Testing Graph Properties with the Container Method

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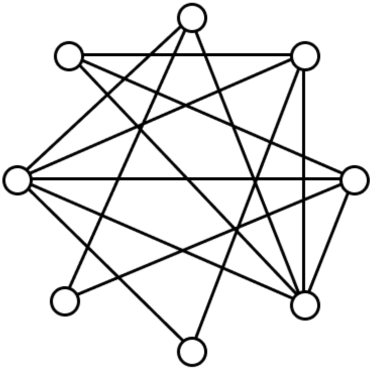
UNIVERSITY OF WATERLOO

Independent Set Problem

Does a graph on n vertices have an independent set of size ρn?



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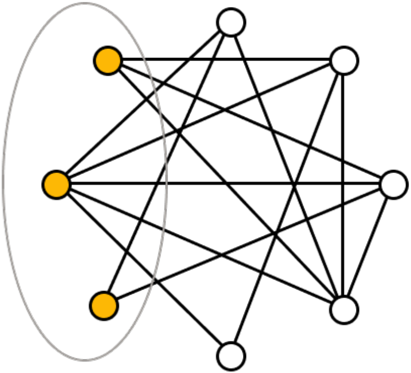

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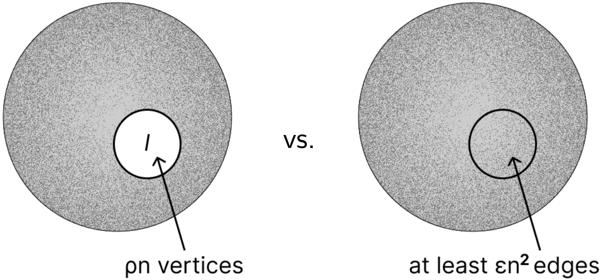
Testing Problem: Distinguish between the cases:

(i) G has a ρn independent set, and

(ii) every induced subgraph of size ρn in G has at least ϵn^2 edges.



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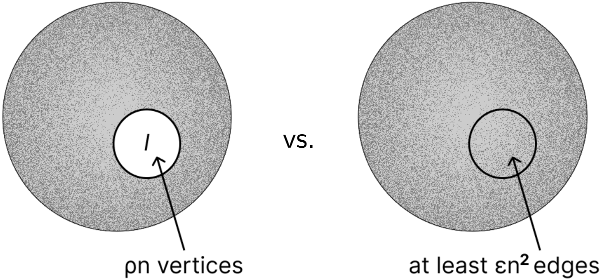

Theorem: Inspecting a random subgraph on $\widetilde{O}(\rho/\epsilon^4)$ vertices is sufficient to distinguish between (i) and (ii).

[Goldreich, Goldwasser, Ron '98]

Definitions

An ϵ-tester for ρ-IndepSet is an algorithm that samples a set S of s random vertices, examines the induced subgraph $G[S]$, and distinguishes between the cases (with high probability):

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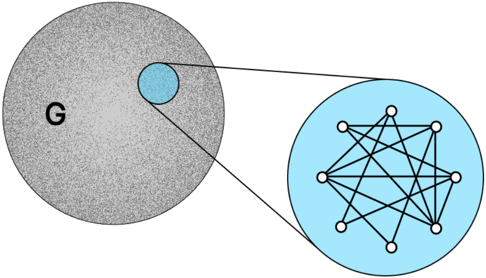

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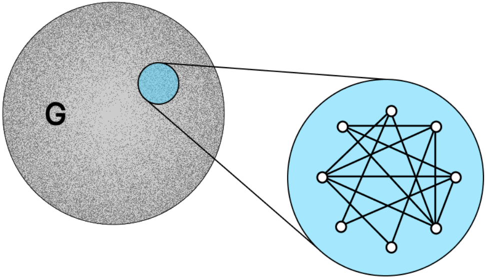

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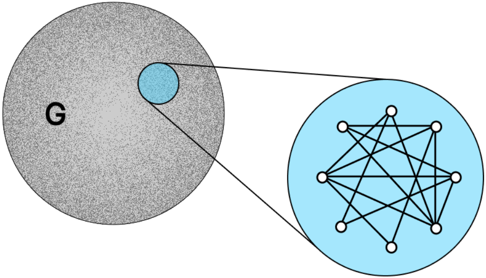

s is the sample complexity of the tester.

Question: What is the minimum sample complexity $\mathcal{S}_{\rho\text{-IndepSet}}(n, \epsilon)$ of ϵ-testing the ρ-IndepSet property?

Results

Prior Results: The sample complexity $\mathcal{S}_{\rho\text{-}\textsc{IndepSet}}(n, \epsilon)$ for ϵ-testing ρ-$\textsc{IndepSet}$ is bounded above by:

- $\widetilde{O}(\rho/\epsilon^4)$ [Goldreich, Goldwasser, Ron '98]
- $\widetilde{O}(\rho^4/\epsilon^3)$ [Feige, Langberg, Schechtman '04]

and it is bounded below by:

- $\widetilde{\Omega}(\rho^3/\epsilon^2)$. [Feige, Langberg, Schechtman '04]

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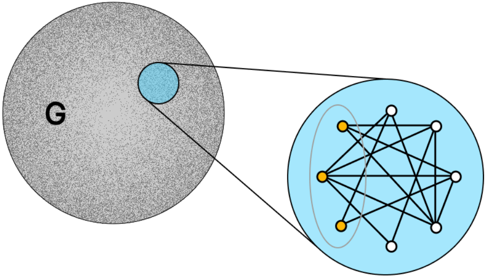
Our Result: $\mathcal{S}_{\rho\text{-}\textsc{IndepSet}}(n, \epsilon) = \widetilde{\Theta}(\rho^3/\epsilon^2)$.

Proof Overview

Testing Algorithm: Take a random sample S of s vertices, check if the induced subgraph $G[S]$ has a ρs independent set.



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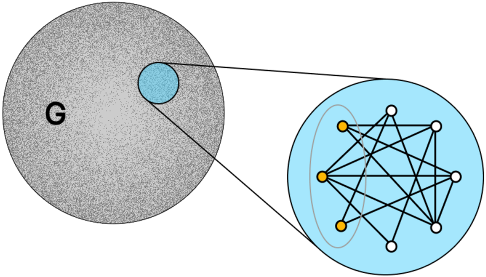
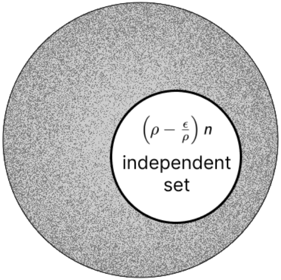


Key Challenge: Show that if G is ϵ-far from having a ρn independent set then a random induced subgraph on $s \approx \rho^3/\epsilon^2$ vertices has a ρs independent set with only small probability.

A Naive Approach

Observation: If G is ϵ-far from having a ρn independent set then the largest independent set in G is of size roughly $\left(\rho - \frac{\epsilon}{\rho}\right) n$.



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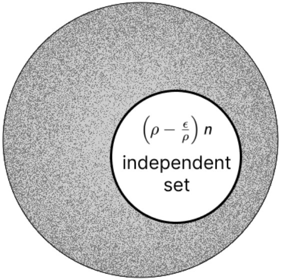

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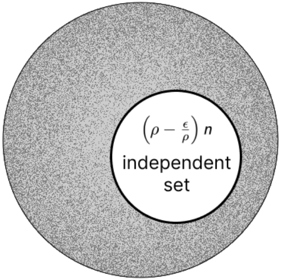

Approach:

- Upper bound the probability of getting ρs vertices in a sample of size s from a single $\left(\rho - \frac{\epsilon}{\rho}\right) n$ independent set.
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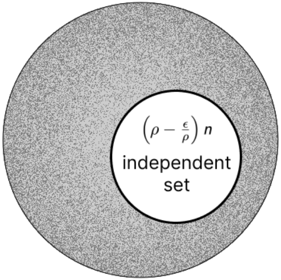

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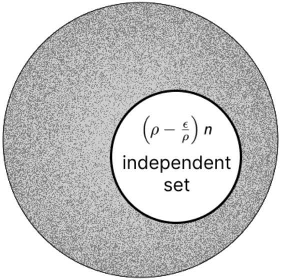

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Solution: The Graph Container Method

Graph Container Method

An Initial Graph Container Lemma

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Lemma. Let $G = (V, E)$ be ϵ-far from ρ-$\textsc{IndepSet}$. Then, there exists a collection $\mathscr{C} \subseteq P(V)$ of containers that satisfies:

1. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$,
2. for every $C \in \mathscr{C}$, $|C| \lesssim \left(\rho - \frac{\epsilon}{\rho}\right) n$, and
3. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$.

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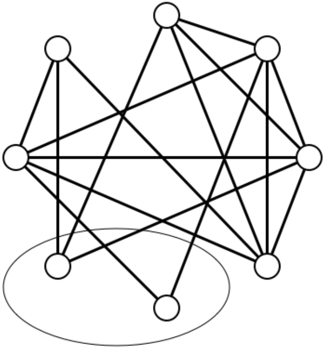
Graph Container Method – Encoding View



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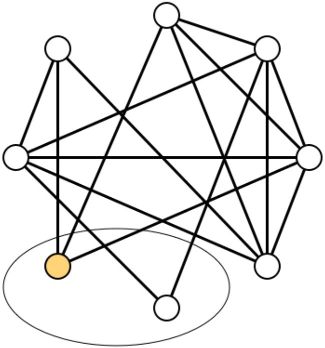

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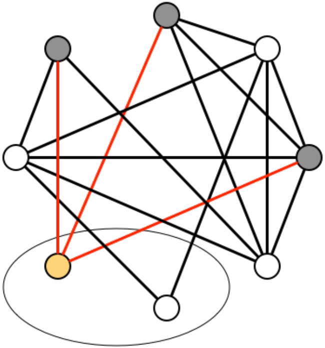

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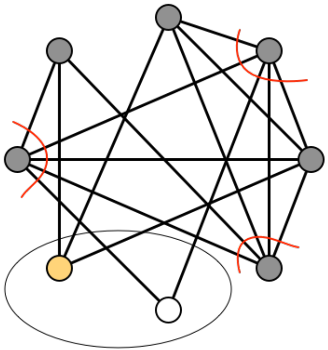

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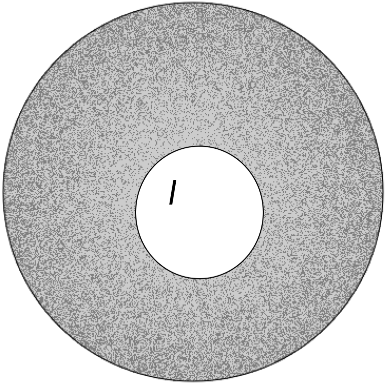
Input: Graph G and an independent set I

- Initialize fingerprint $F = \emptyset$ and container $C = V$
- Repeatedly select a vertex $v \in I$ with highest degree in $G[C]$ and add it to F. Remove $N(v)$ and all vertices with higher degree than v from C.



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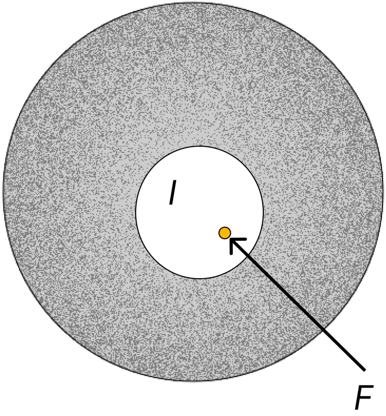

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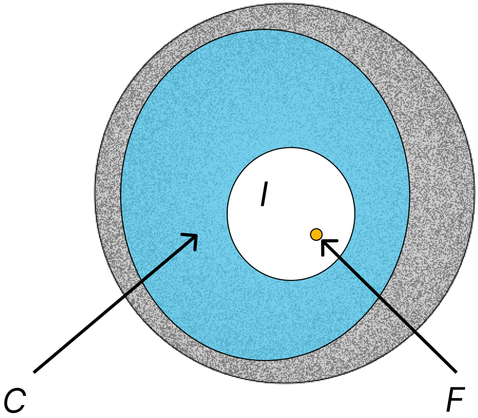

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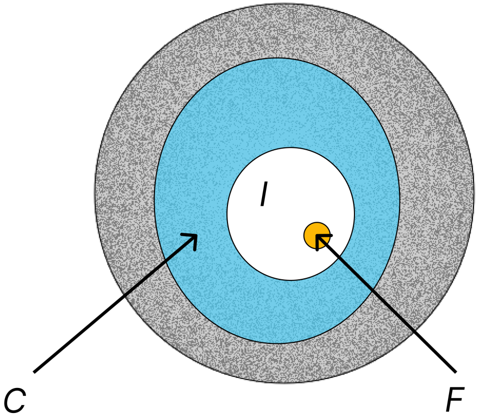

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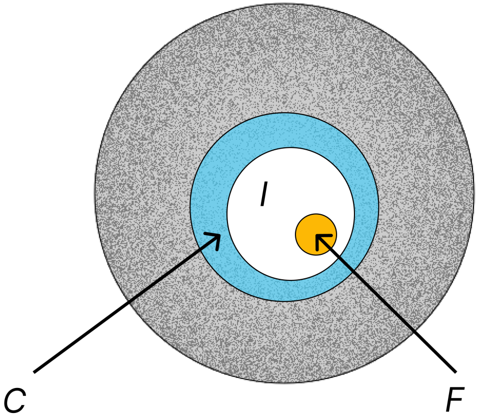

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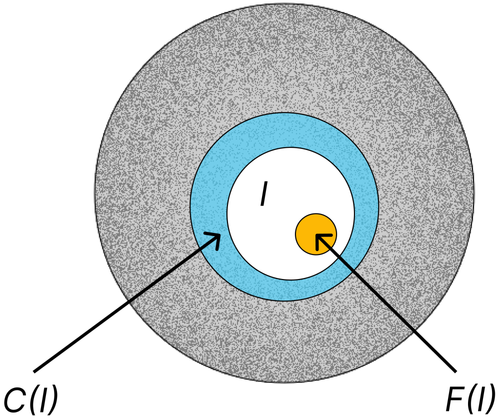

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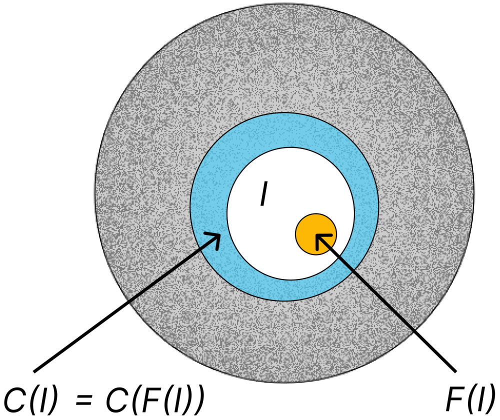

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Graph Container Lemma - Restated

Lemma. Let $G = (V, E)$ be ϵ-far from ρ-$\textsc{IndepSet}$. Then, there exists a collection $\mathscr{C} \subseteq P(V)$ of containers that satisfies:

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For example, at most $\binom{n}{O(1)}$ containers of size roughly $\left(\rho - \frac{\epsilon}{\rho}\right) n$.

Proof of the Main Theorem

Theorem. $\mathcal{S}_{\rho\text{-}\textsc{IndepSet}}(n, \epsilon) = \widetilde{O}(\rho^3/\epsilon^2)$.

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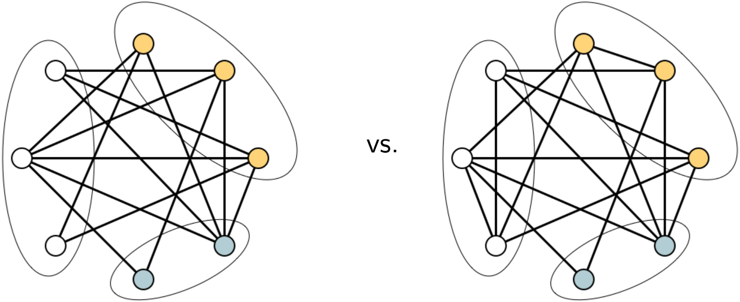
$$
\begin{aligned}
&\Pr[S \text{ contains } \rho s \text{ independent set}] \\
&\le \sum_{F \in \mathscr{F}} \Pr[S \text{ contains a } \rho s \text{ independent set with fingerprint } F] \\
&\le \sum_{F \in \mathscr{F}} \Pr[S \text{ contains } F \text{ and at least } \rho s \text{ vertices from } C(F)].
\end{aligned}
$$

Testing Colorability

Testing k-Colorability



Goal: Distinguish between

(i) G is k-colorable, and

(ii) G is ϵ-far from being k-colorable.

Results: Testing k-Colorability

Prior Results. The sample complexity $\mathcal{S}_{k\text{-}\textsc{Color}}(n, \epsilon)$ for testing k-colorability is bounded above by:

- $O(k^2/\epsilon^3)$ [Goldreich, Goldwasser, Ron '98]
- $O(k/\epsilon^2)$ [Alon, Krivelevich '02]
- $O(k^4/\epsilon)$ [Sohler '12]

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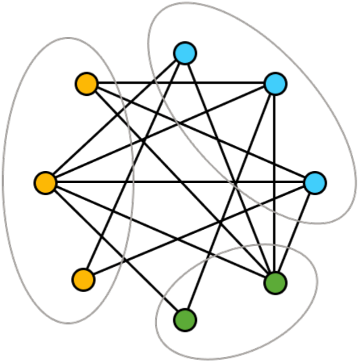
Theorem. $\mathcal{S}_{k\text{-}\textsc{Color}}(n, \epsilon) = \widetilde{O}(k/\epsilon)$.

Container Method for k-Colorability

Any k-colorable subgraph can be partitioned into k disjoint independent sets.



Run container procedure on each independent set, and take the union of the containers.

Summary: Property Testing and the Container Method

New graph container lemmas can be used to show:

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Thank you!