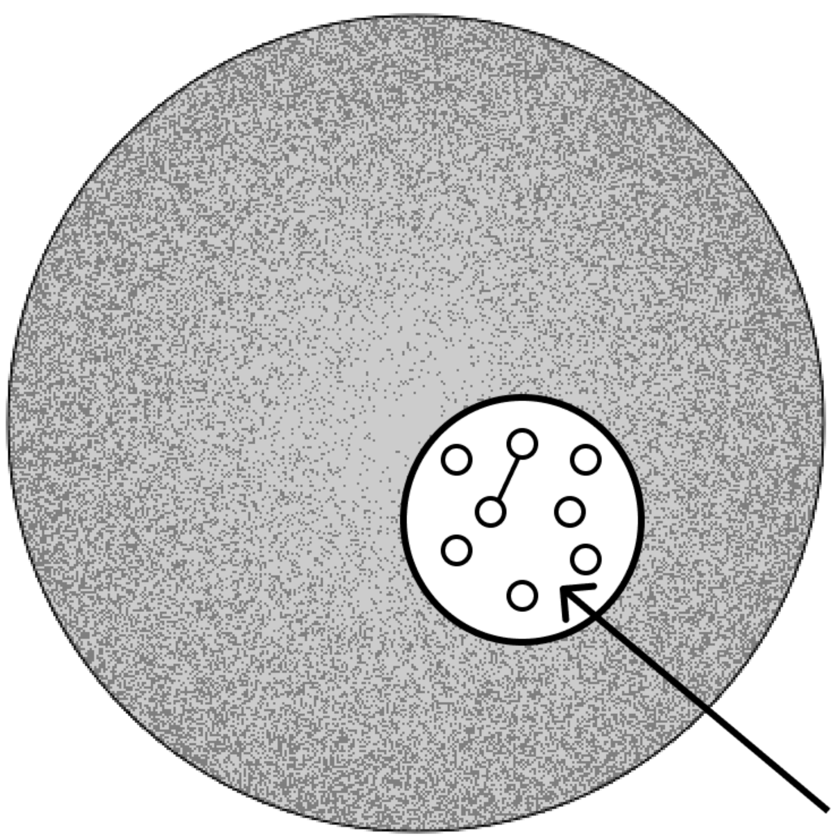

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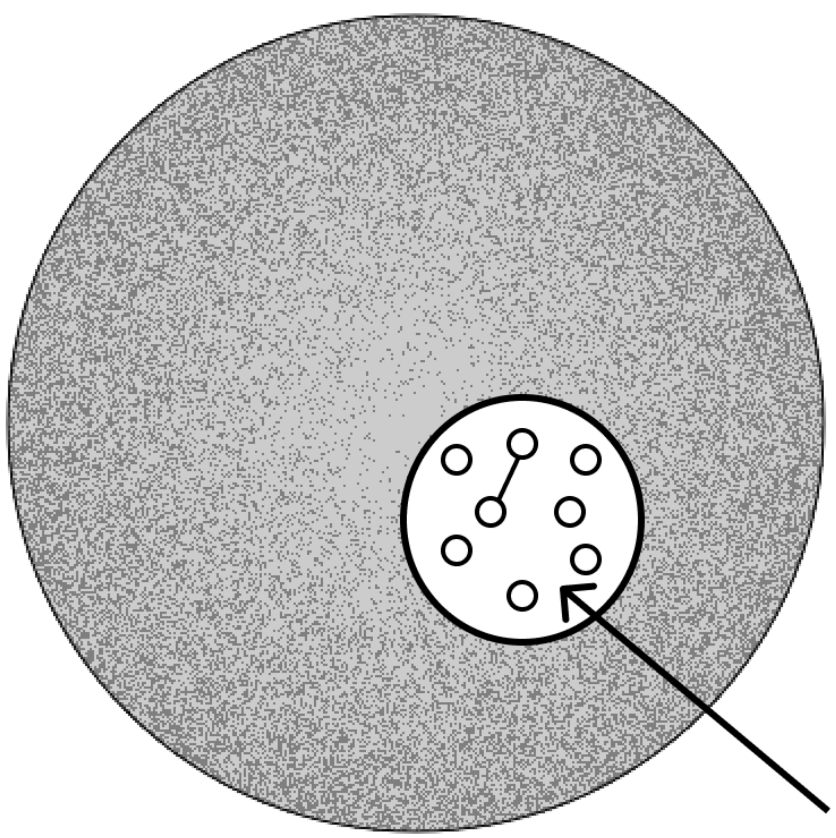


ρn vertices with 1 edge

Answer: Testing algorithms have no guarantee on this type of input!

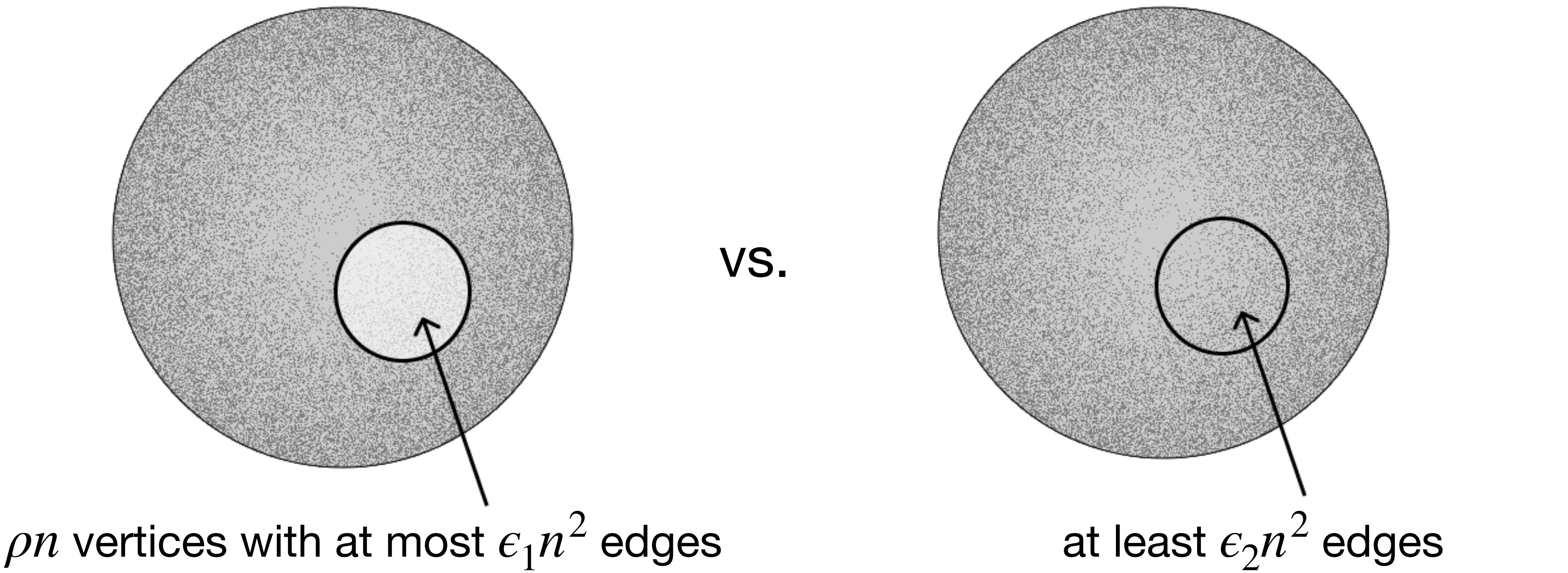
Ideally we would like the algorithm to accept this type of graph.

Tolerant Testing Independent Sets

Problem: For $\epsilon_1 < \epsilon_2$, distinguish between the cases:

(i) G has an induced subgraph of size ρn with fewer than $\epsilon_1 n^2$ edges (ϵ_1-close)

(ii) Every induced subgraph of size ρn has at least $\epsilon_2 n^2$ edges (ϵ_2-far)



An algorithm that, with high probability, distinguishes between (i) and (ii) is called an $(\epsilon_1, \epsilon_2)-$tester.

[Parnas, Ron, Rubinfeld '06]

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Remarks:

- Generalizes the standard testing problem ($\epsilon_1 = 0$)
- In general ϵ_1 may be a function of ϵ_2
- In some other settings (bounded degree model, boolean strings), there is exponential gap between the query complexity of $\epsilon-$testing and $(\tilde{\Theta}(\epsilon), \epsilon)-$tolerant testing.

[Fischer, Fortnow '05]

[Goldreich, Wigderson '22]

Main Result

Theorem: There is a $\left(\frac{\epsilon}{\text{polylog}(1/\epsilon)}, \epsilon\right)-$ tolerant tester for the ρn independent set property with sample complexity $\tilde{O}(\rho^3/\epsilon^2)$.

[This Work]

Main Result

Theorem: There is a $\left(\frac{\epsilon}{\mathrm{polylog}(1/\epsilon)}, \epsilon\right)-$ tolerant tester for the ρn independent set property with sample complexity $\tilde{O}(\rho^3/\epsilon^2)$.

[This Work]

Remarks:

- Matches the (optimal) sample complexity bound for $\epsilon-$testing
- Generalizes container method approach of Blais, Seth '23
- Best prior result is from a general result for all graph partition properties, which gives sample complexity of roughly $(1/\epsilon)^{12}$

[Fiat, Ron '21]

[Goldreich, Goldwasser, Ron '98]

Outline of Talk

- Main technique to prove theorem: graph container method
    - What is the container method?
    - How to use the container method to prove testing results

[Blais, Seth ’23]

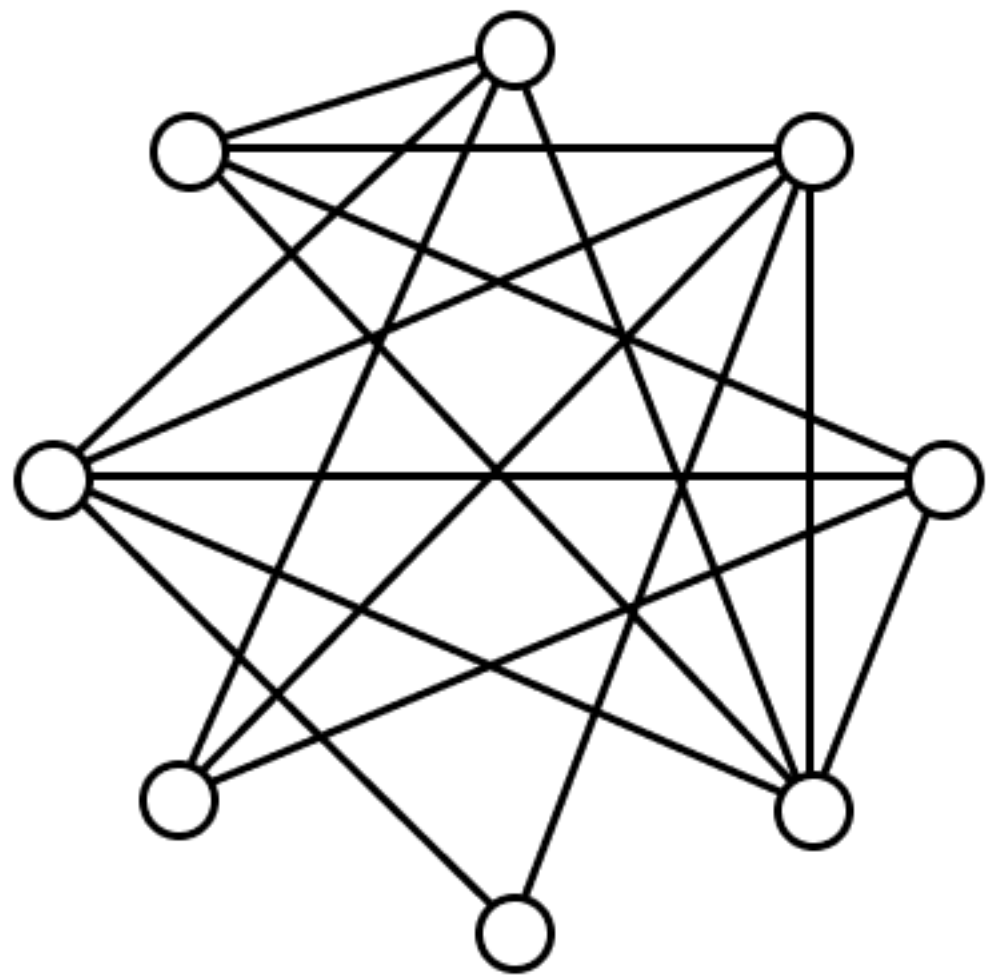
- A new graph container lemma for sparse subgraphs

[This Work]

    - Proof ideas of new container lemma
    - Another application of new container lemma

What is the Container Method?

Answer: A tool for characterizing independent sets in some graphs.



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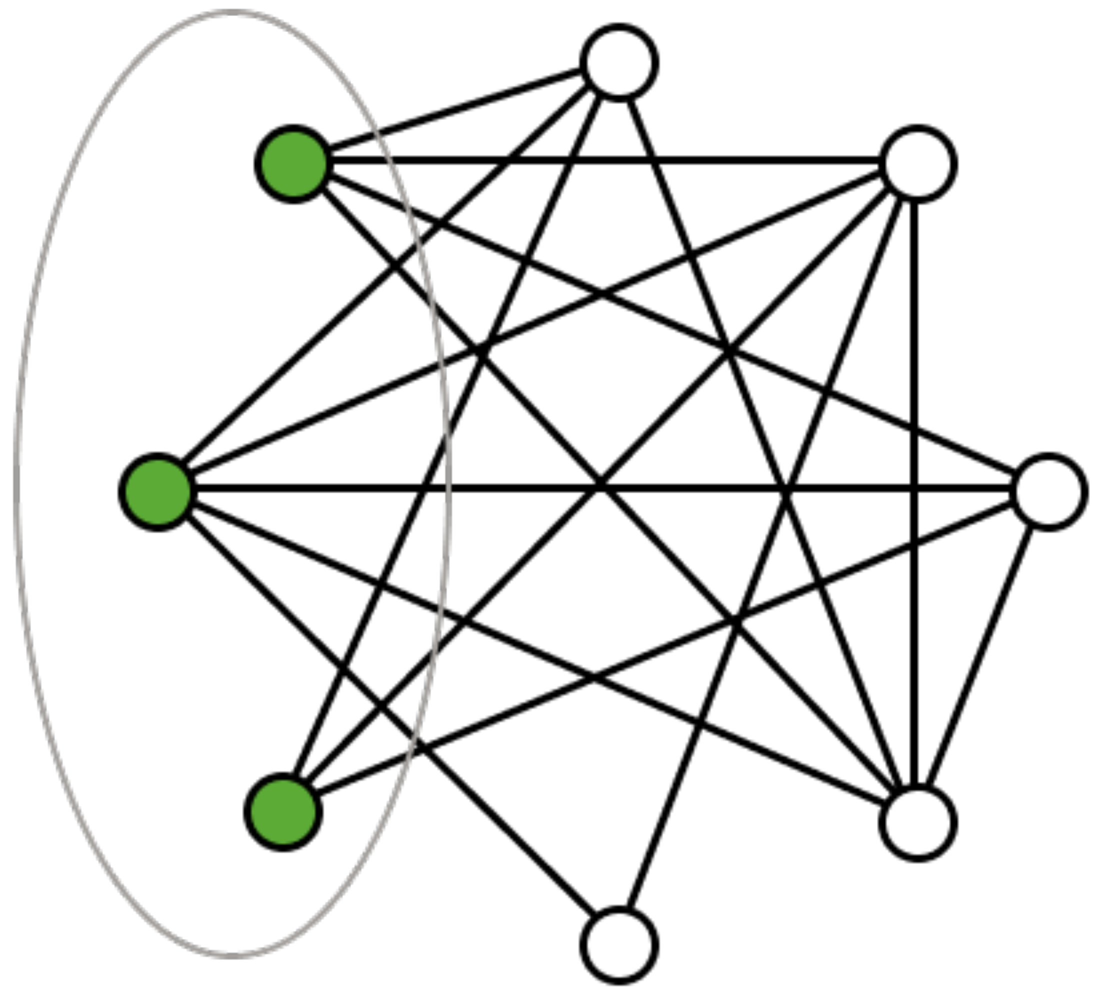

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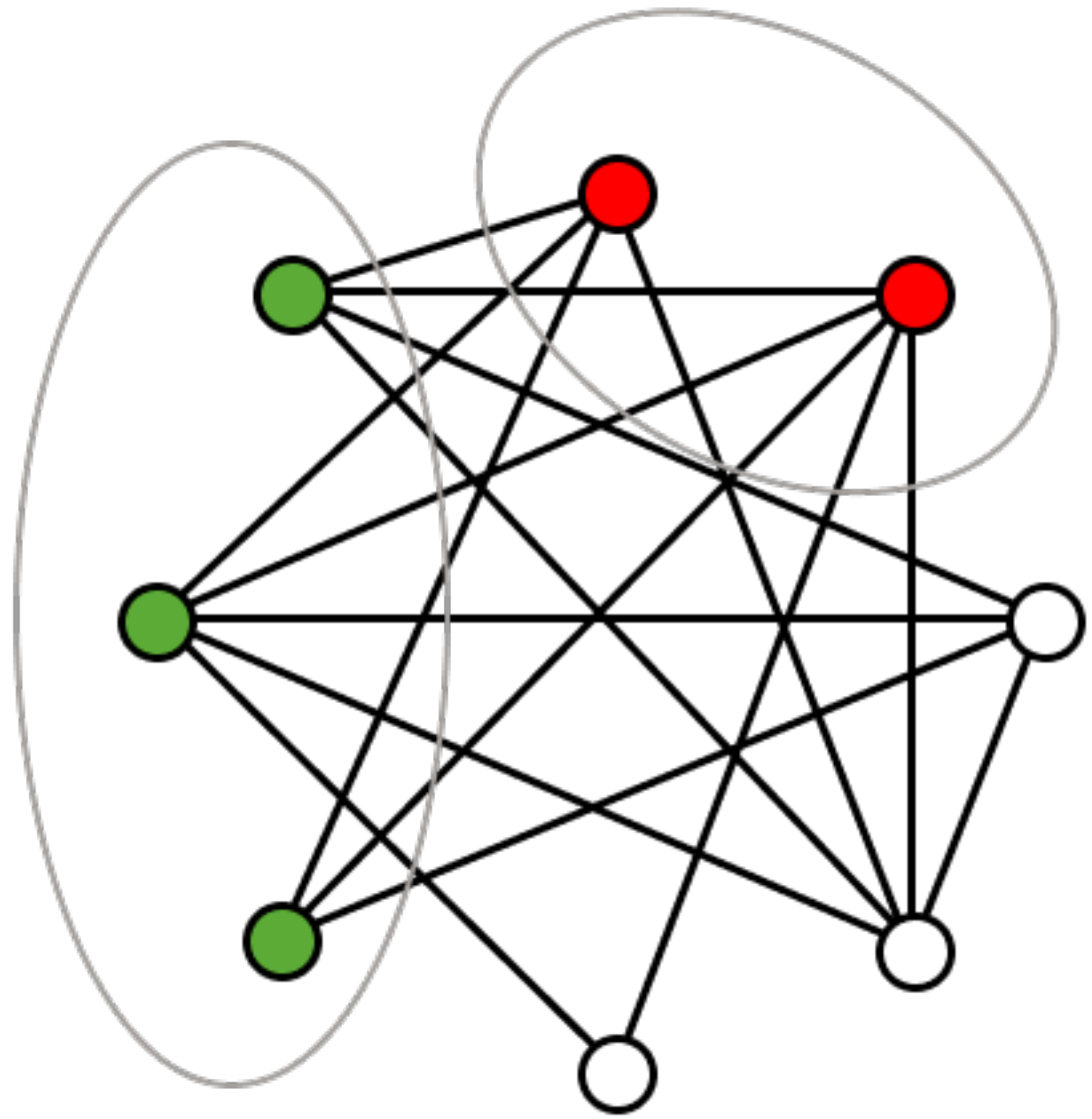

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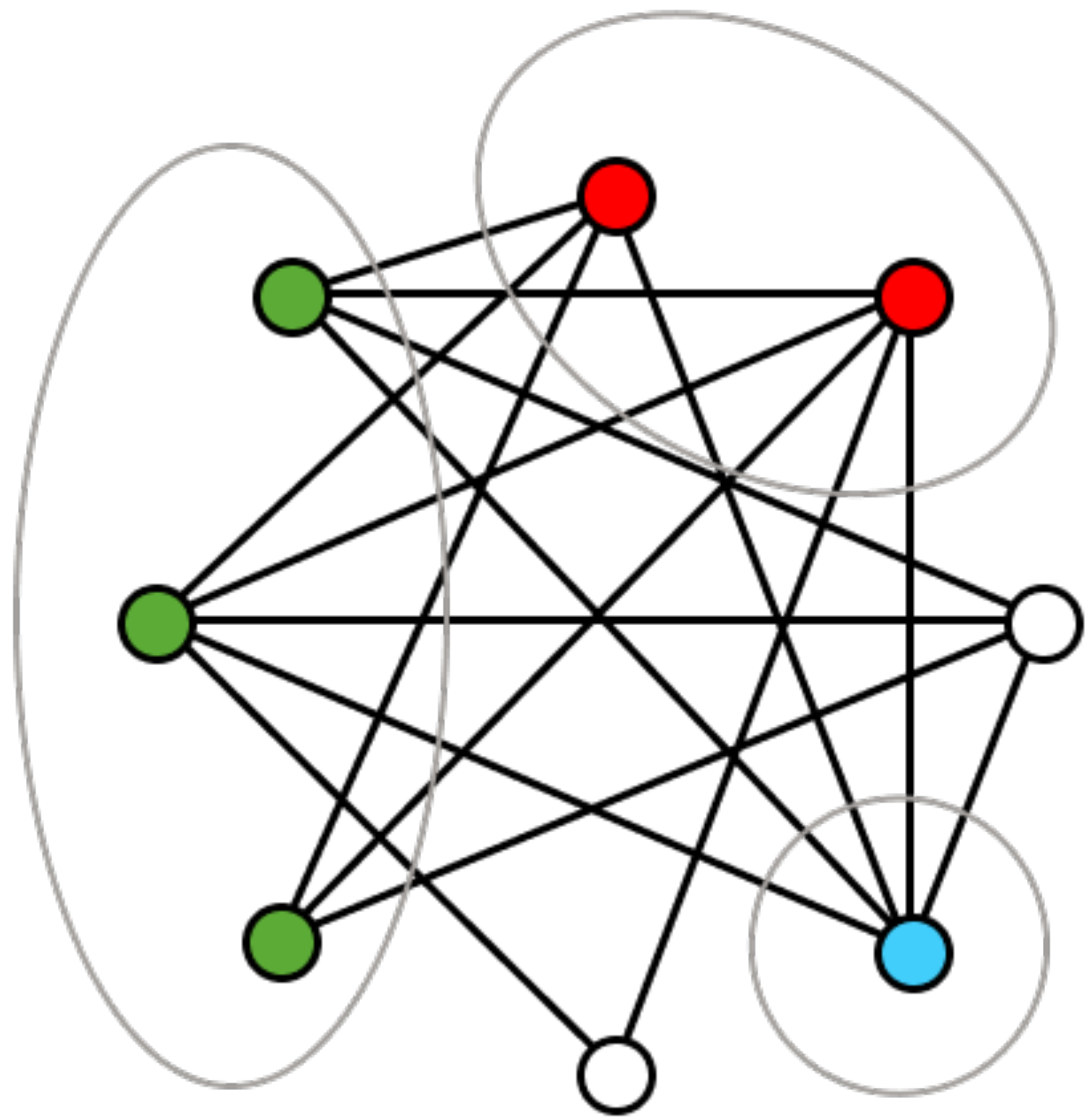

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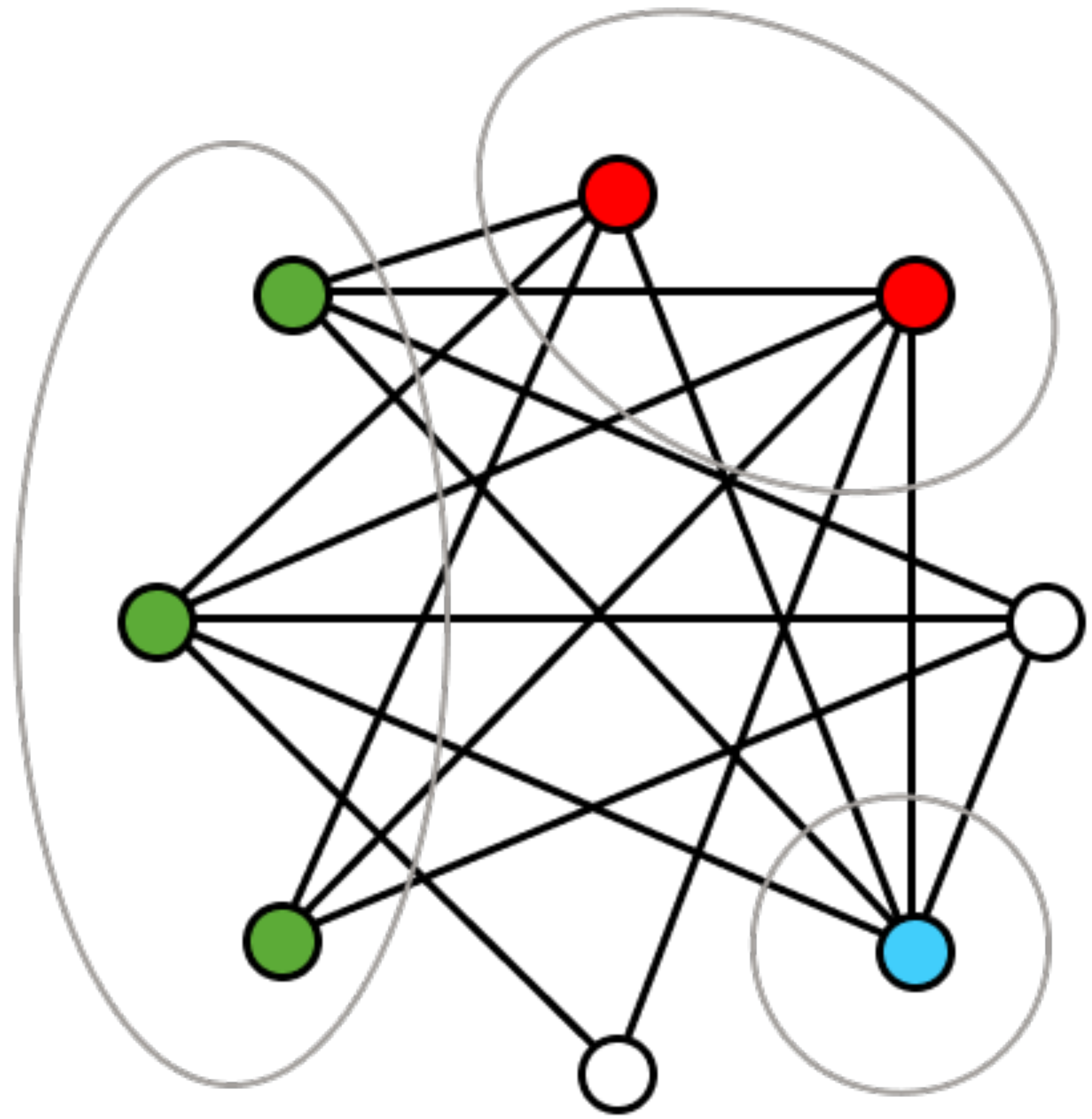

Informal Idea: For any graph satisfying some “nice” conditions, all independent sets in the graph can be covered by a small number of containers (each container is a subset of vertices).

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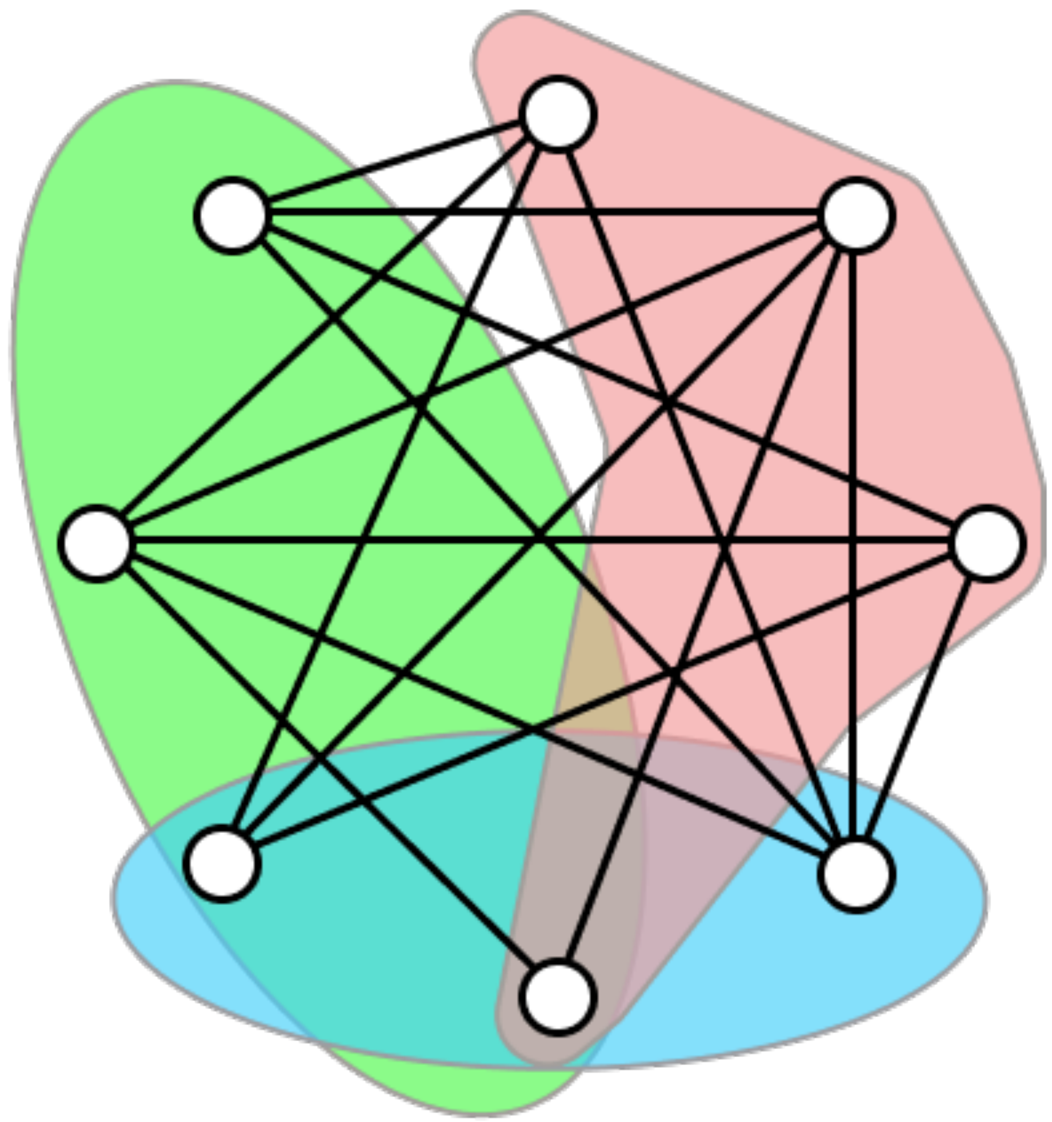

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An Initial Graph Container Lemma

Lemma: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,
2. for every $C \in \mathscr{C}$, $|C| \lesssim (1 - \epsilon)\rho n$.
3. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$.

[Kleitman, Winston '82]

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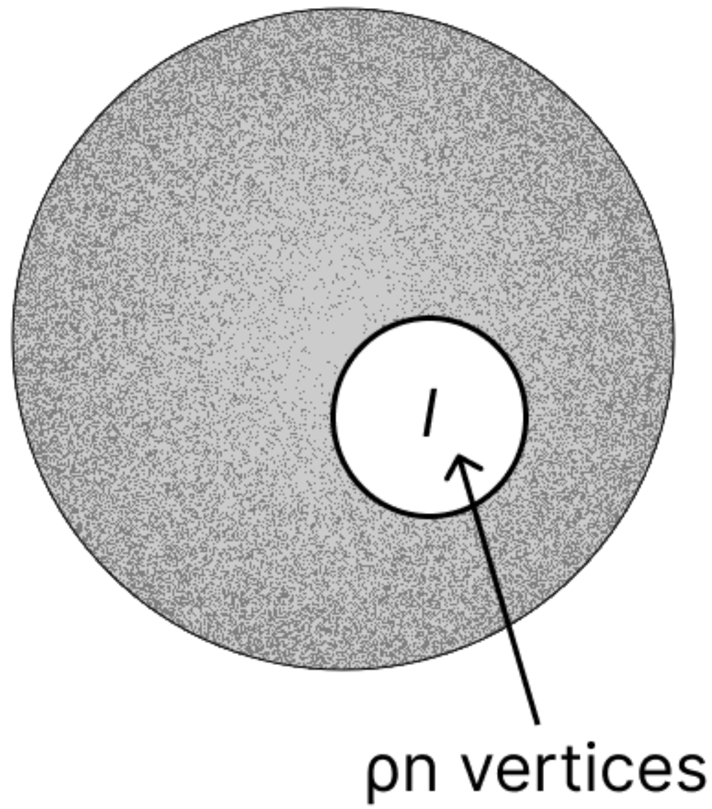
Note: for survey of combinatorial applications see "Counting Indepedent Sets in Graphs" by Samotij or "The method of hypergraph containers" by Balogh, Morris, and Samotij.

Application to Non-Tolerant Testing



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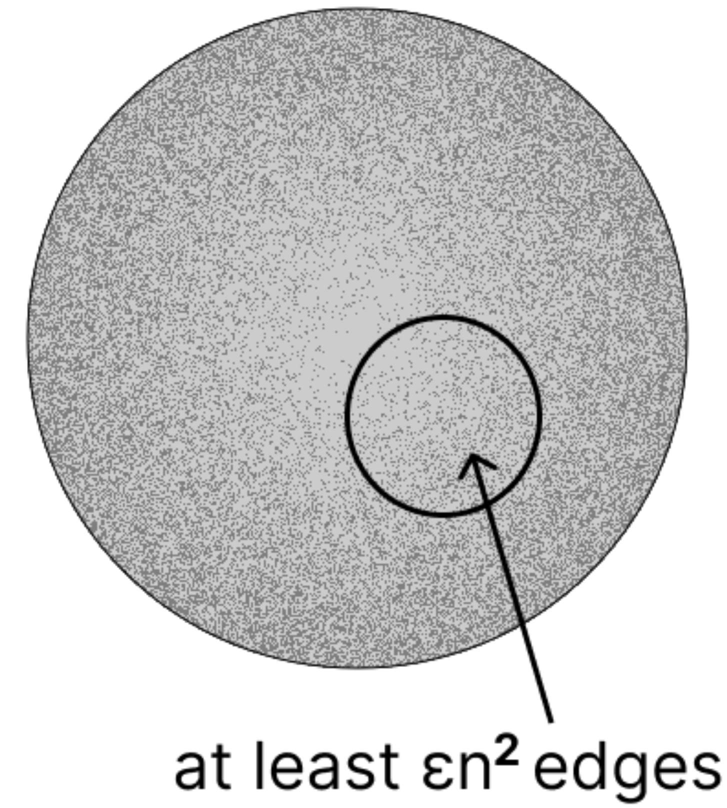

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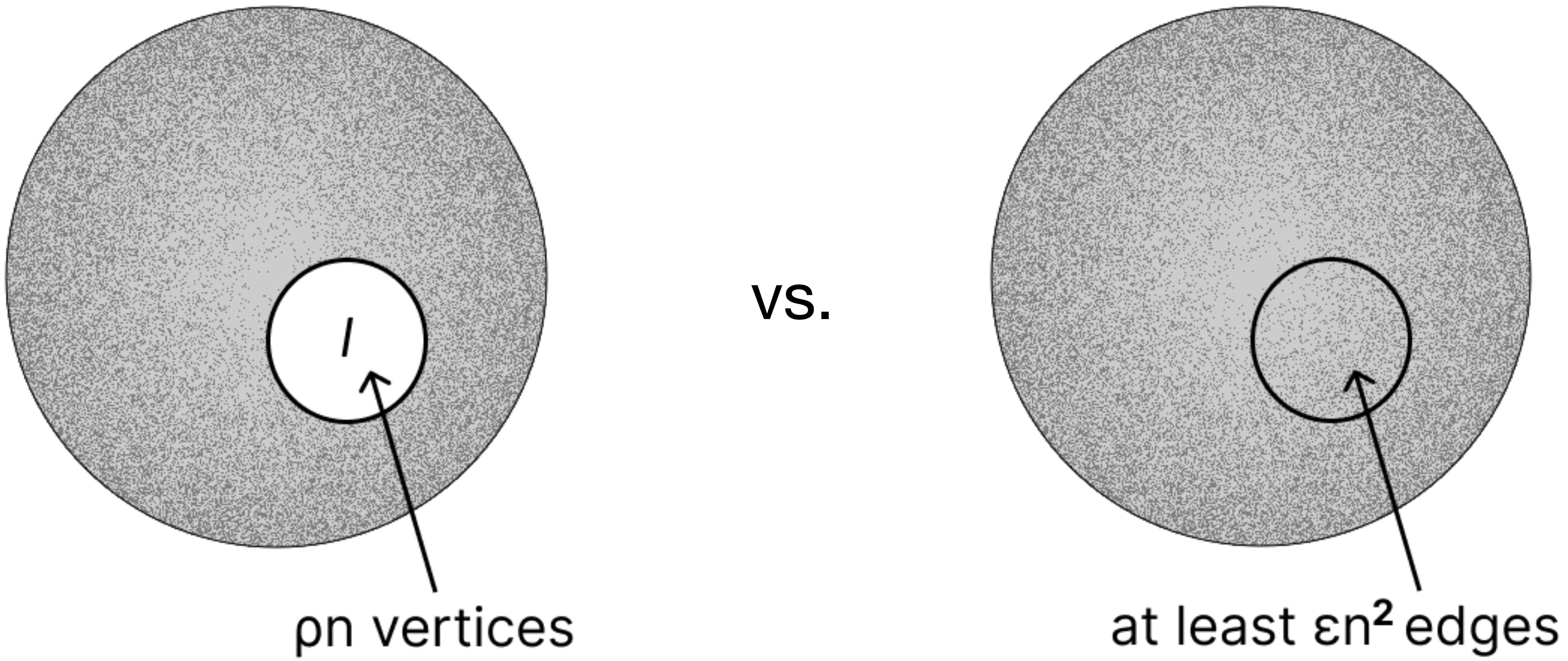

Testing Algorithm: Take a random sample S of $s \sim \rho^3/\epsilon^2$ vertices, check if the induced subgraph $G[S]$ has a ρs independent set.

[Blais, Seth ’23]

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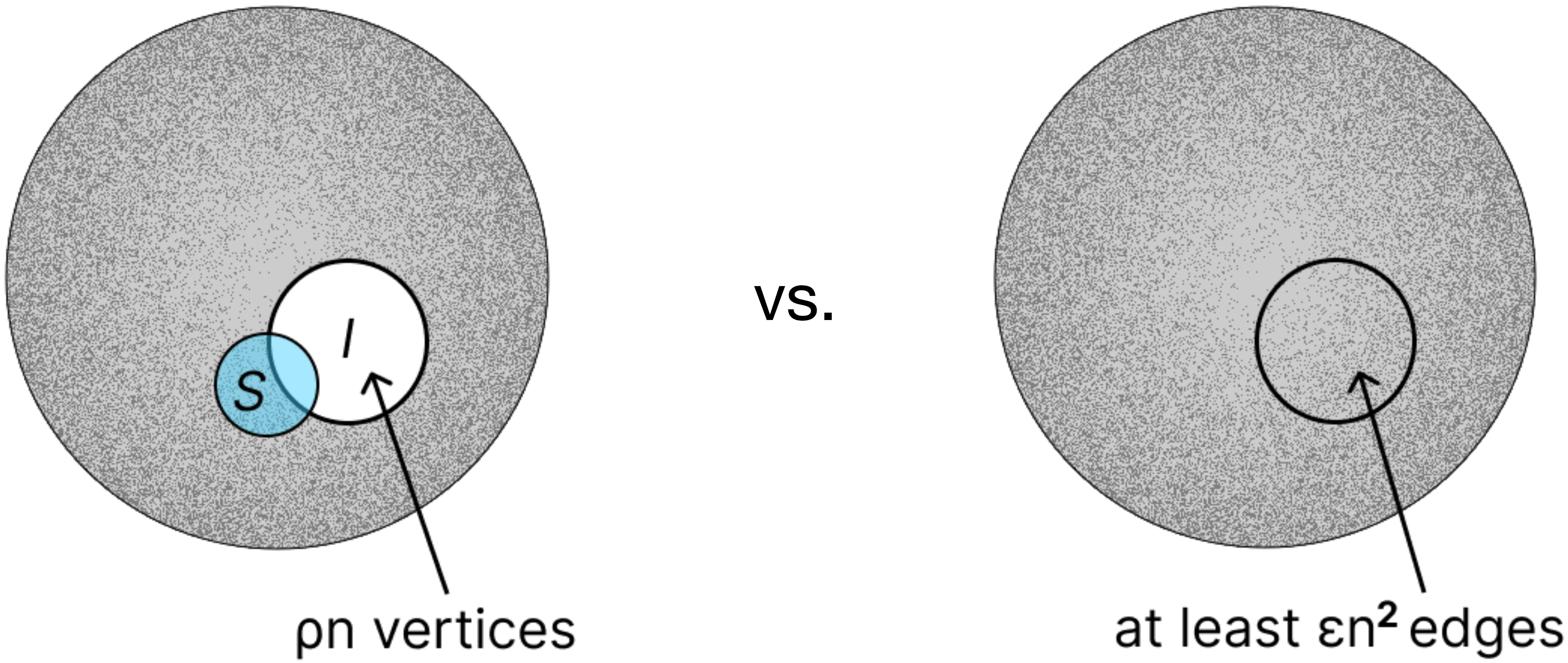

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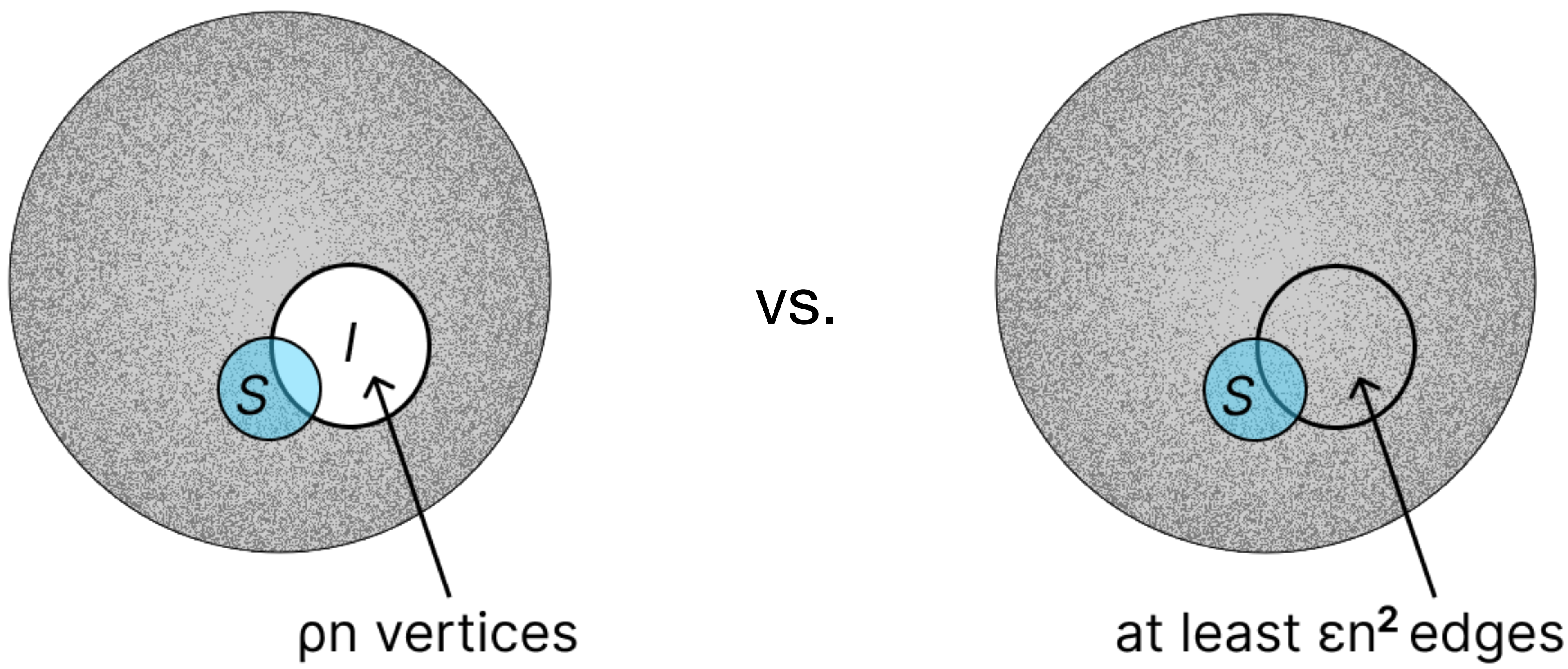

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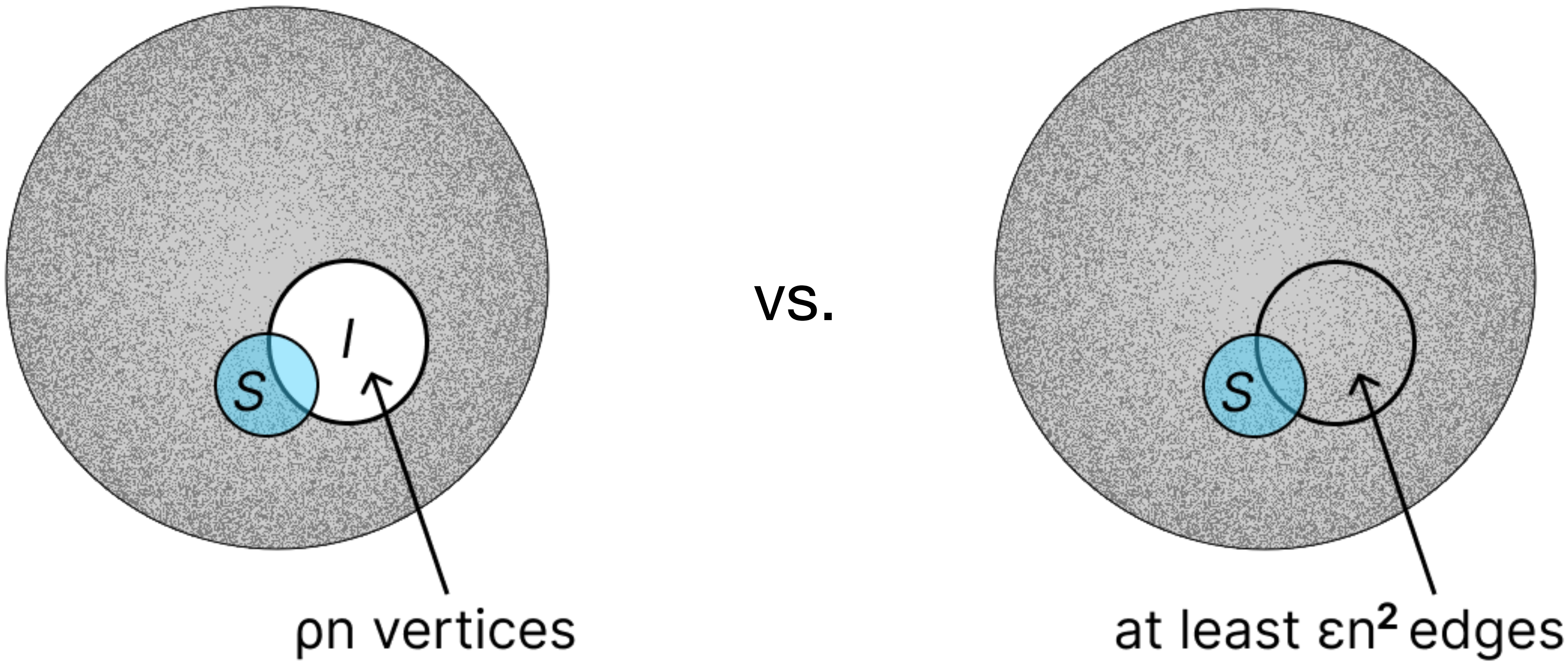

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Use Container Lemma*: Every independent set is contained in a container! Each container is of size at most $|C| < (1-\epsilon)\rho n$ and there are at most $\binom{n}{1/\epsilon}$ of them.

* [Blais, Seth '23] proved a stronger container lemma compared to the lemma of [Kleitman, Winston '82]

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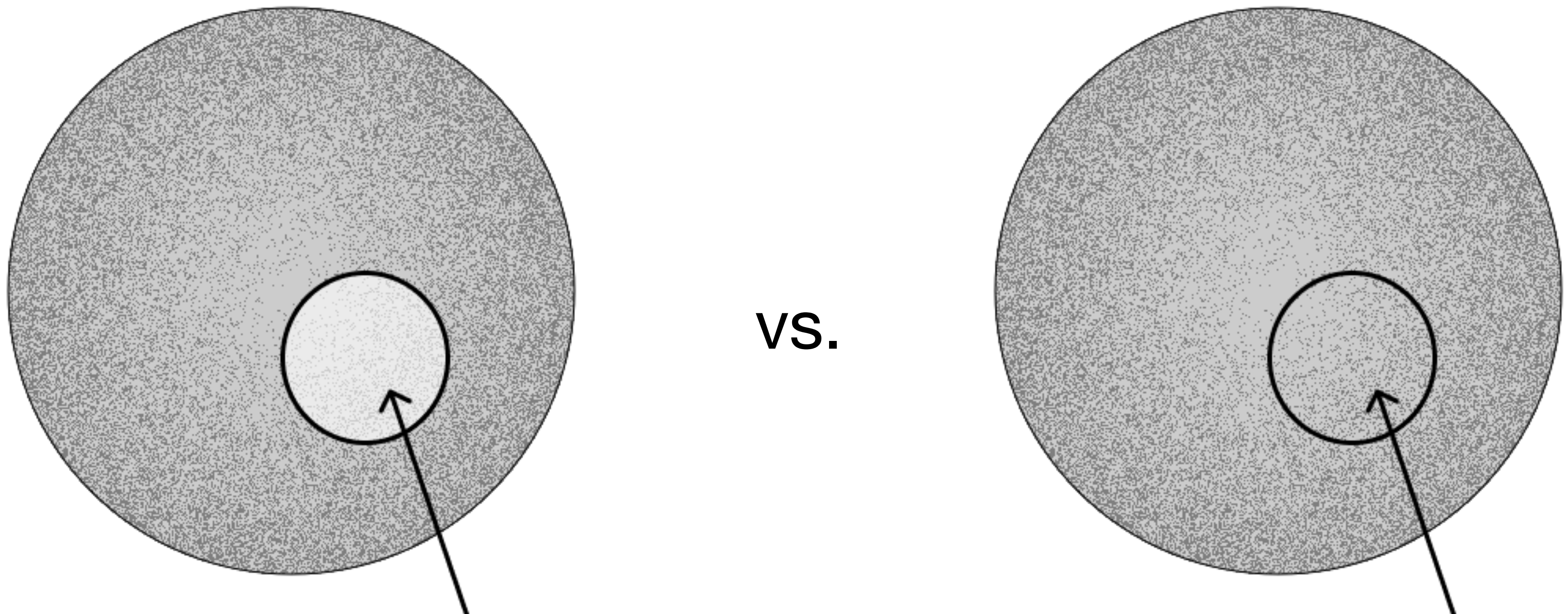
Towards a Tolerant Tester



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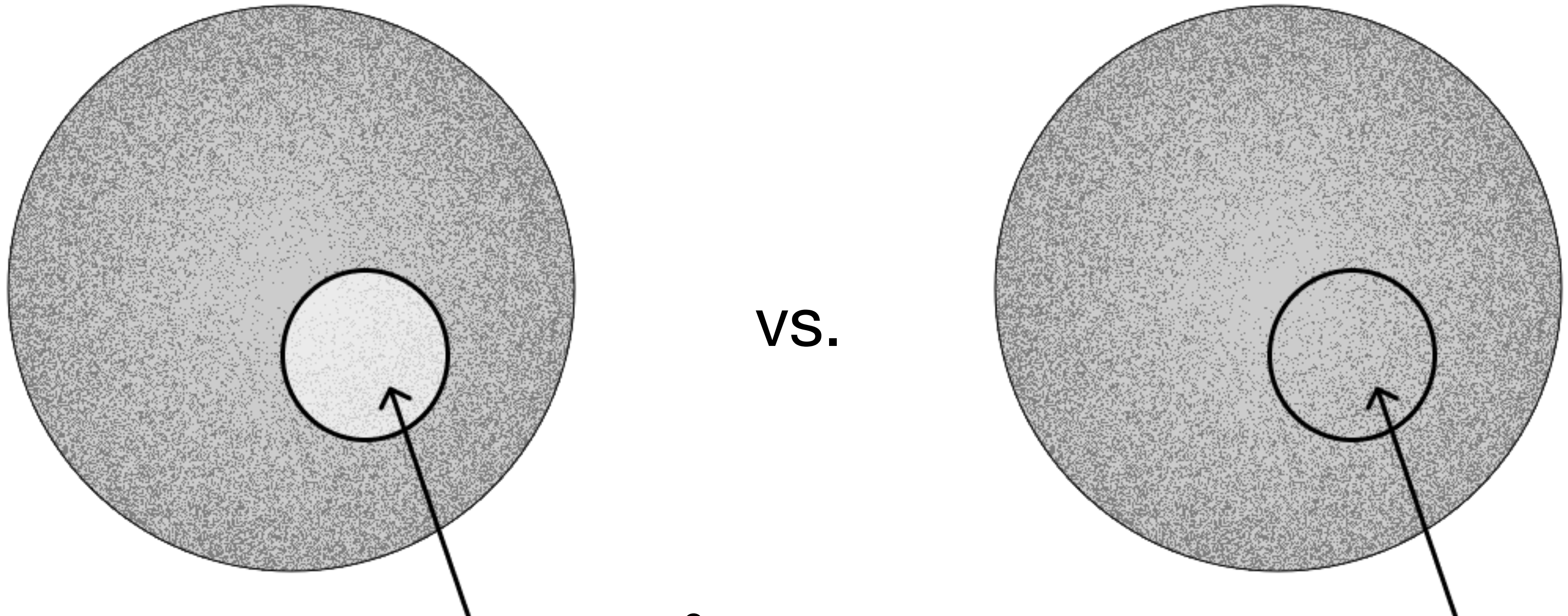

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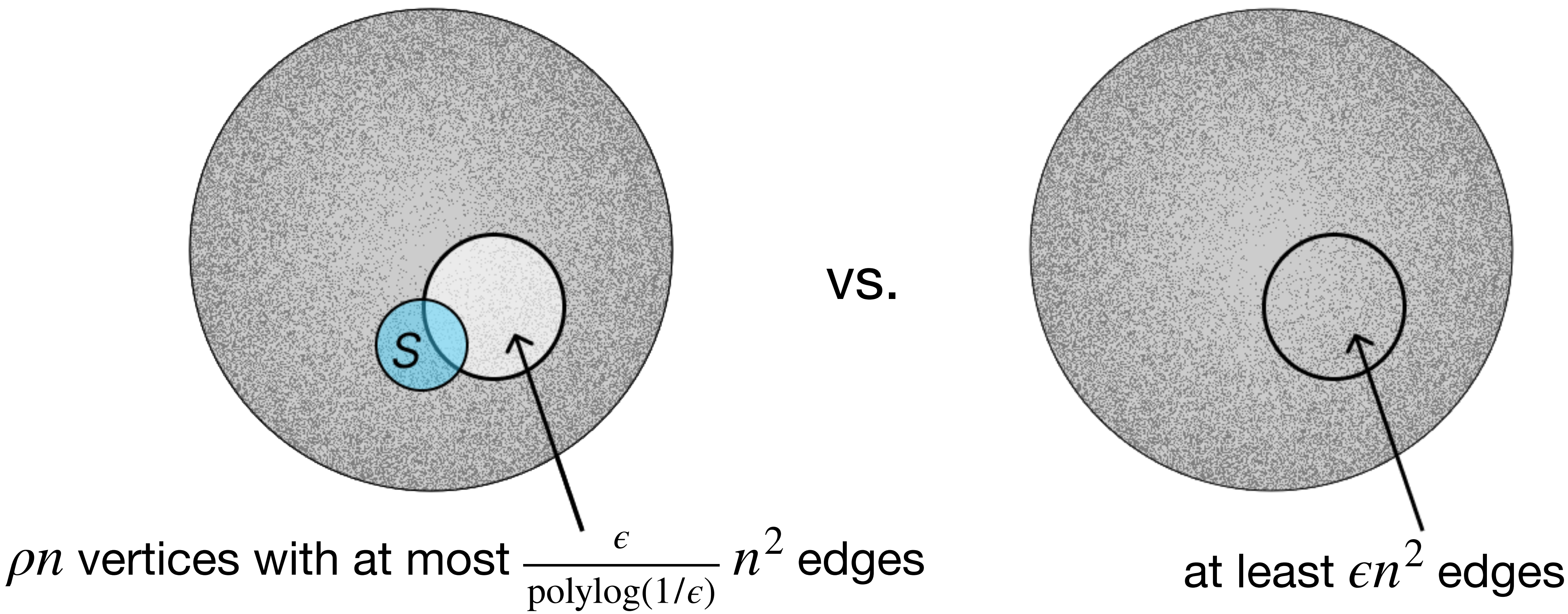

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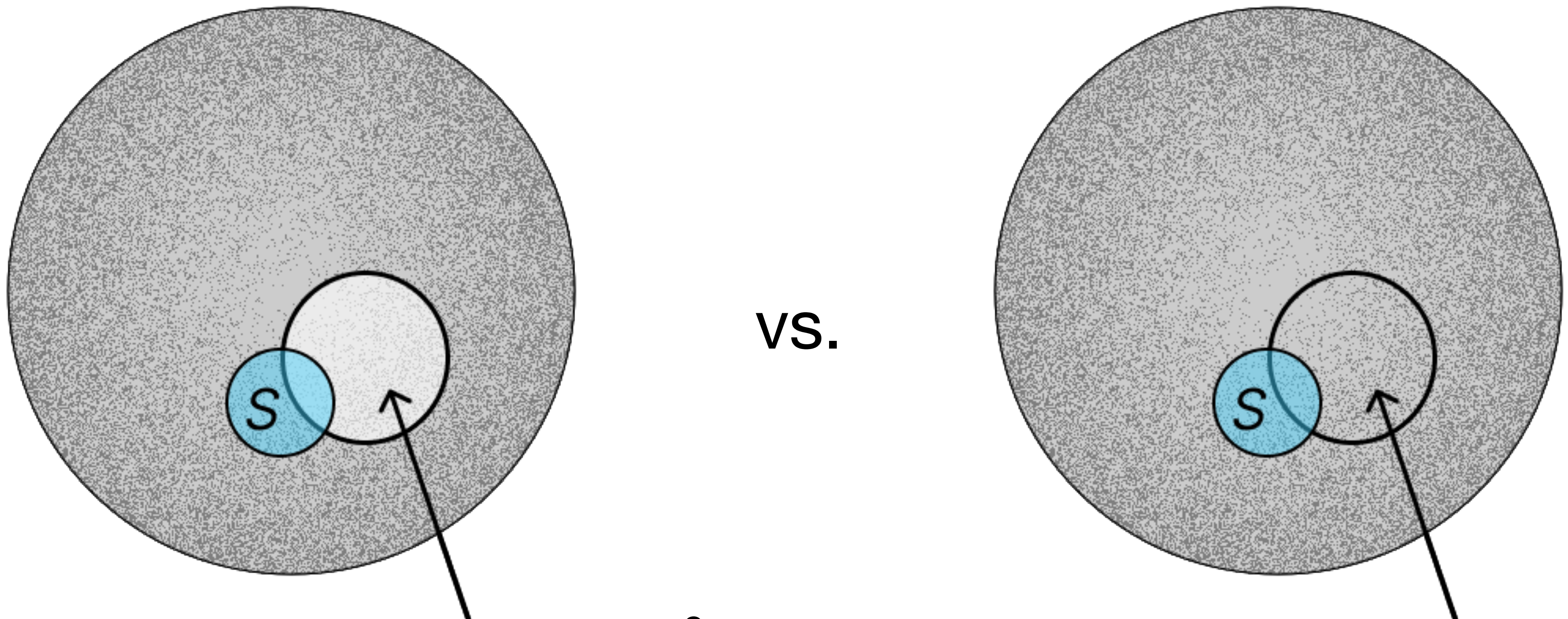

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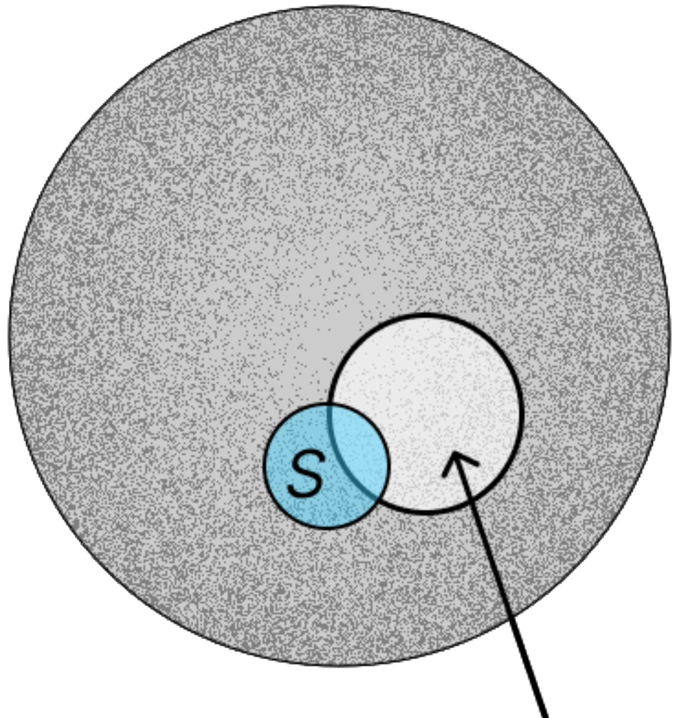

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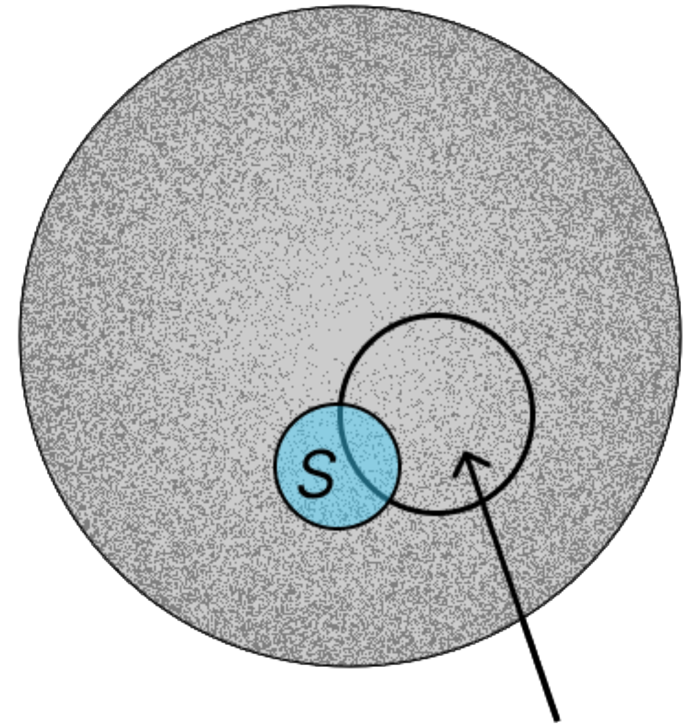

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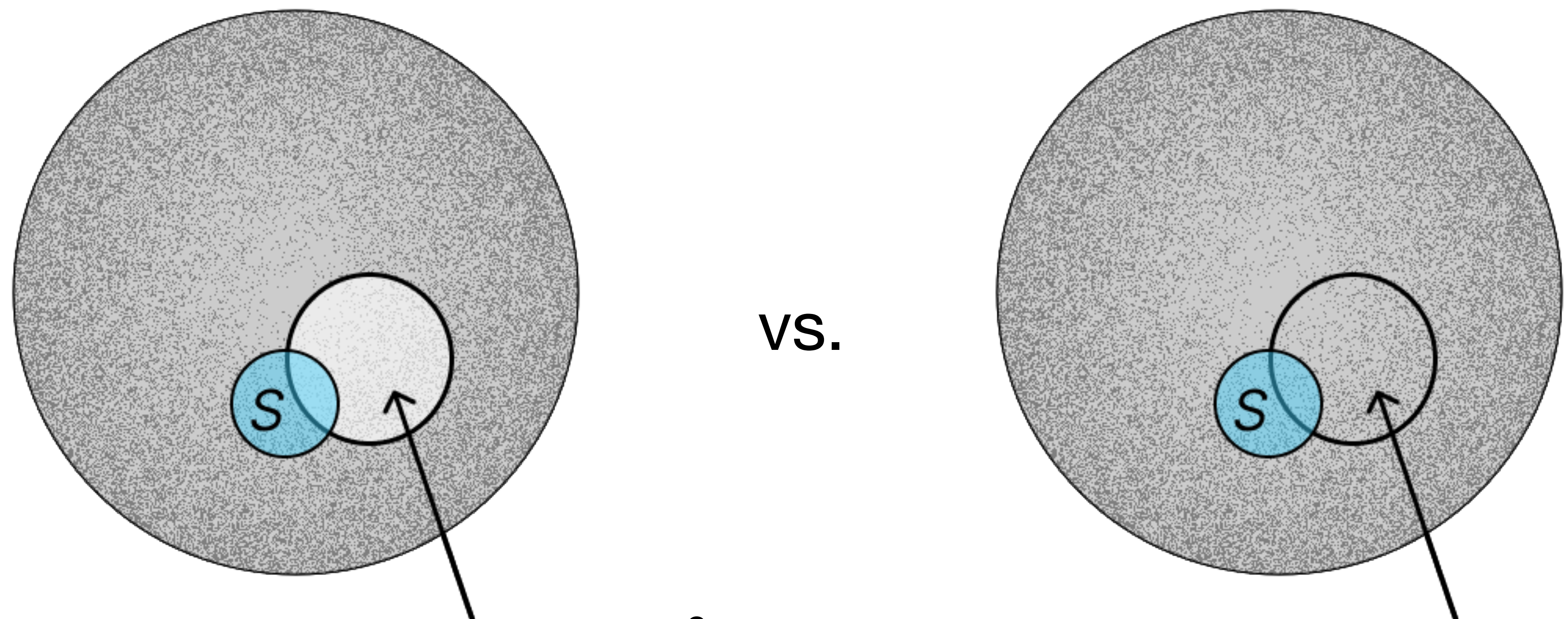

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Induced subgraph has fewer than $\frac{\epsilon}{\text{polylog}(1/\epsilon)} s^2$ edges

A Container Lemma For Sparse Sets

Desired Lemma?: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,
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(more details in a few slides)

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No prior results of this form. Only similar results on "sparse subgraphs" apply to subgraphs with smaller density or bounded max degree.

[Nenadov '24] [Saxton, Thomason '15]

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Informally: each sparse set is "mostly" contained in a container

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Using this new container lemma instead of standard container lemma used by Blais, Seth ('23) we can the prove theorem.

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,
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$$|C \cap J| \geq \left(1 - \frac{\alpha}{2}\right)|J|.$$

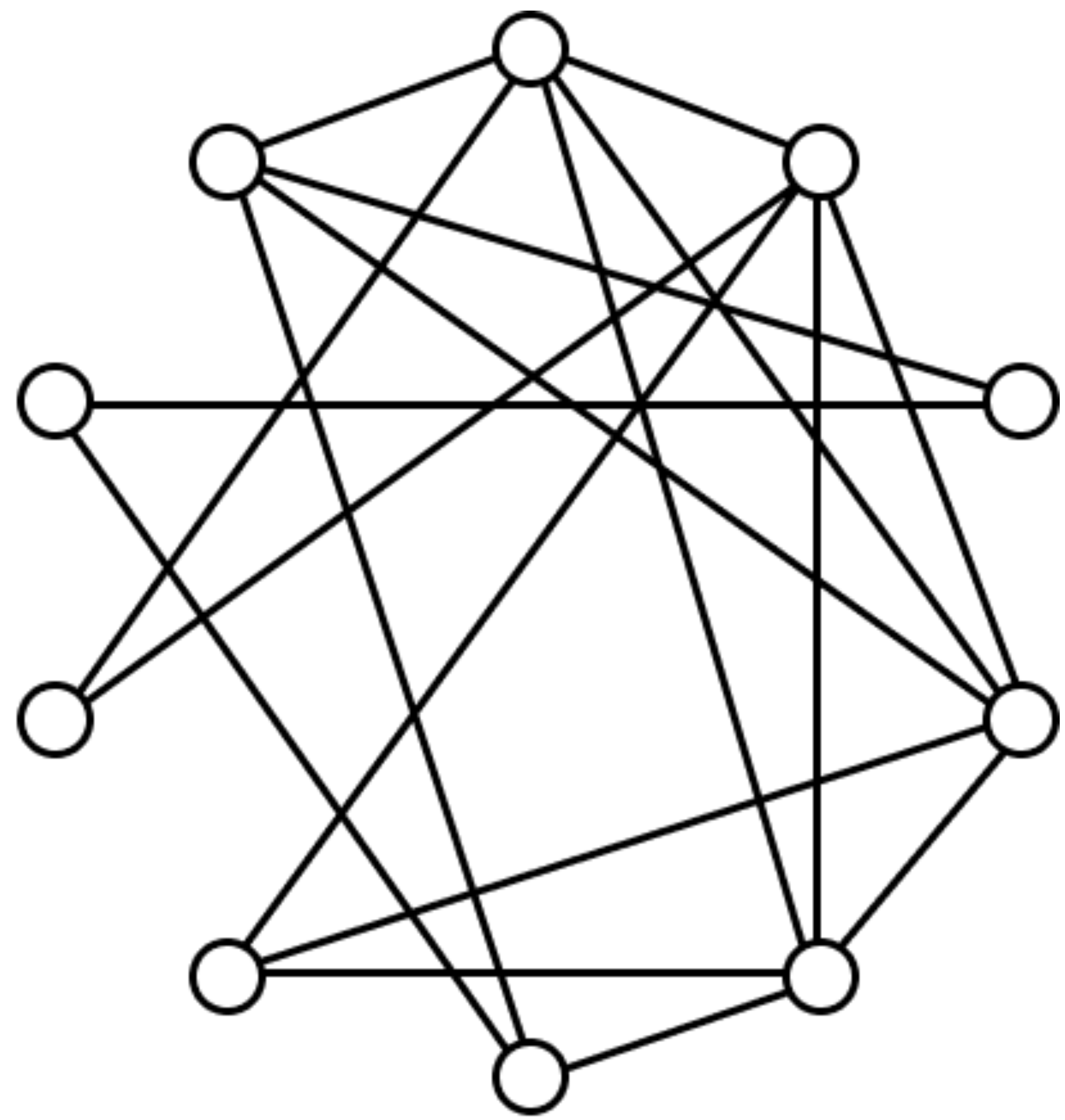
Informally: each sparse set is "mostly" contained in a container

* Omitting some details

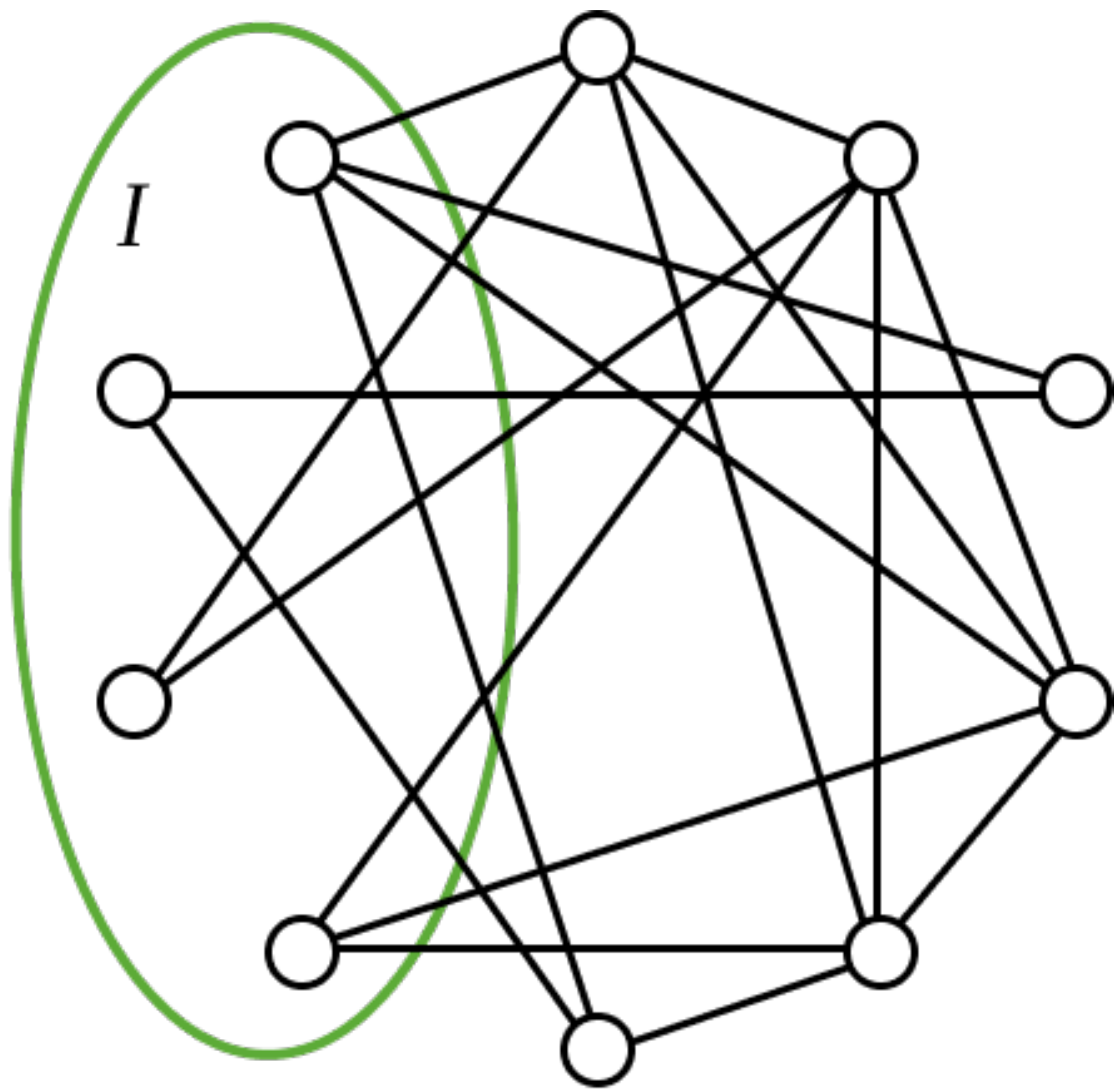
Warmup: How to prove Container Lemma for Independent Sets - An Encoding Argument

Encoding Independent Sets

Encoding Independent Sets



Encoding Independent Sets



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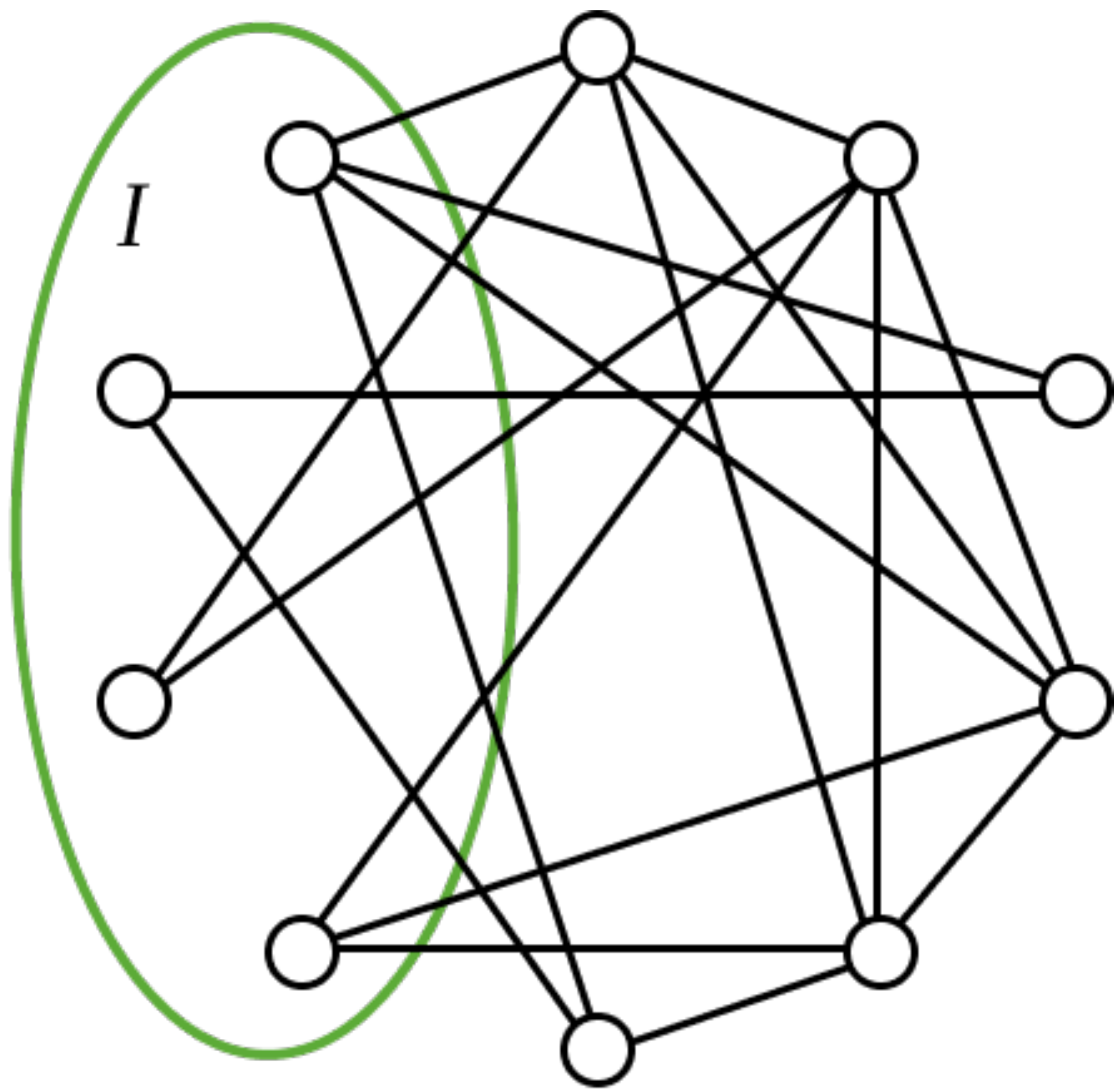

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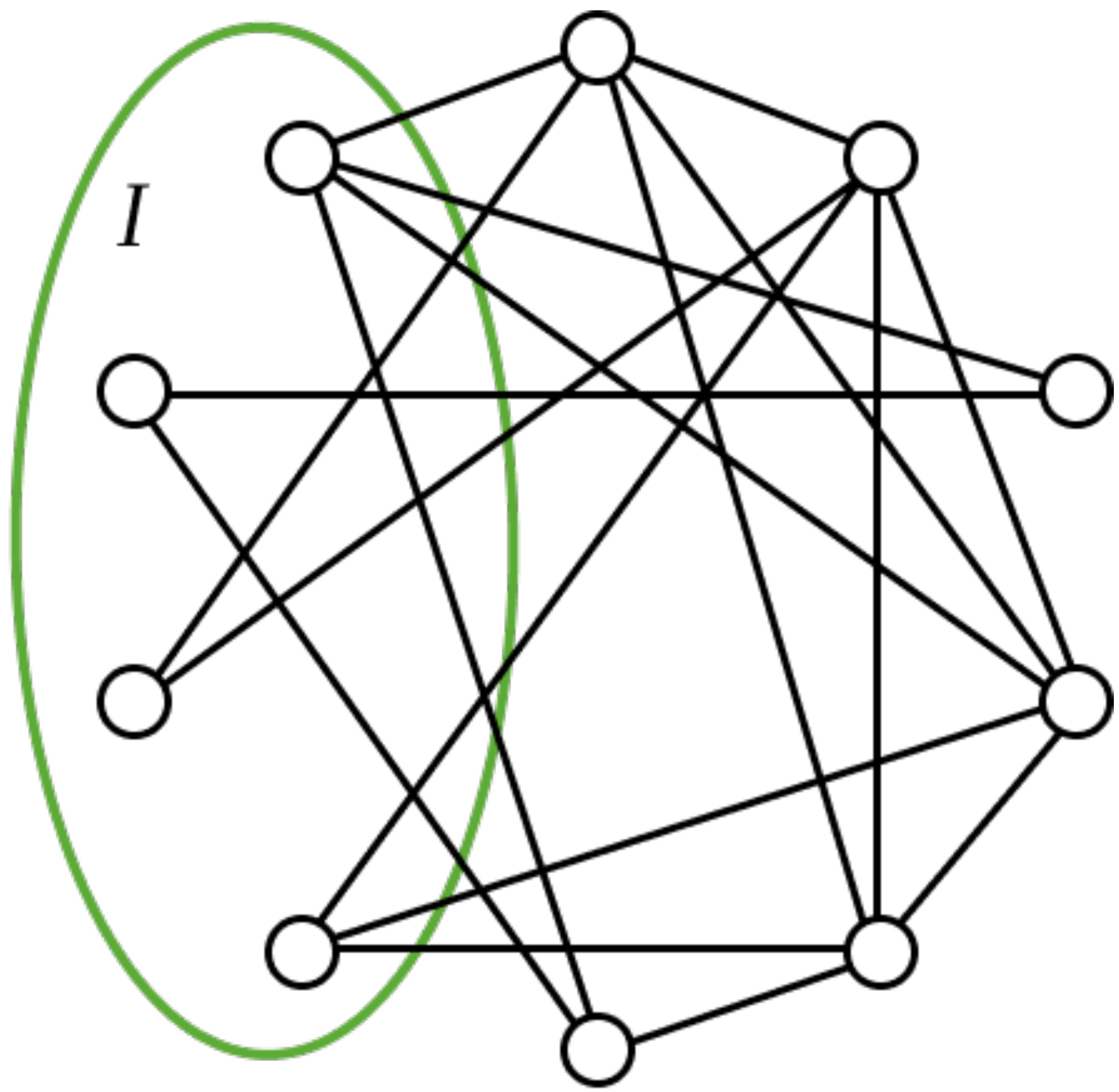

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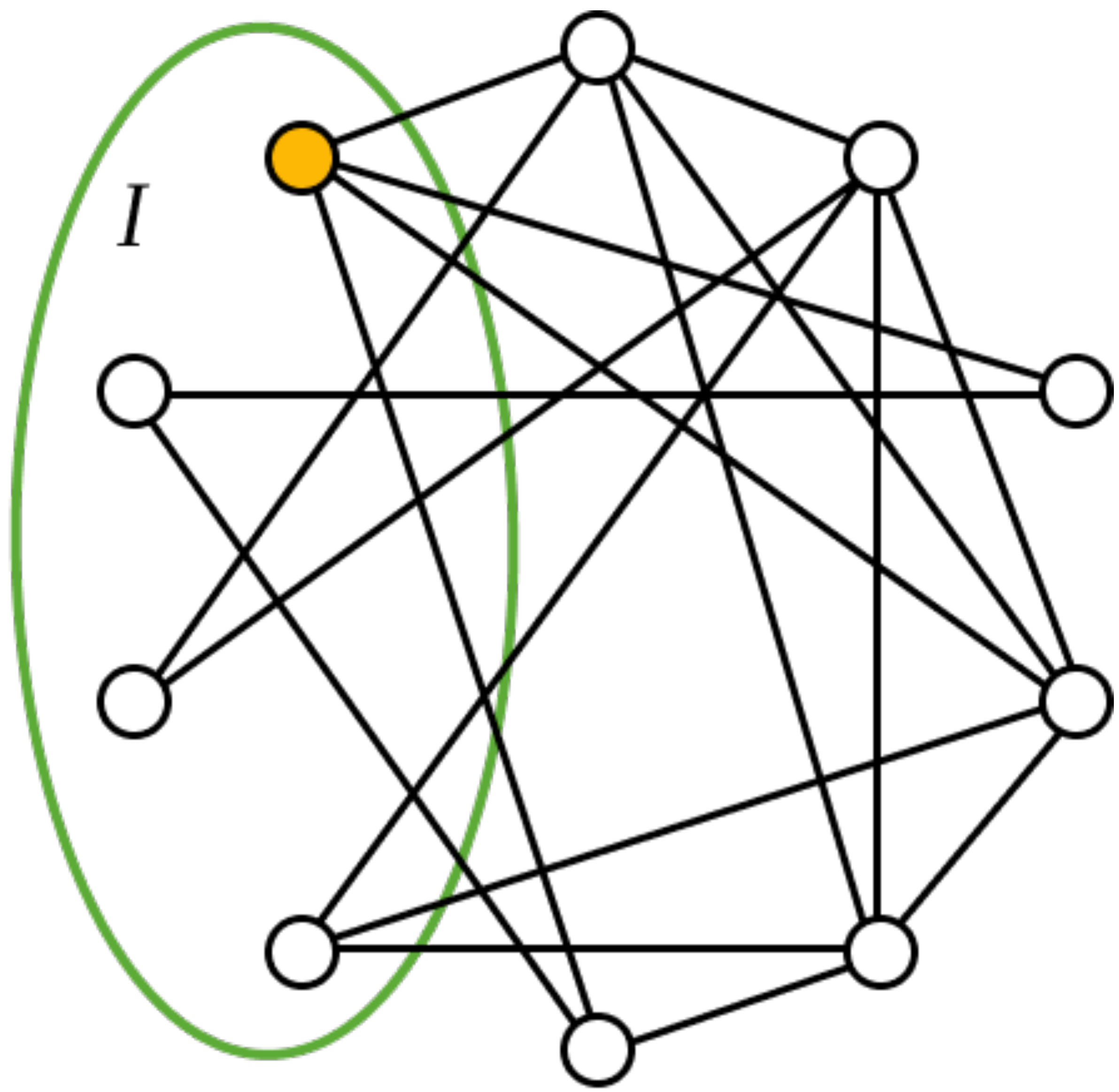

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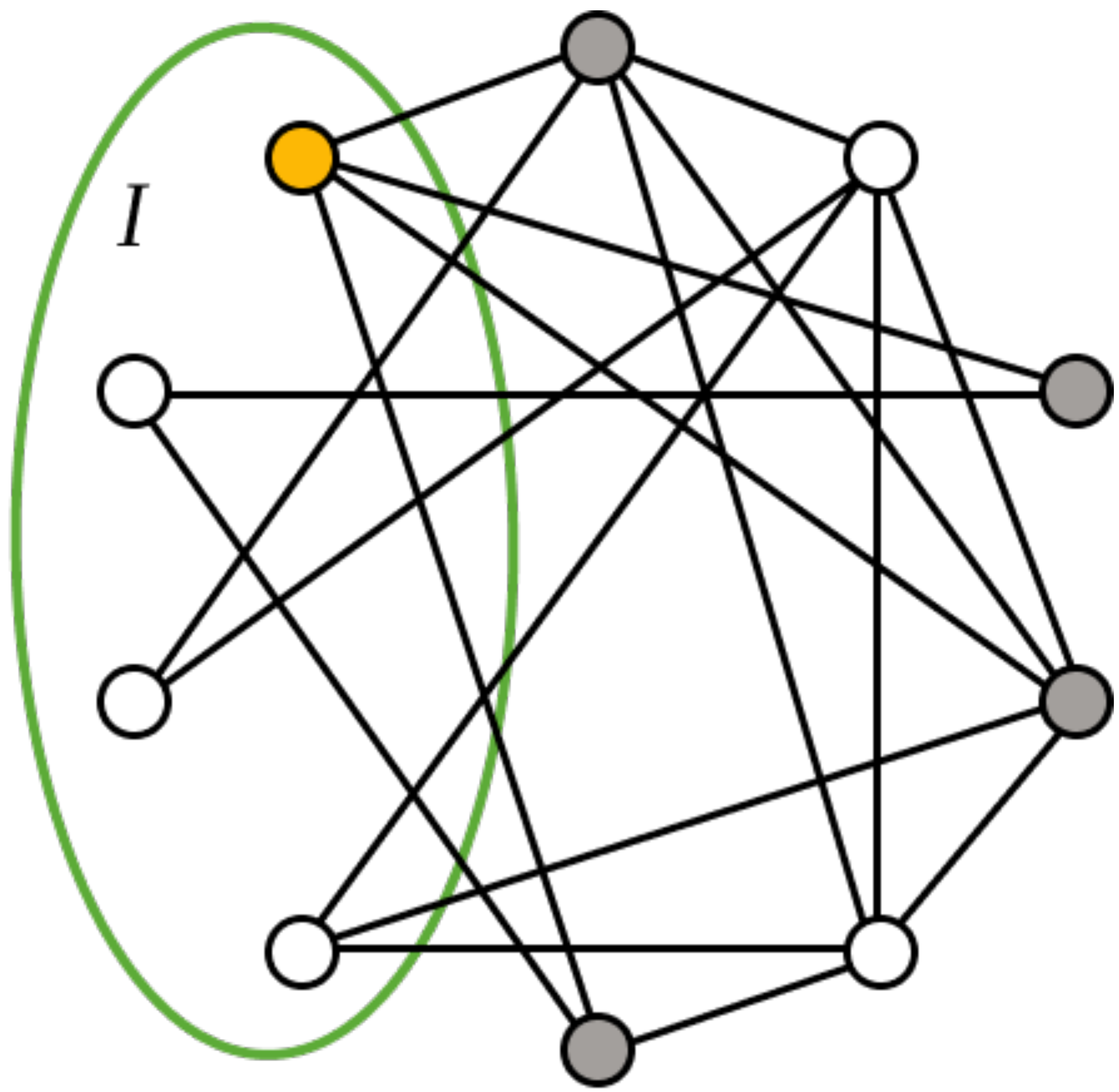

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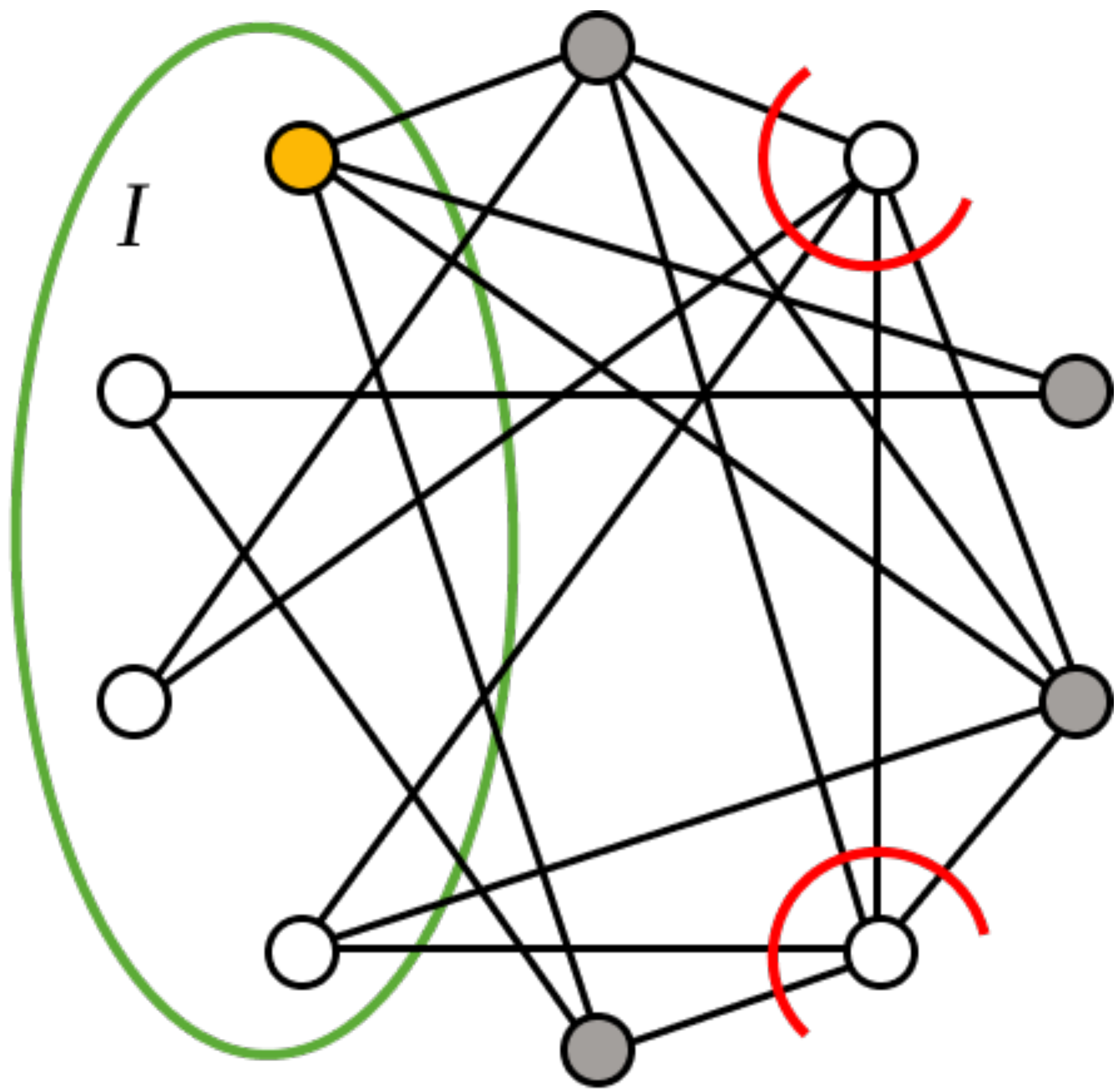

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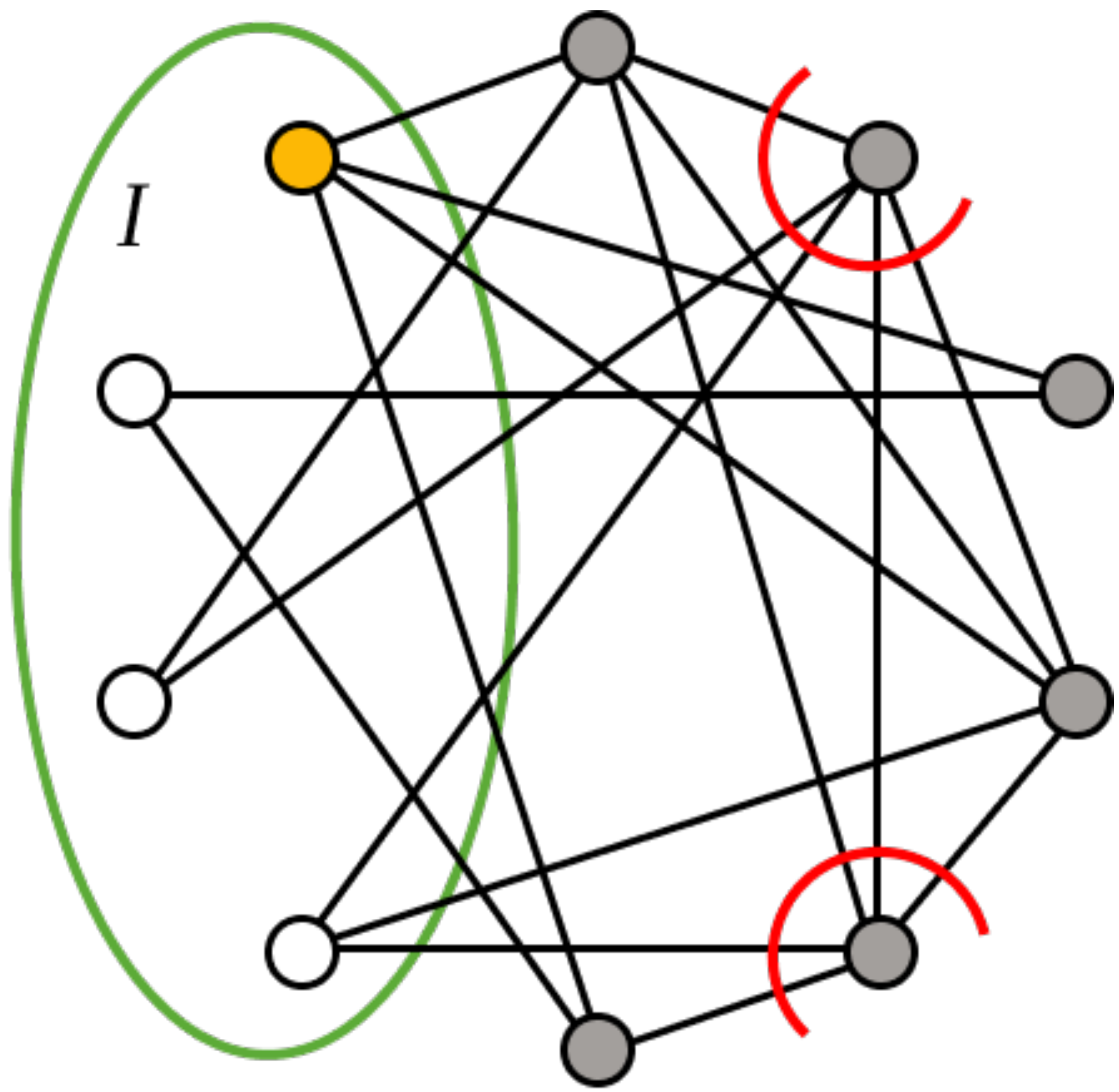

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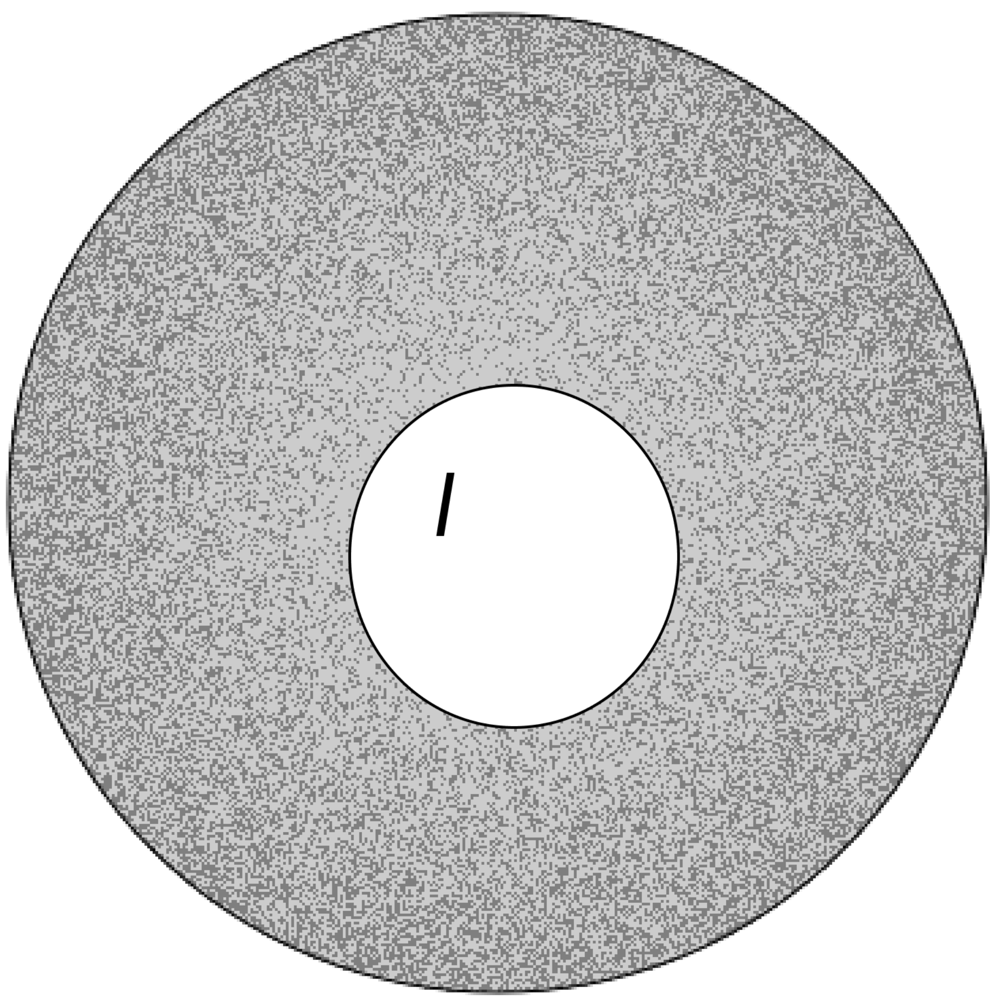

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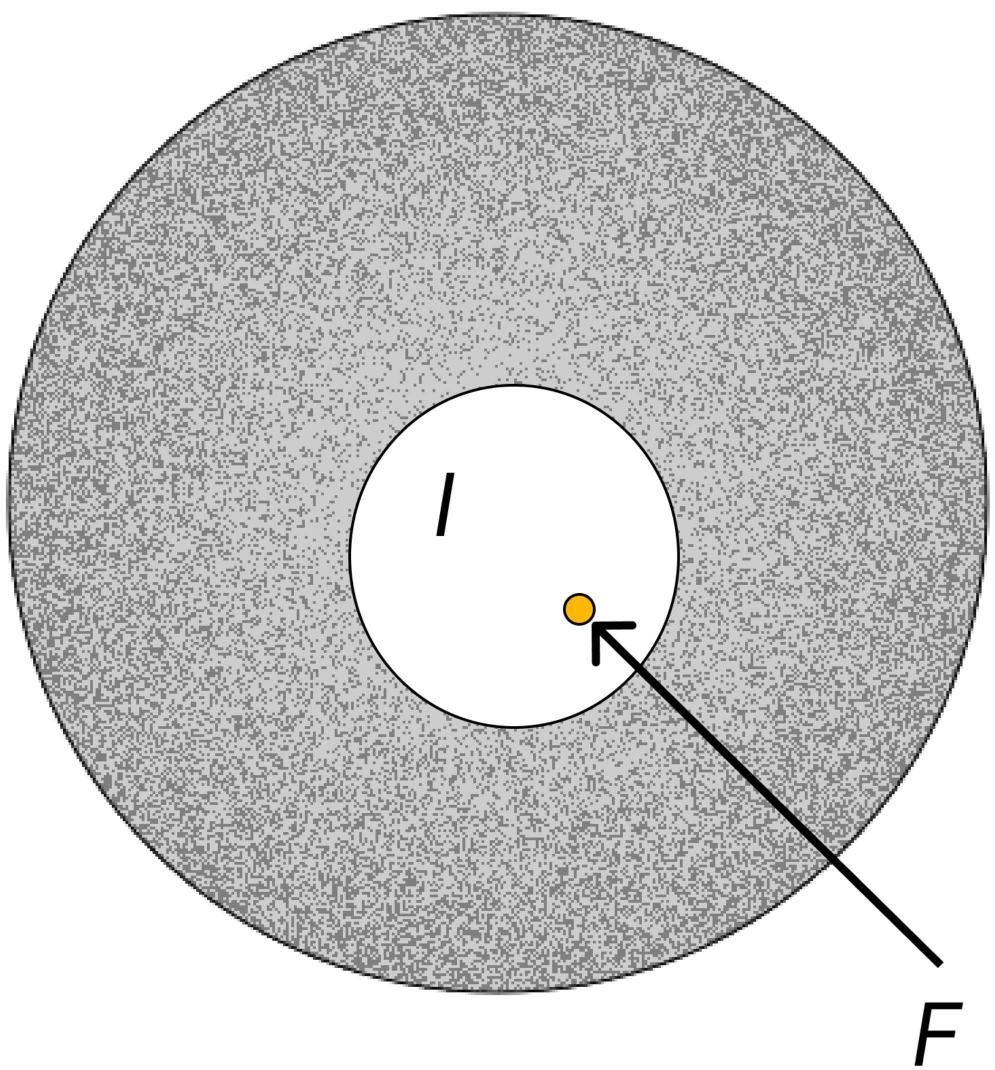

1st Iteration

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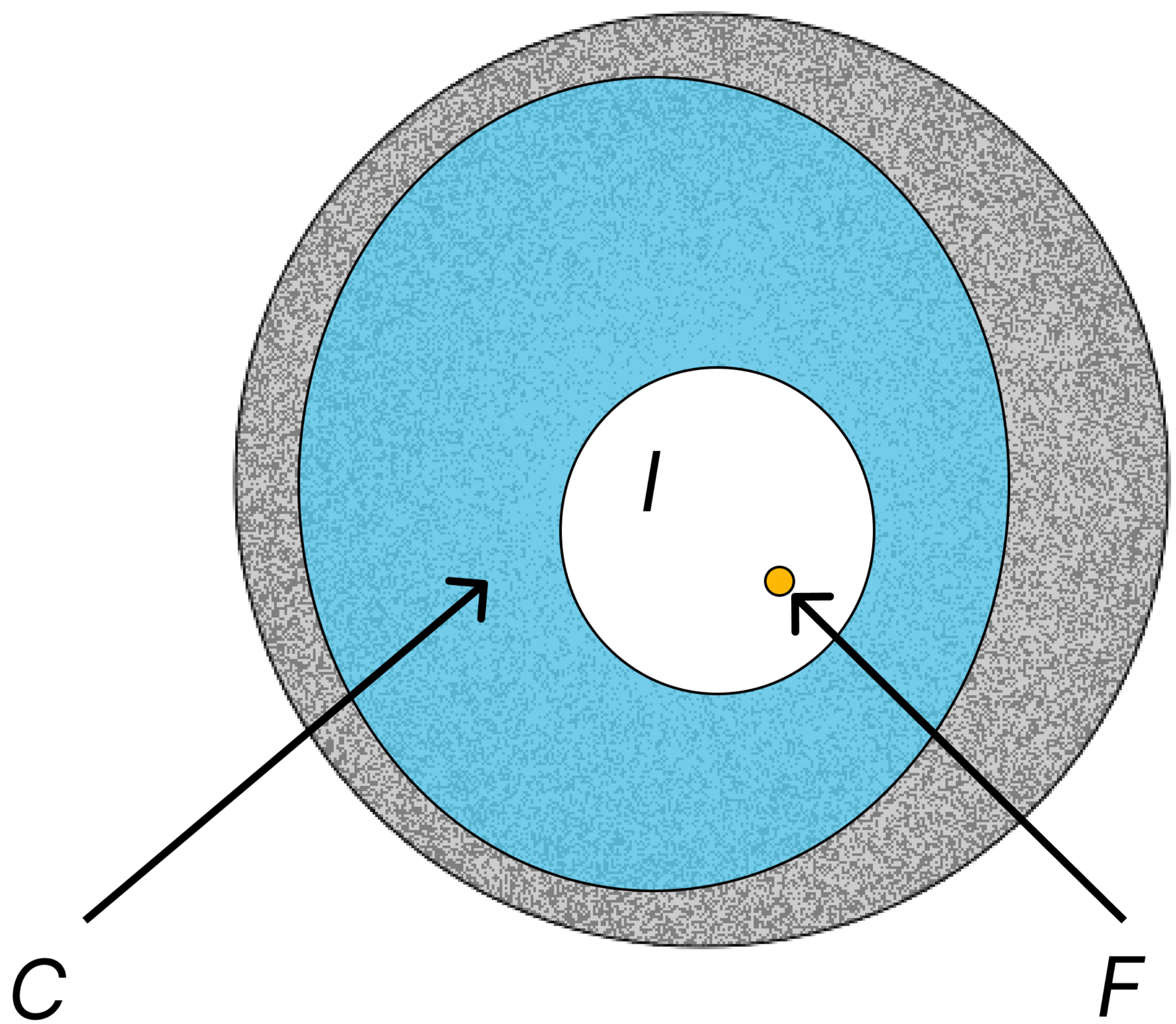

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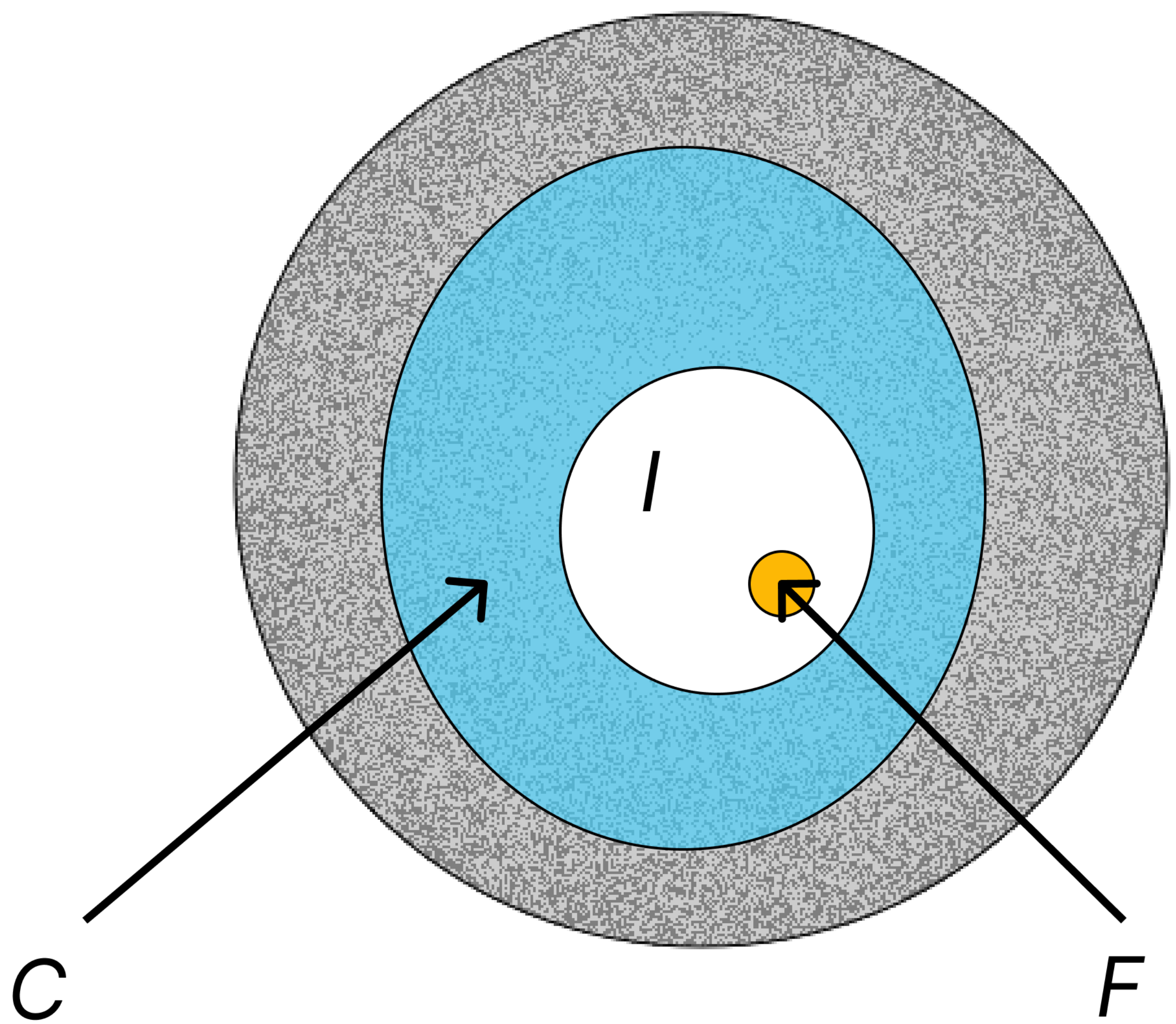

2nd Iteration

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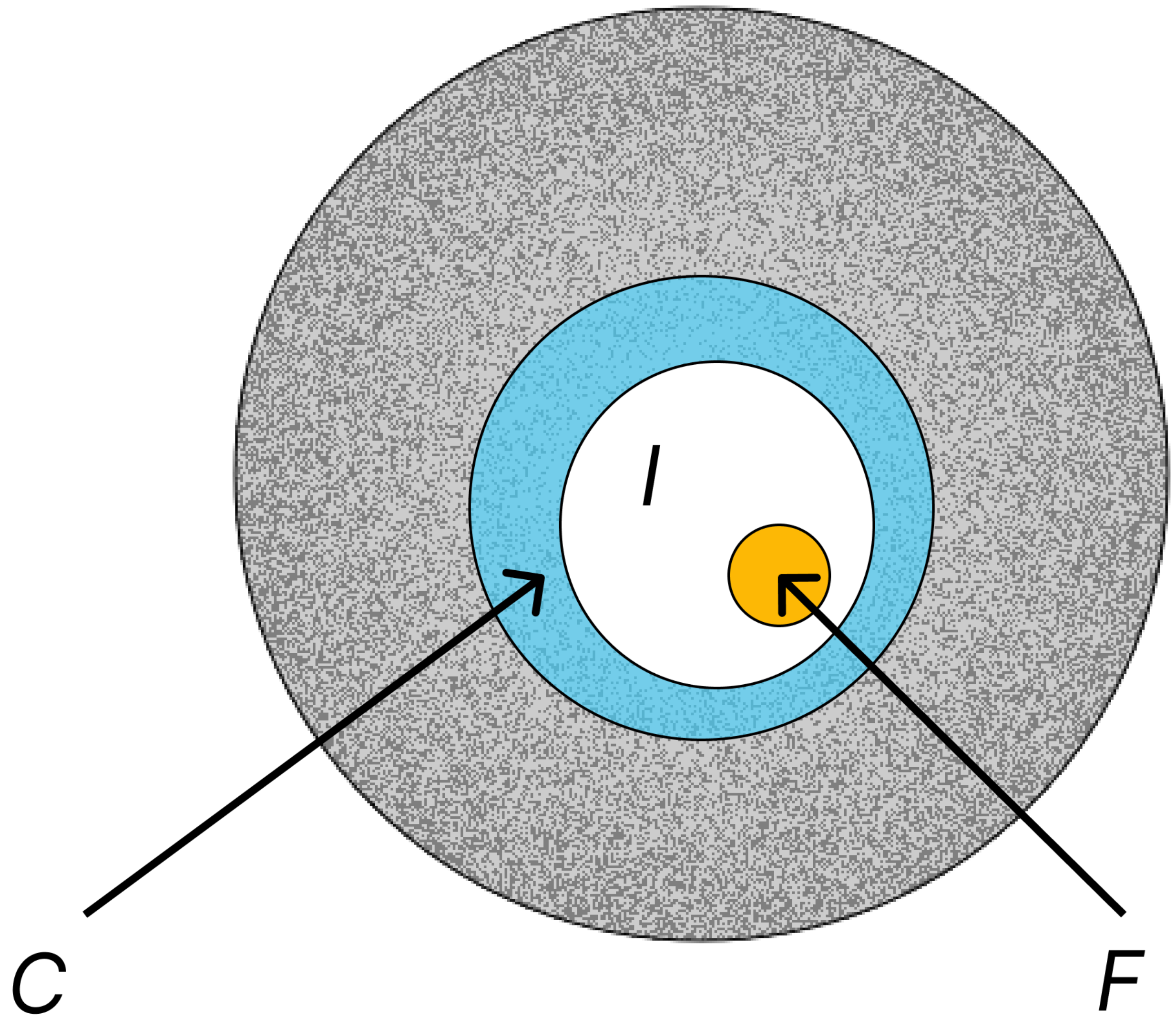

Final Iteration

Container Generator Algorithm

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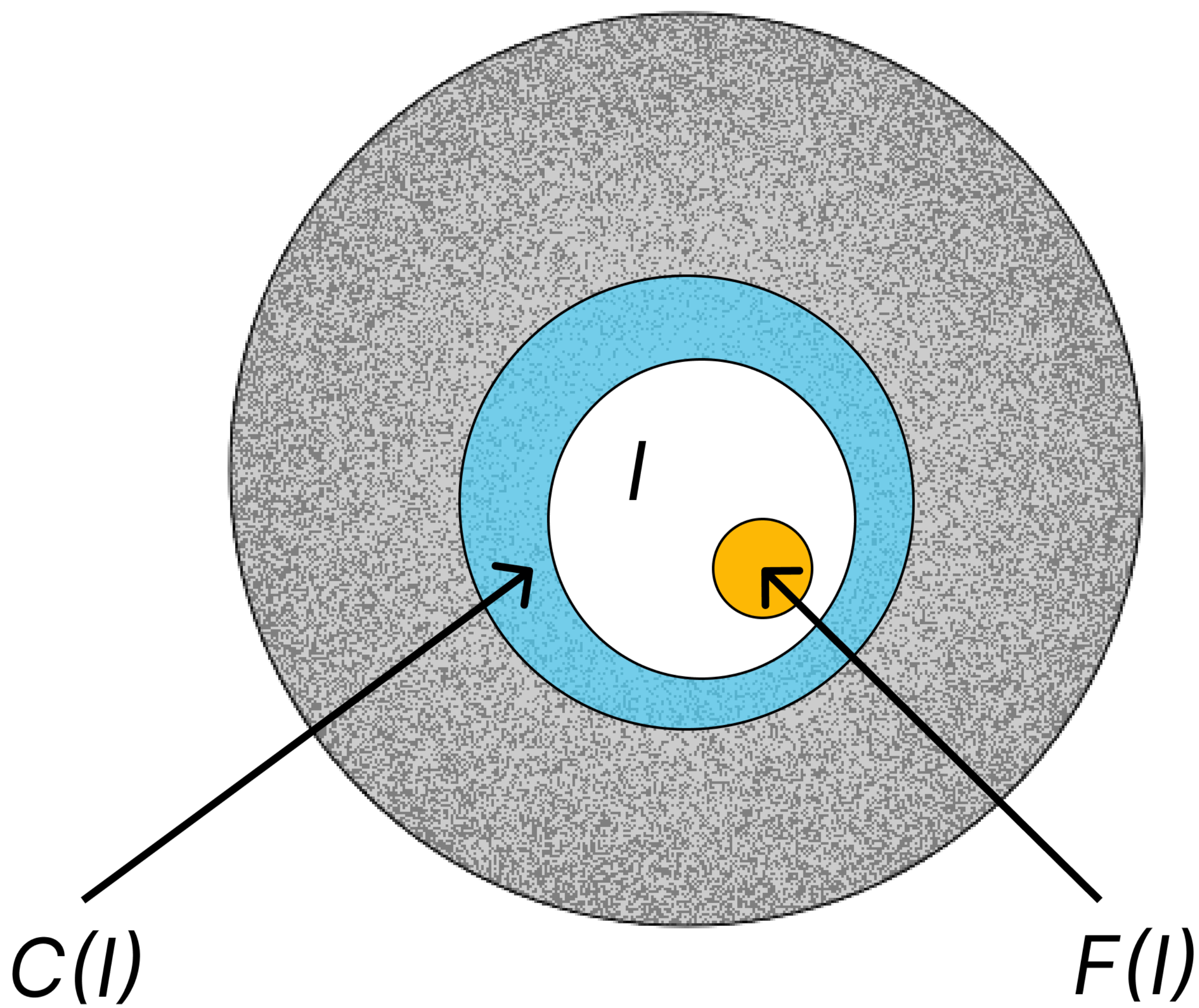

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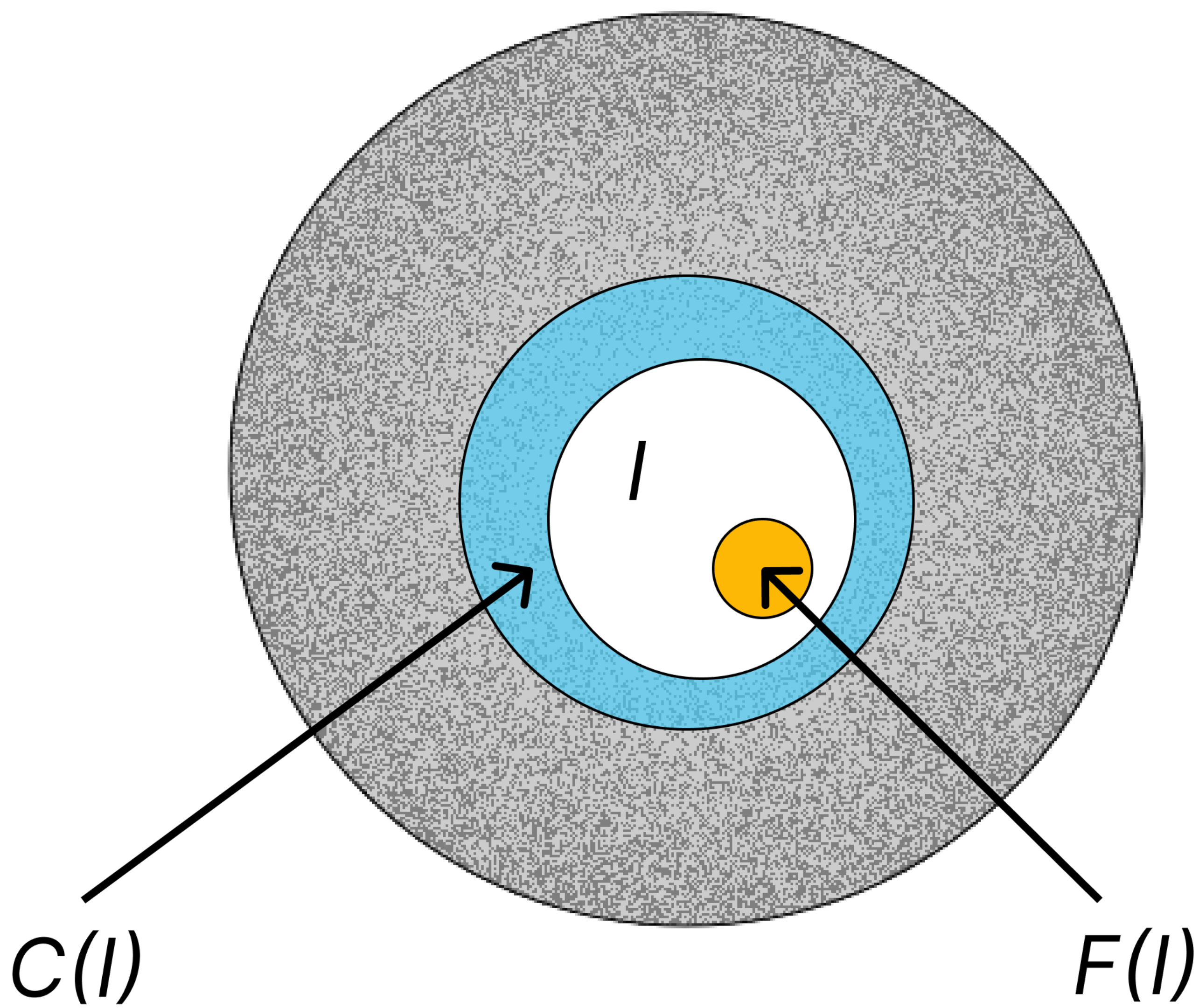

Final Iteration

Key Observations:

- $I \subseteq C(I)$
- $C(I) = C(F(I))$

Container Lemma for Independent Sets - Restated

Lemma: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,
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3. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$.

[Kleitman, Winston '82]

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$$\text{Let } \mathscr{C} = \{C(I) : I \text{ is an independent set in } G\}$$
$$= \{C(F(I)) : I \text{ is an independent set in } G\}$$

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$$\begin{aligned} \text{Let } \mathscr{C} &= \{C(I) : I \text{ is an independent set in } G\} \\ &= \{C(F(I)) : I \text{ is an independent set in } G\} \\ &= \{C(F) : F = F(I) \text{ for some independent set } I \text{ in } G\} \end{aligned}$$

How to prove new Container Lemma for Sparse Sets - An Encoding Argument

A Container Lemma For Sparse Sets (restated)

Lemma*: For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,
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Informally: each sparse set is "mostly" contained in a container

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Encoding Sparse Sets

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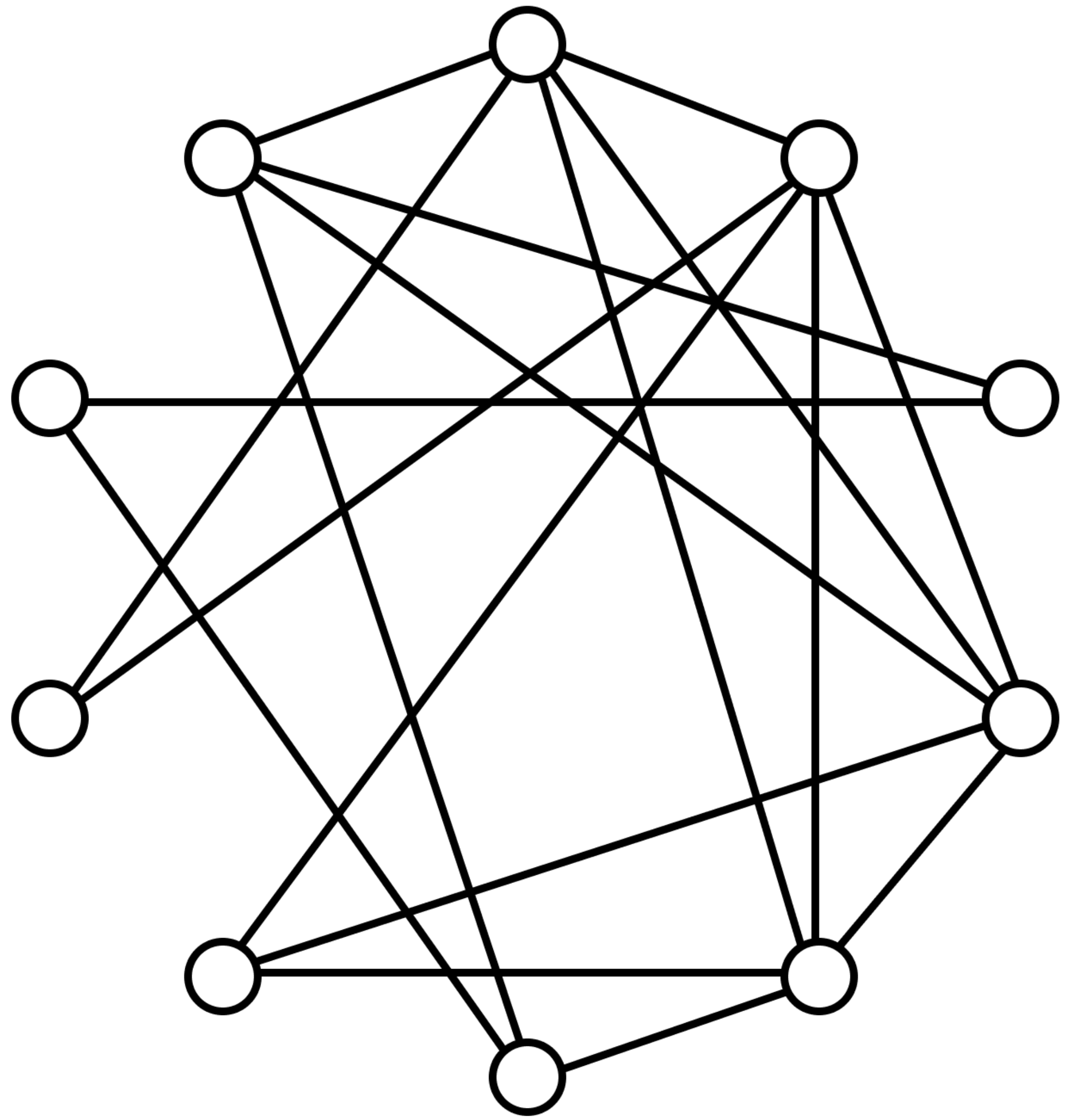

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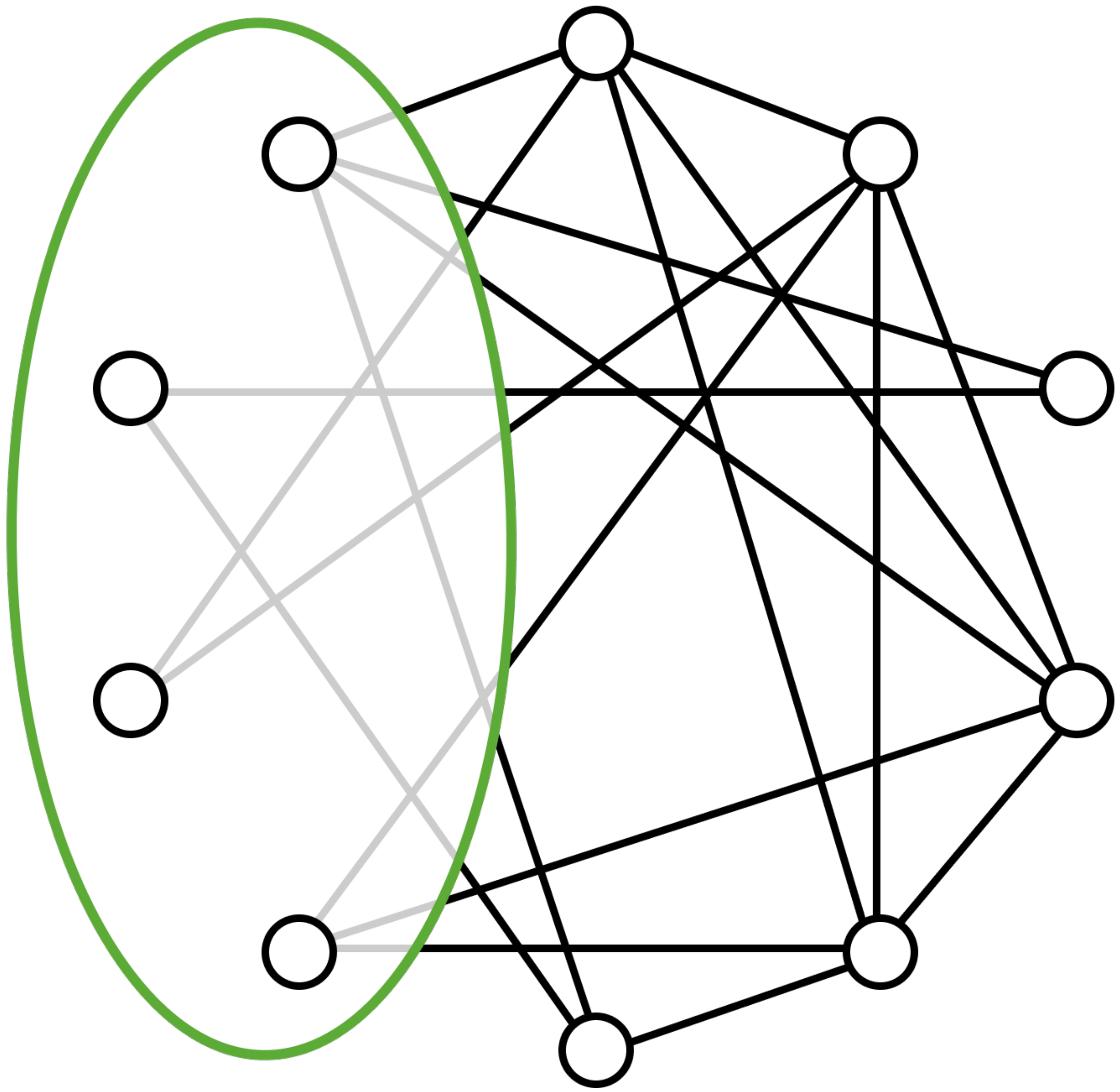

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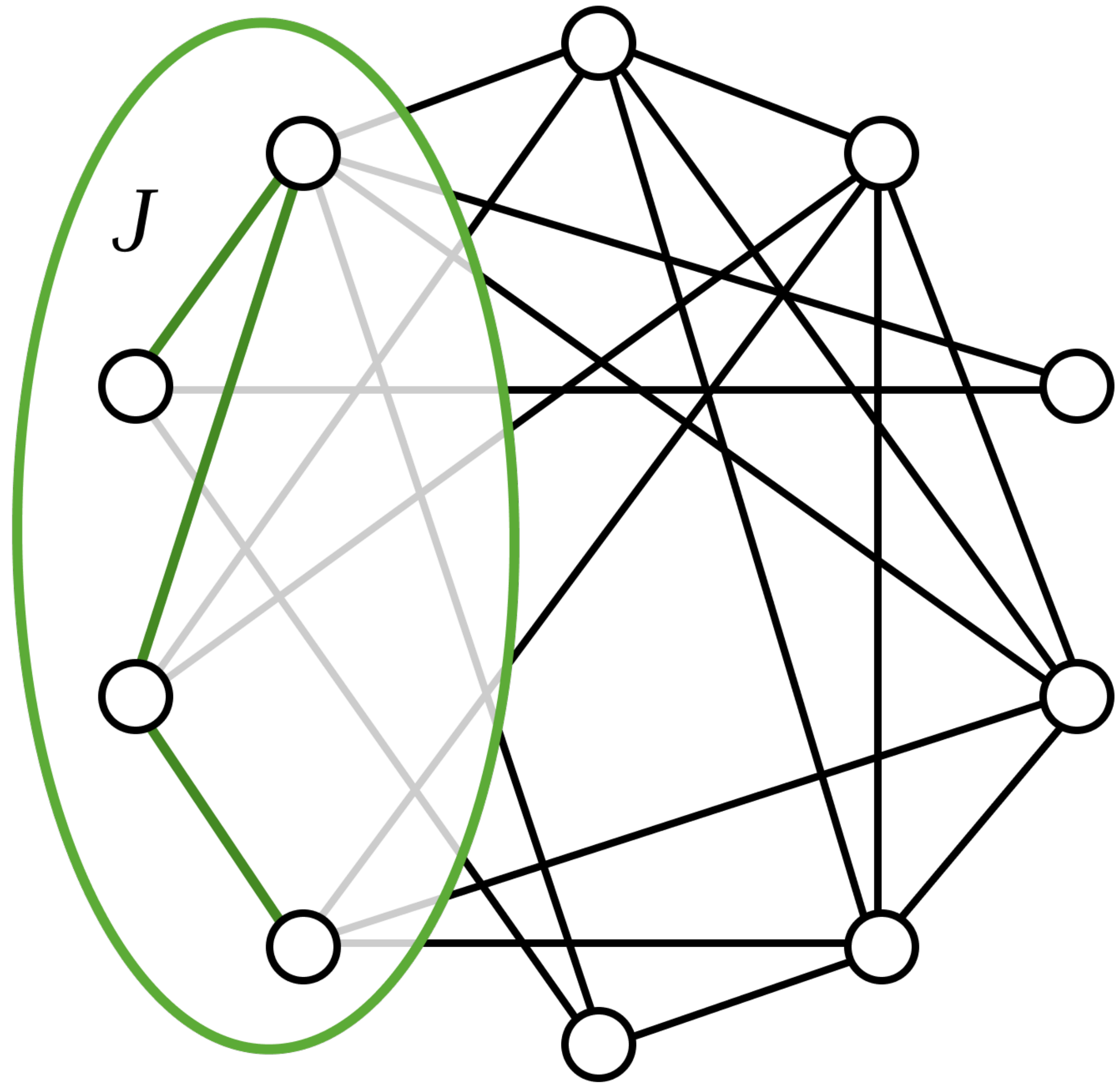

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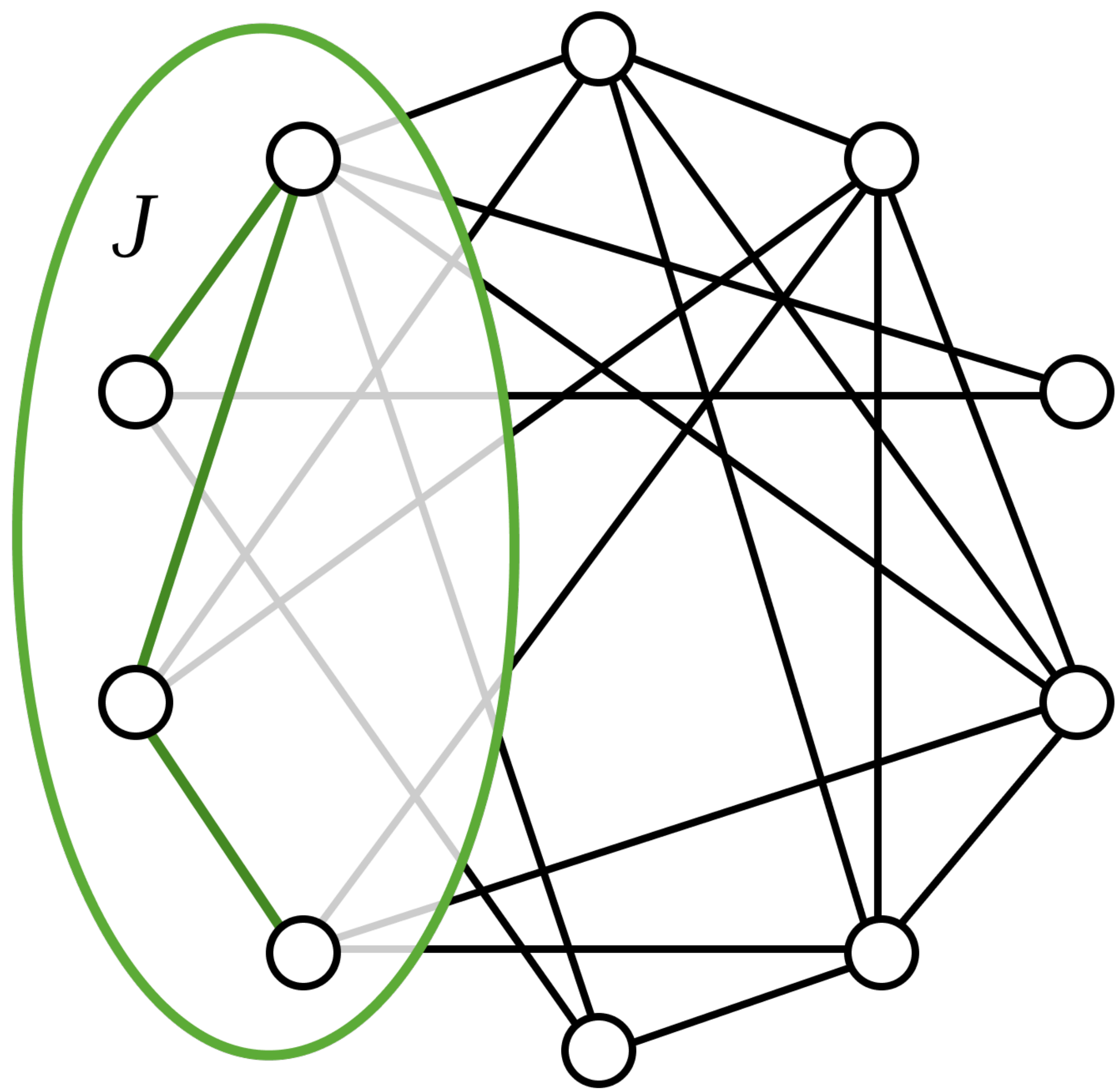

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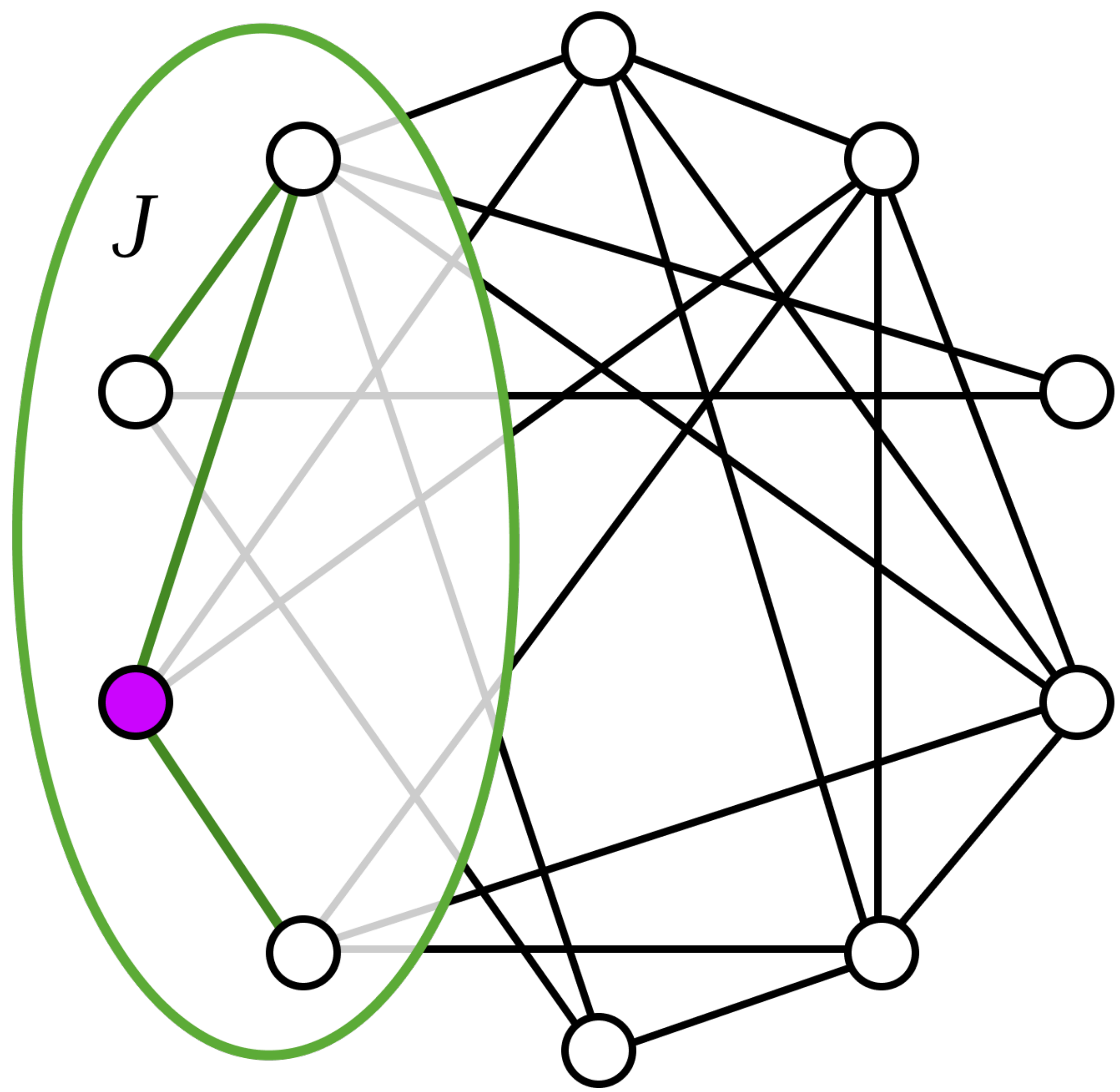

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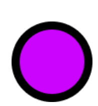
Answer: Send  and remove higher degree vertices

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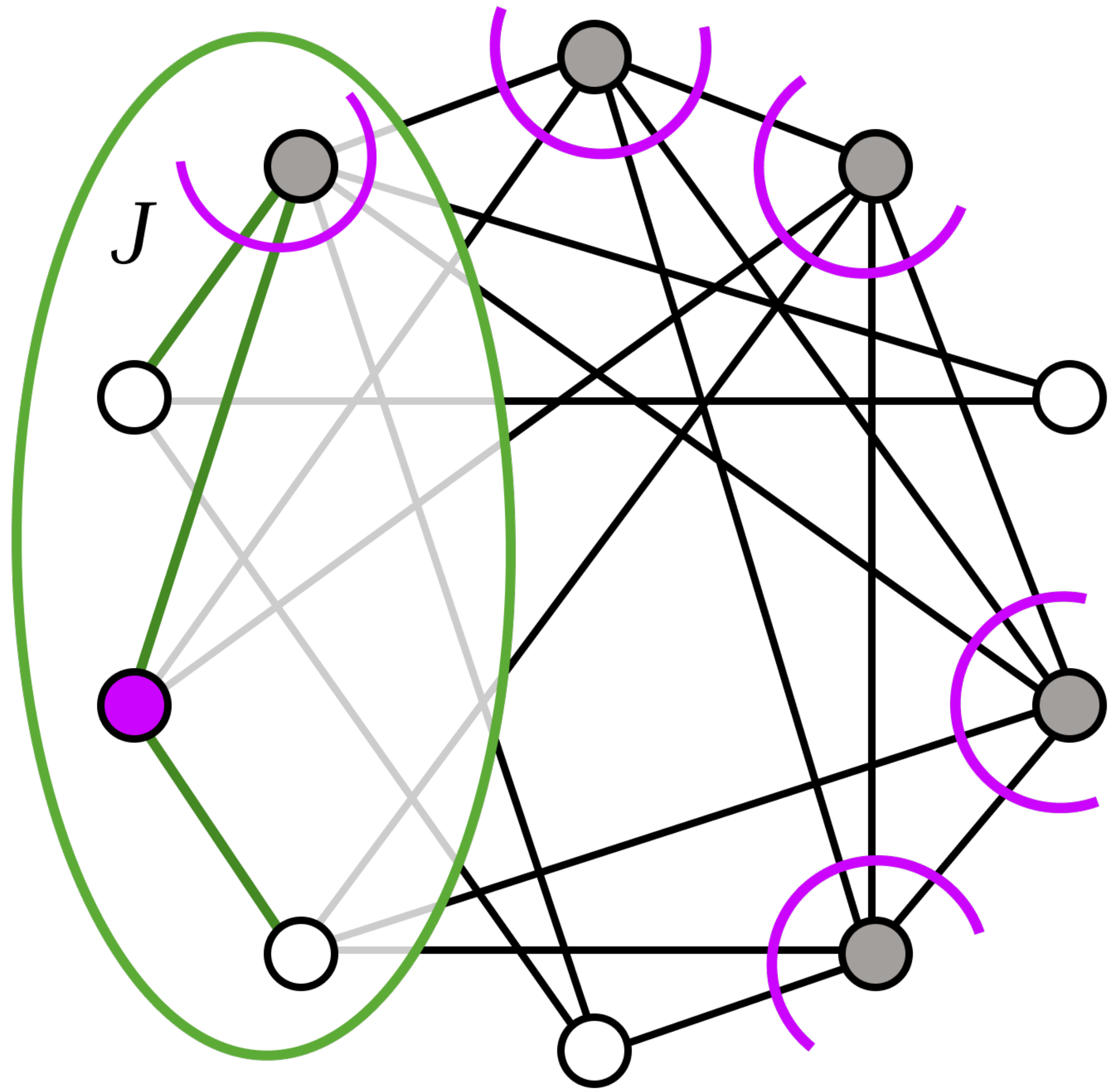

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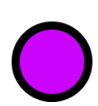
Answer: Send  and remove higher degree vertices

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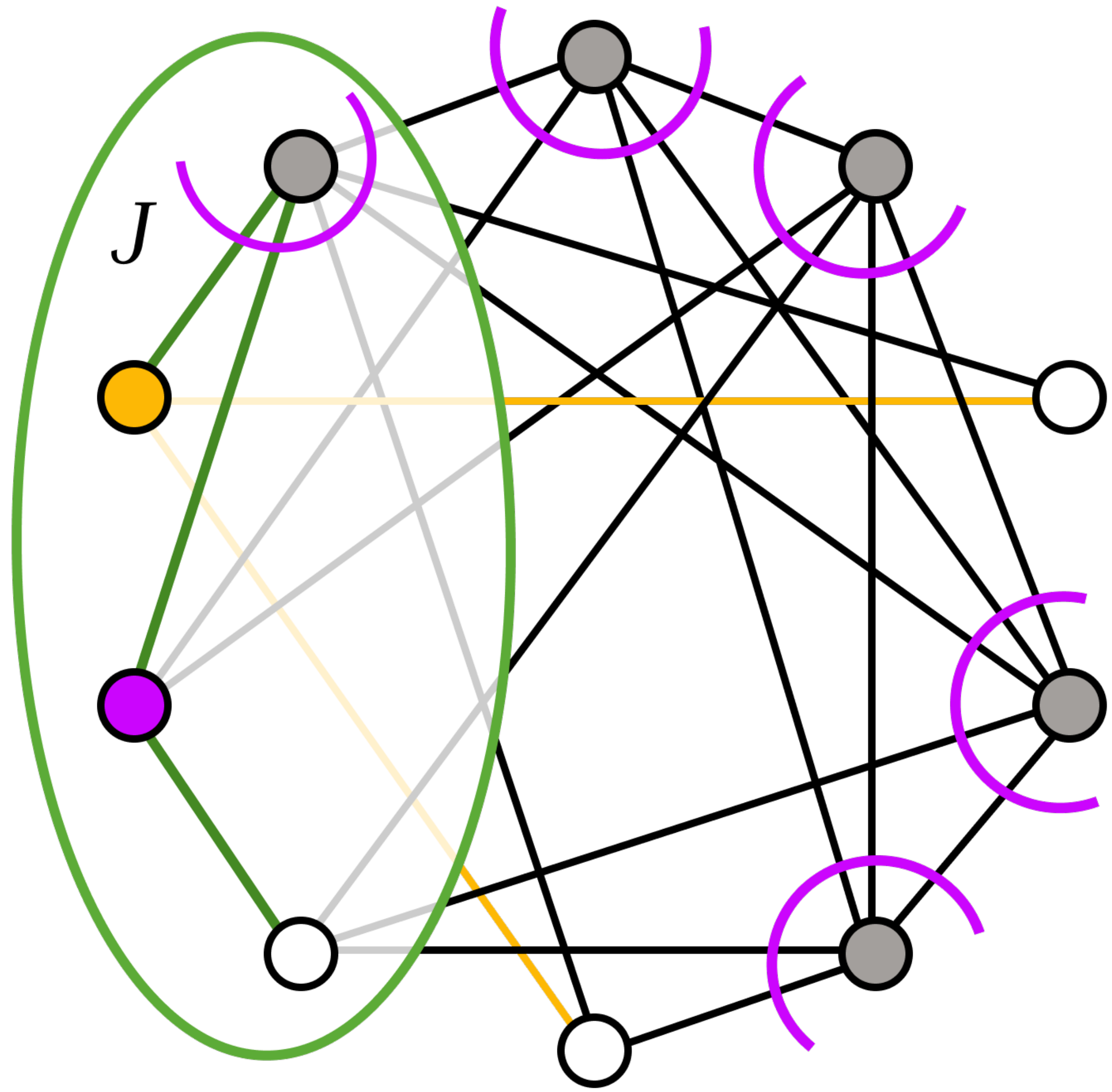

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Answer: Send  and remove higher degree vertices

Send  and remove neighbours

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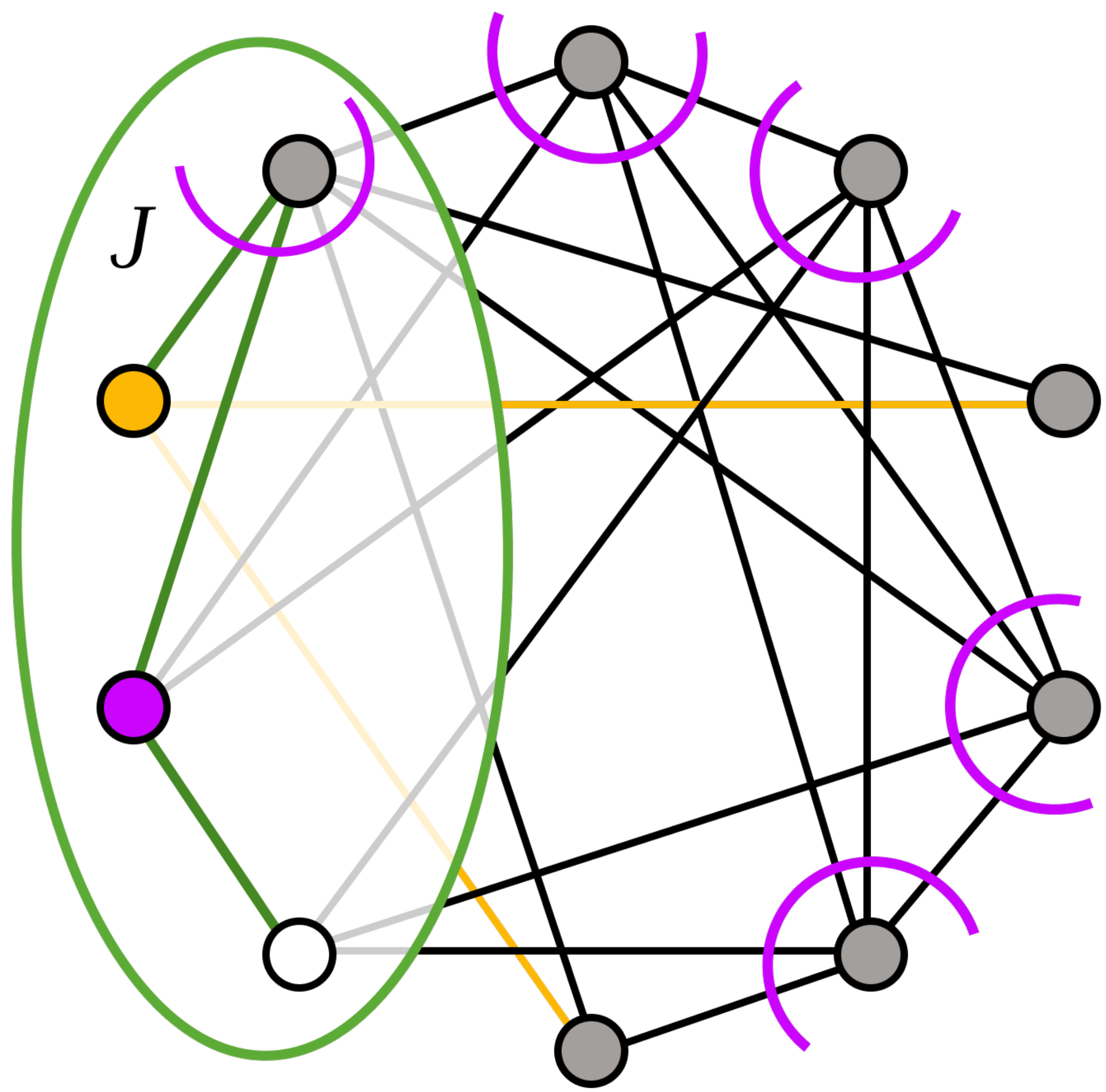

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Container Generator Algorithm - Sparse Sets

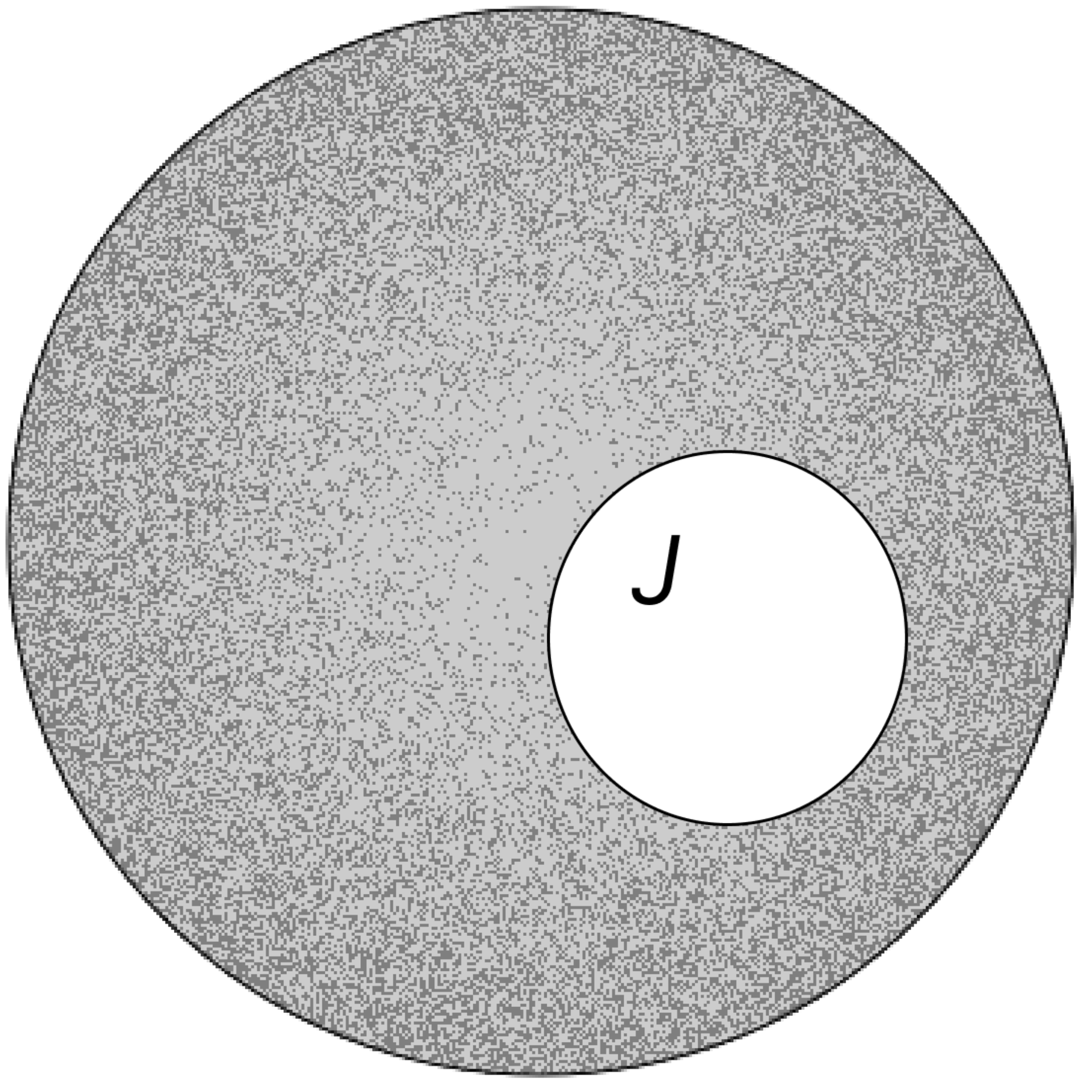
Input: Graph G and a sparse set J

- Initialize fingerprint $F = \varnothing$ and container $C = V$
- Repeat while $G[C]$ has more than ϵn^2 edges
  - select a vertex $v \in J$ and an operation of "remove neighbours" or "remove higher degree vertices" that maximizes the following:

$$\frac{\text{\# vertices removed from } C \text{ by operation}}{\text{\# vertices in } J \text{ removed from } C \text{ by operation}}$$

  - apply operation to container, add v and operation to F

Container Generator Algorithm - Sparse Sets



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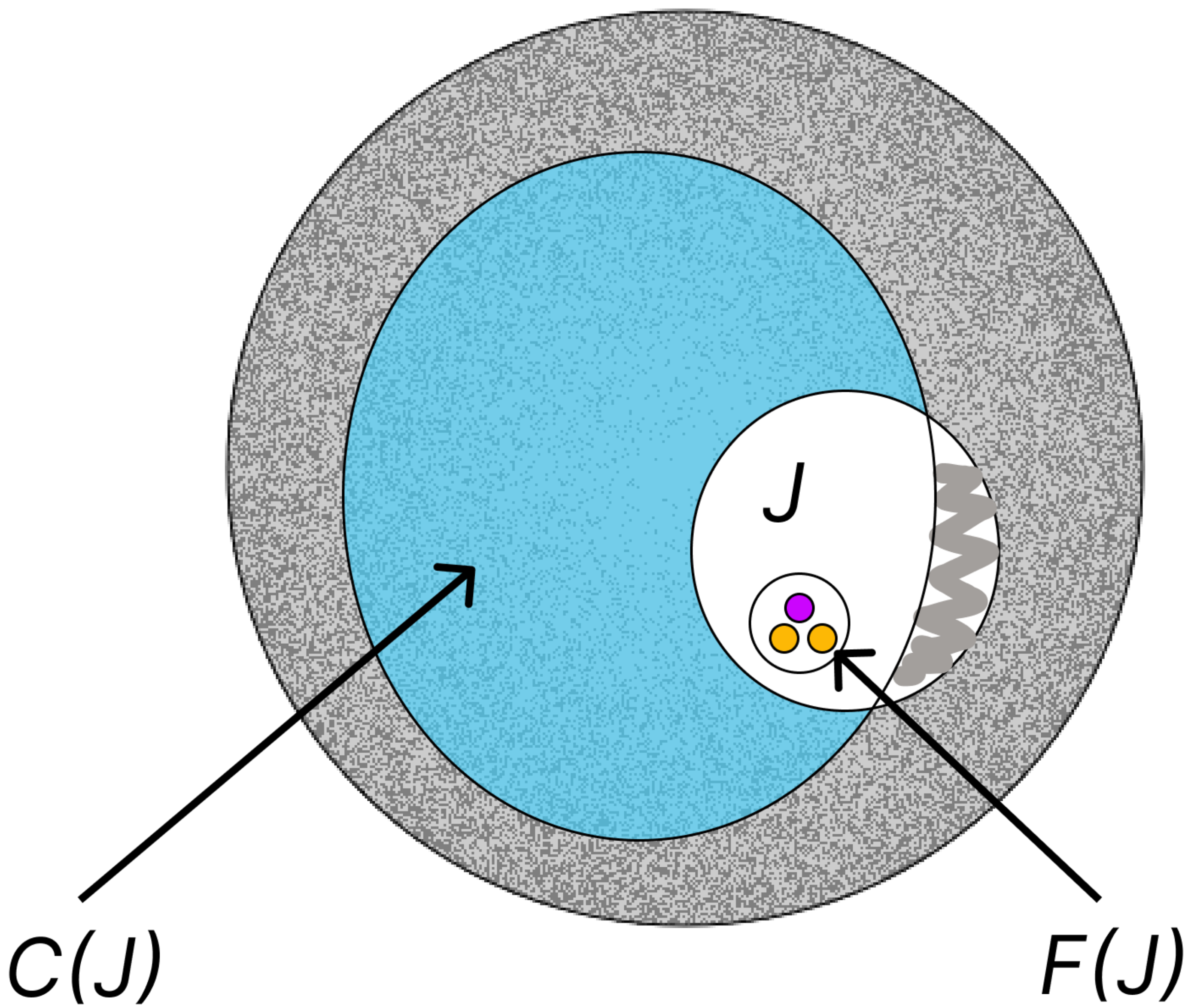

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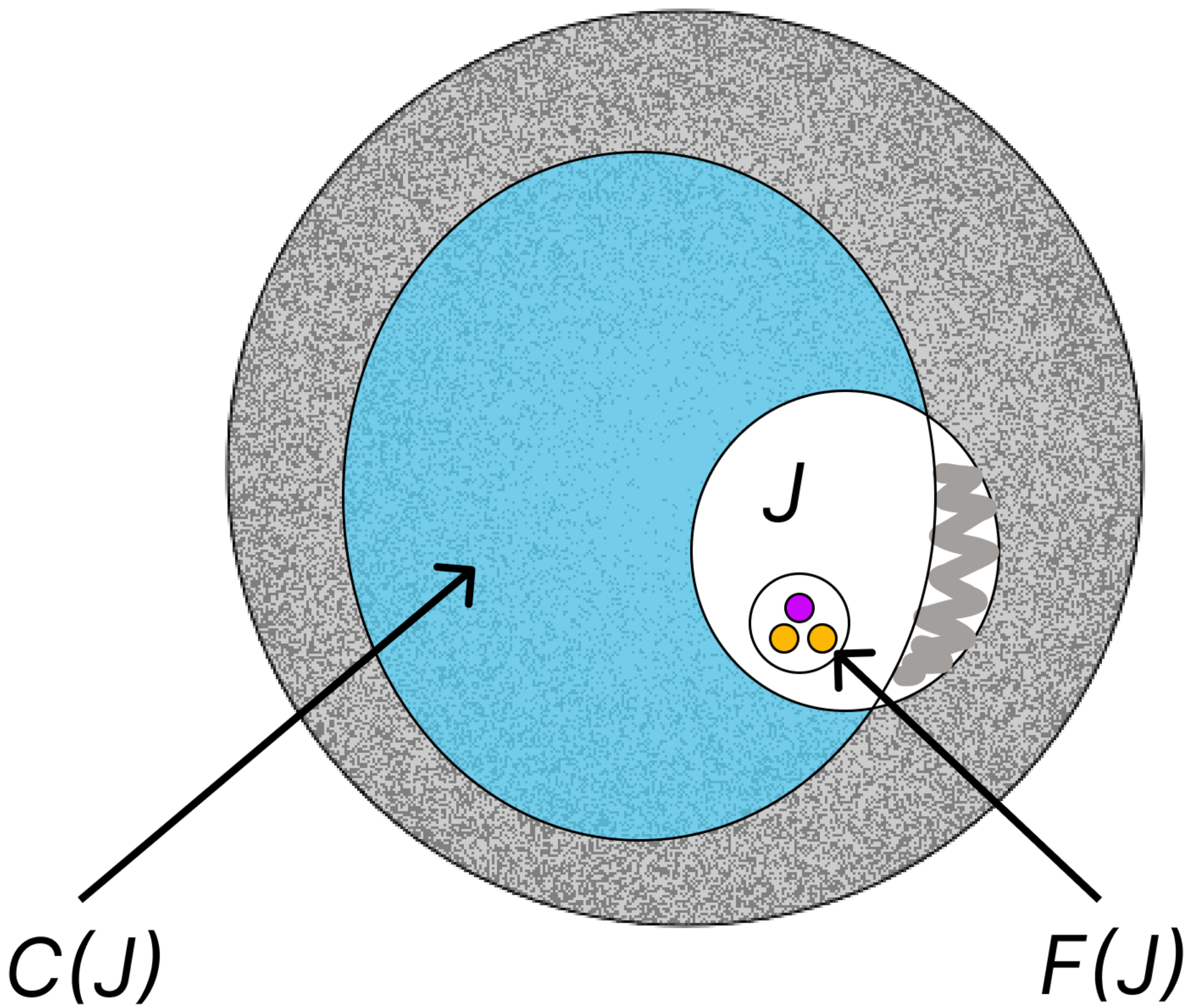

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Observations:

- $C(J)$ can be reconstructed from $F(J)$
- J is NOT fully contained in $C(J)$

Intuition: Procedure makes Progress

Claim: In every step of procedure where $G[C]$ has more than ϵn^2 edges:

$$\frac{\text{\# vertices removed from } C \text{ by operation}}{\text{\# vertices in } J \text{ removed from } C \text{ by operation}} \gtrsim \log(1/\epsilon) \cdot \frac{n}{|J|}$$

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Informally: in every step the procedure removes “much” more from the current container C than from J (relatively).

For example, if $\frac{n}{2}$ vertices are removed from C then only $\frac{|J|}{2\log(1/\epsilon)}$ vertices of J are removed from C.

Proof of Claim: Some Examples

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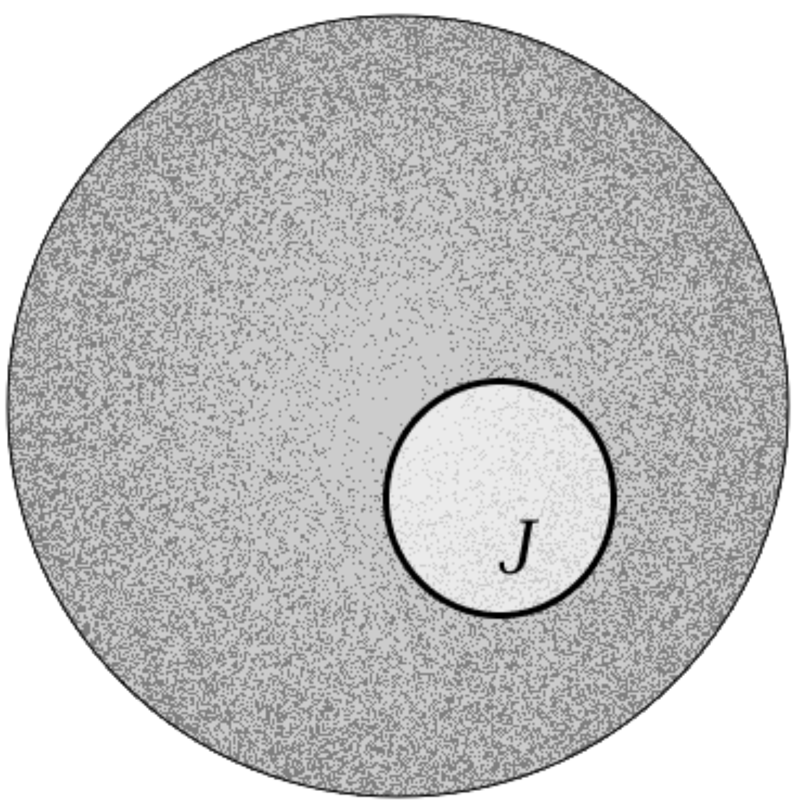

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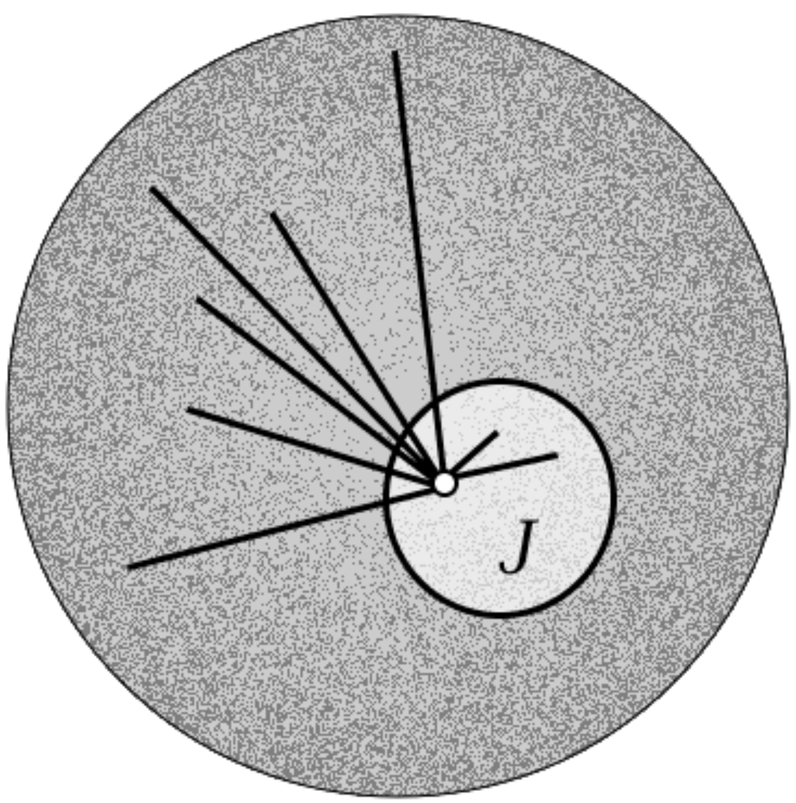

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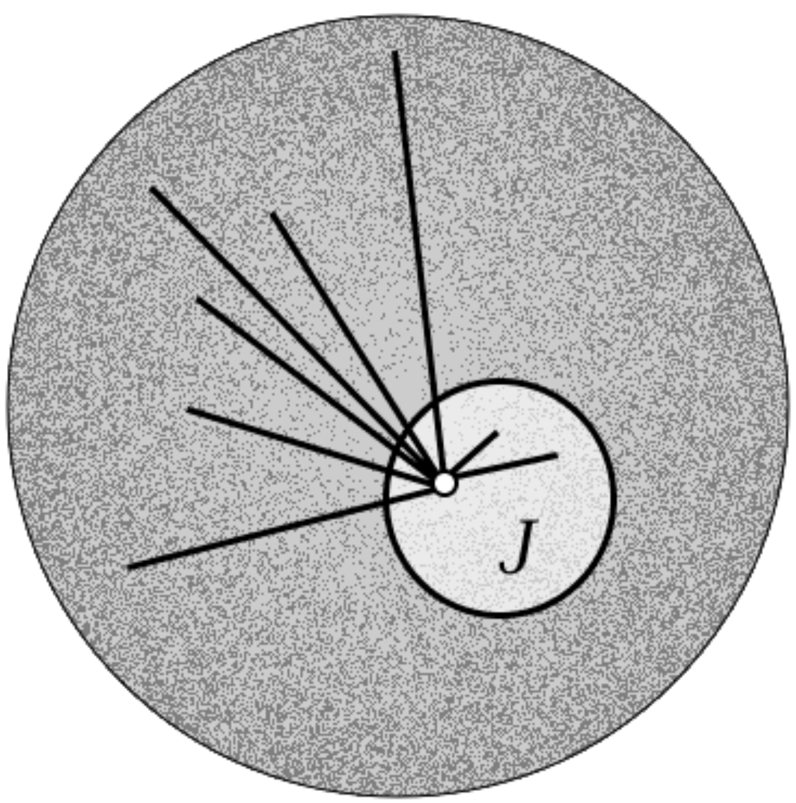

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Such a vertex exists because this is the average degree in $G[J]$

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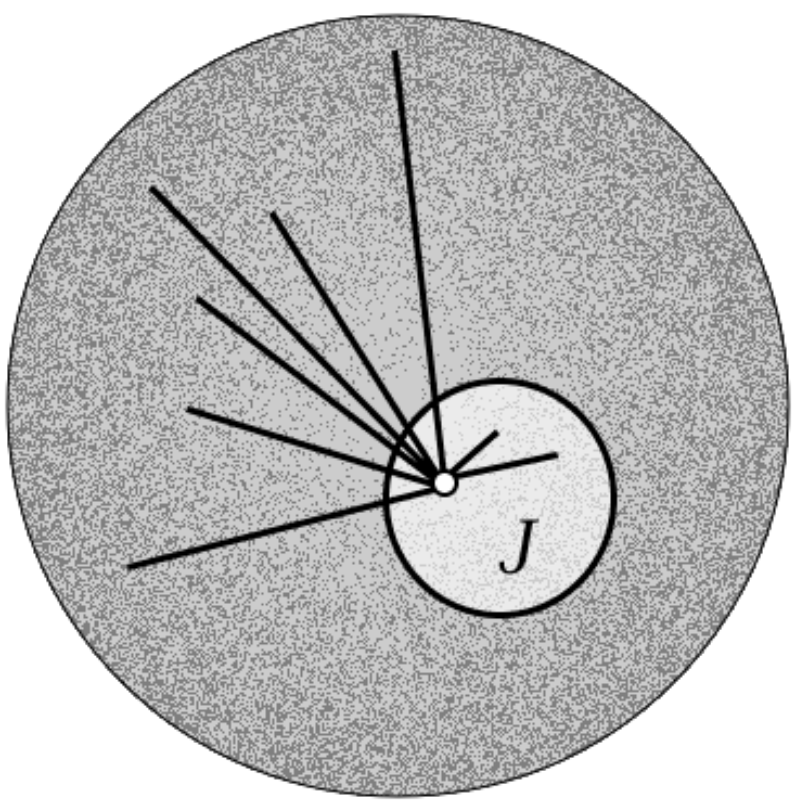

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- Using "remove neighbours" operation, can remove ϵn vertices from C and less than $\frac{\epsilon}{\log^2(1/\epsilon)}|J|$ vertices of J from C.

Proof of Claim: Some Examples

Claim: In every step of procedure where $G[C]$ has more than ϵn^2 edges:

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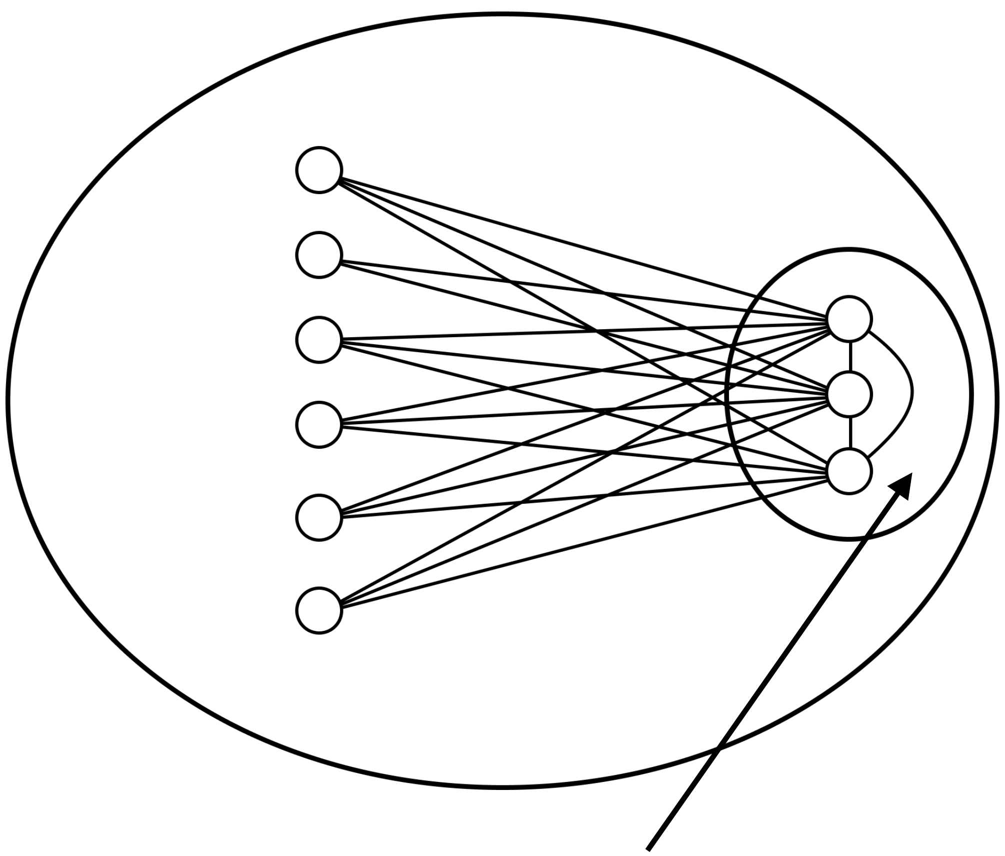
Example 2: Suppose $G[C]$ has ϵn vertices with degree n.



ϵn vertices adjacent to everything

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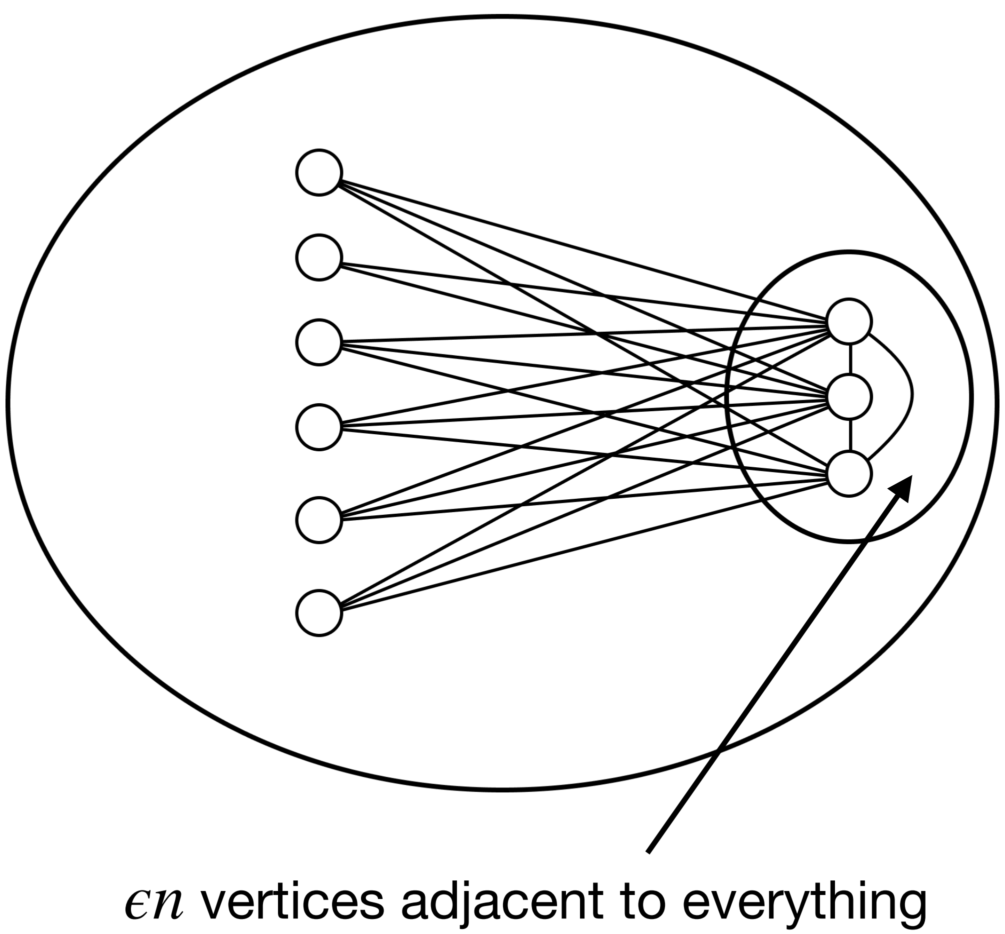

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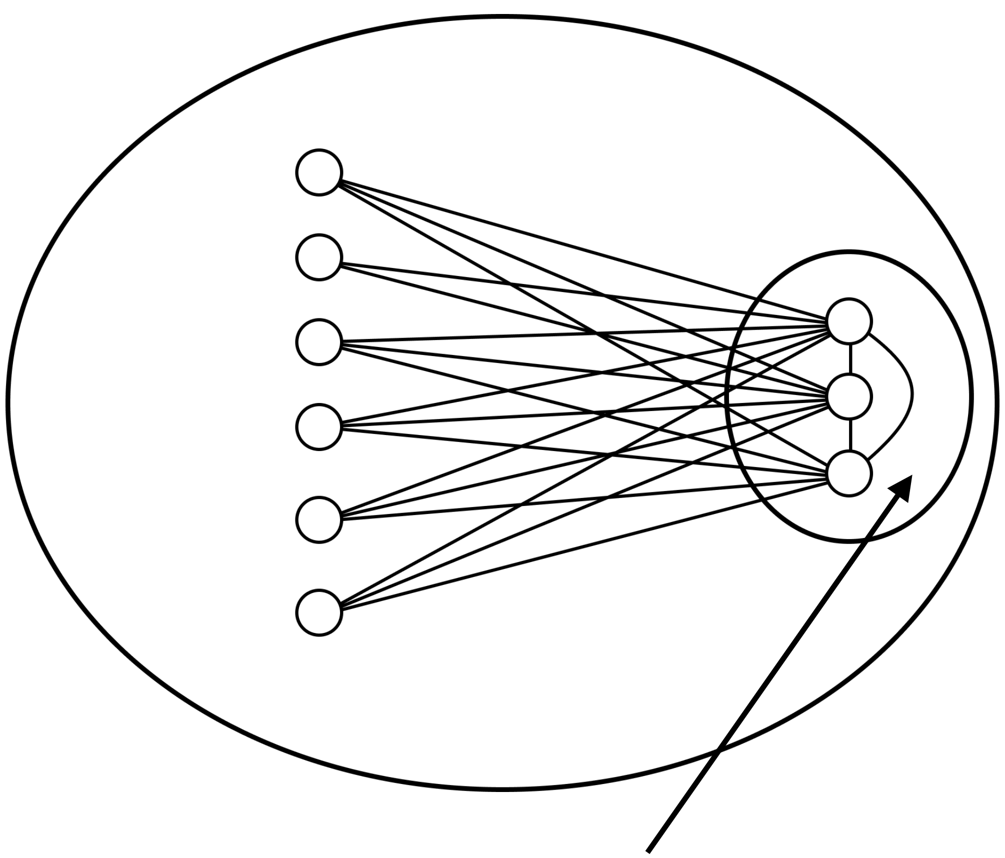

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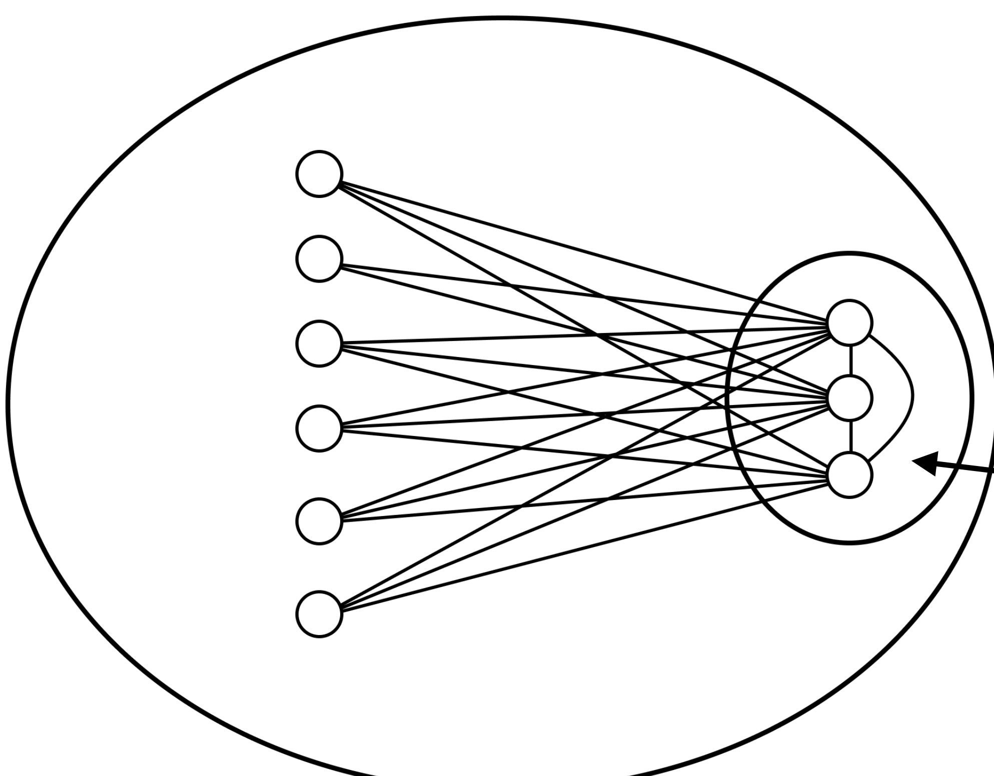

$|U|$ vertices each with degree $\gtrsim \frac{\epsilon n^2}{|U|}$

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□

Towards Proving Stronger Lemma

Lemma (Restated): For any ϵ, ρ let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq P(V)$ of containers that satisfies:

1. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$,
2. for every $C \in \mathscr{C}$, $|C| < \rho n$.
3. For every set $J \subseteq V$ such that $G[J]$ has fewer than $\frac{\epsilon}{\text{polylog}(1/\epsilon)} |J|^2$ edges, there exists $C \in \mathscr{C}$ and α such that $|C| \leq (1-\alpha)\rho n$ and
$$|C \cap J| \geq \left(1 - \frac{\alpha}{2}\right)|J|.$$

- Full proof involves showing that when container is large (close to ρn) the container procedure makes faster progress OR we can shrink the container at the end of the process (full details in paper)

Another Application of new Container Lemma

Counting Independent Sets in Regular Graphs

Theorem: Let G be a $d-$regular graph. Then the number of independent sets in G is at most $2^{\frac{n}{2}\left(1+\frac{1}{d}\right)}$, and there exists a $d-$regular graph achieving this maximum ($\frac{n}{2d}$ copies of $K_{d,d}$).

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Question: What about sparse sets in a $d-$regular graph? Can we use the new container lemma in place of the standard container lemma used by Sapozhenko?

Counting Sparse Sets in Regular Graphs

Theorem: Let G be a $d-$regular graph. Let $k \geq \text{polylog}(n)$. Then the number of induced subgraphs in G with edge density less $\frac{1}{k}\frac{d}{n}$ is at most

$$2^{\frac{n}{2}\left(1+O\left(\frac{\text{polylog}(n)}{d}\right)+O\left(\frac{\text{polylog}(n)}{k^{1/3}}\right)\right)}.$$

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- Observe that the edge density of G is roughly $\frac{d}{n}$ and there are at least $\frac{1}{2}2^n$ induced subgraphs with edge density at least $\frac{4d}{n}$
- [Zhao '10] says that there are at most $2^{\frac{n}{2}\left(1+\frac{1}{d}\right)}$ induced subgraphs with edge density 0.
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- $\frac{n}{2d}$ copies of $K_{d,d}$ gives lower bound of $2^{\frac{n}{2}\left(1+\frac{1}{d}+\frac{1}{k}\right)}$

Summary and Open Questions

Open Questions

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Thank you!