Property Testing and the Container Method

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Outline

1. What is the Container Method?
2. How to use Container Method to analyze property testers.
3. New container lemmas for new property testing results.

What is the Container Method?

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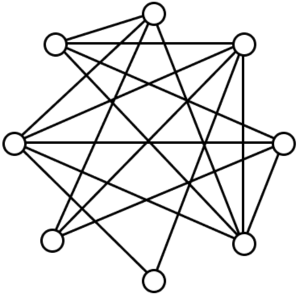
Answer: a tool for characterizing independent sets in some graphs.



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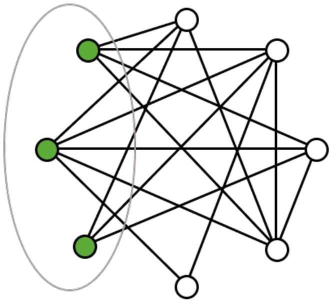

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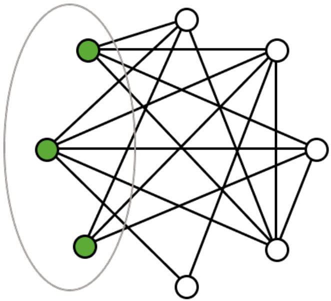

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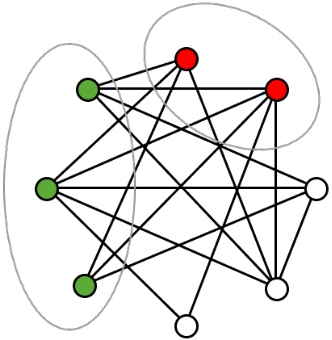

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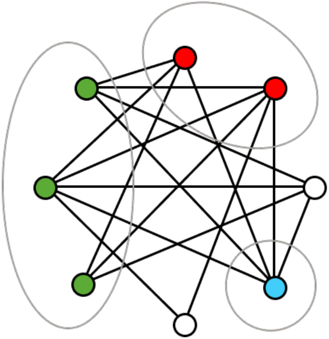

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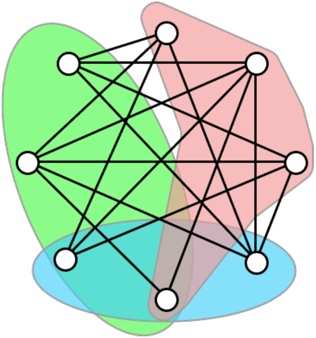

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An Initial Graph Container Lemma

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Lemma. For any ϵ, ρ, let $G = (V, E)$ be a graph such that every induced subgraph on ρn vertices has at least ϵn^2 edges. Then, there exists a collection $\mathscr{C} \subseteq P(V)$ of containers that satisfies:

1. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$,
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3. $|\mathscr{C}| \lesssim \binom{n}{1/\epsilon}$. [Kleitman, Winston '82]

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Examples:

- Random graphs
- d-regular graphs
- graphs that are ϵ-far from having a ρn independent set

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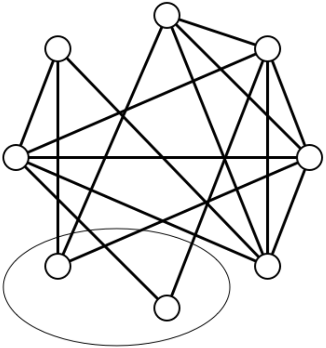
[1] For survey of combinatorial applications see "Counting Indepedent Sets in Graphs" by Samotij or "The method of hypergraph containers" by Balogh, Morris, and Samotij.

Graph Container Method – Encoding View



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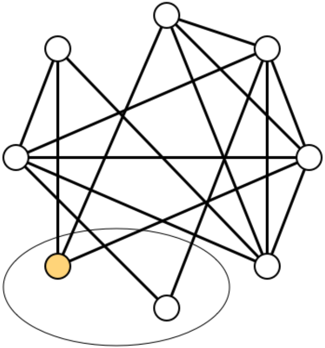

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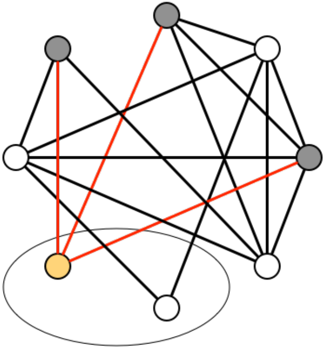

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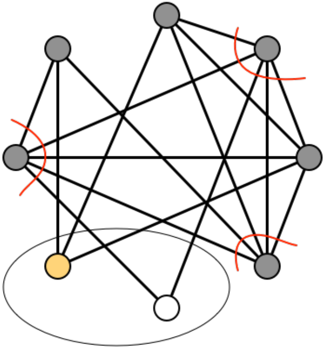

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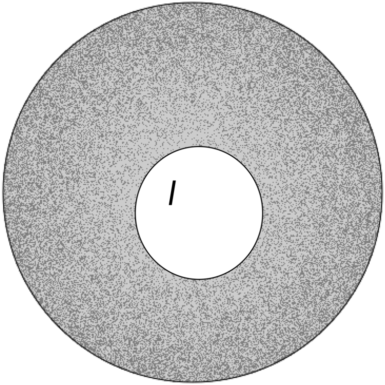
Output: Container $C(I)$ and Fingerprint $F(I)$



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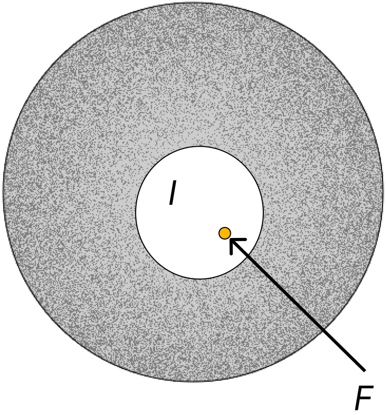

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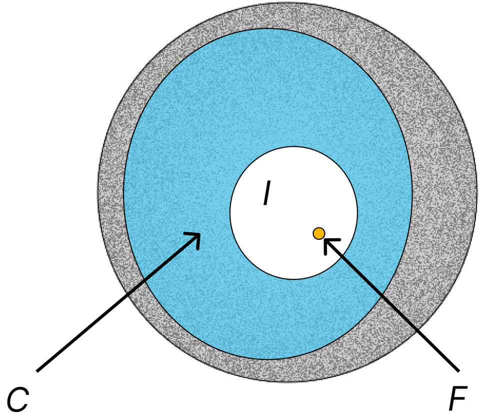

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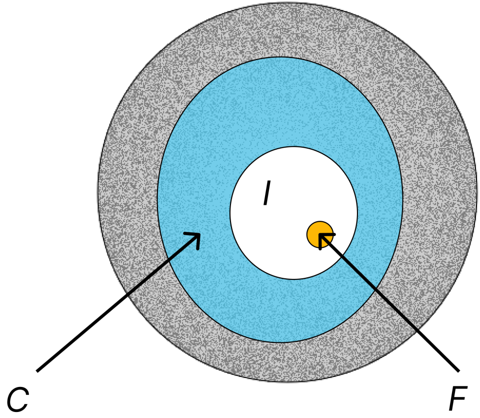

2nd Iteration

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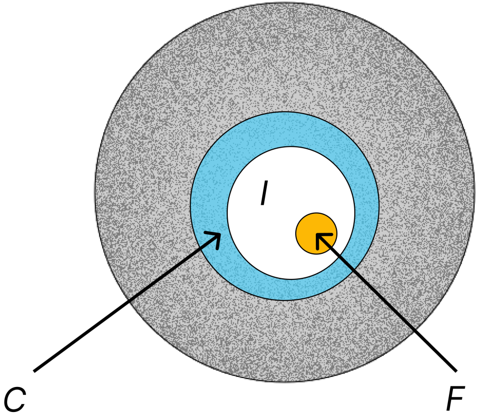

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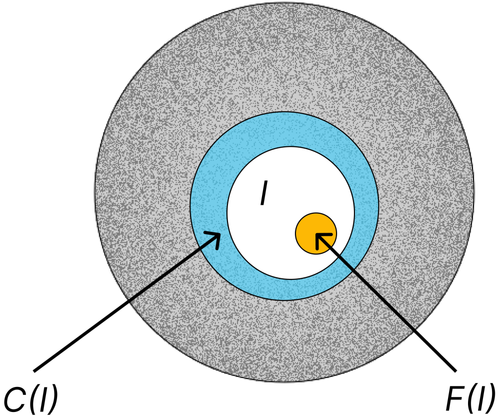

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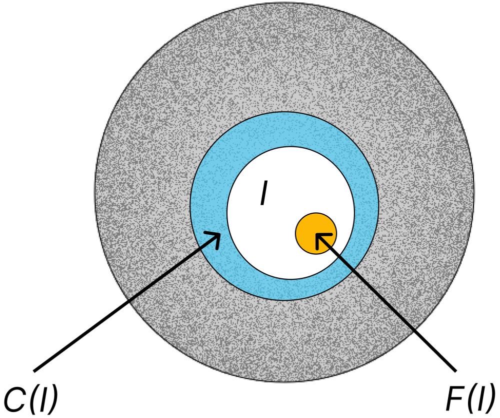

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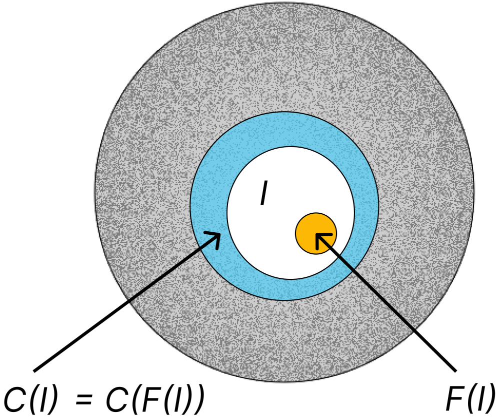

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$$\begin{aligned}\text{Let } \mathscr{C} &= \{C(I) : I \text{ is an independent set in } G\} \\ &= \{C(F) : F = F(I) \text{ for some independent set } I \text{ in } G\}\end{aligned}$$

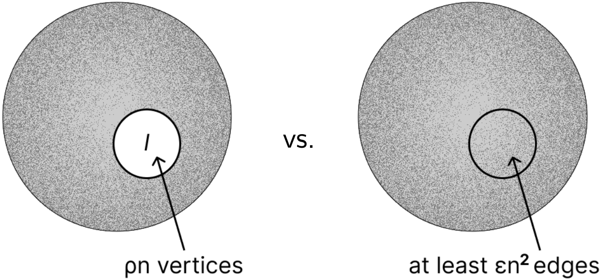
How to use the Container Method to Analyze Property Testers

Graph Property Testing - Example

Testing Independent Sets: Distinguish between the cases:

(i) G has a ρn independent set, and

(ii) every ρn-induced subgraph in G has at least ϵn^2 edges (ϵ-far).

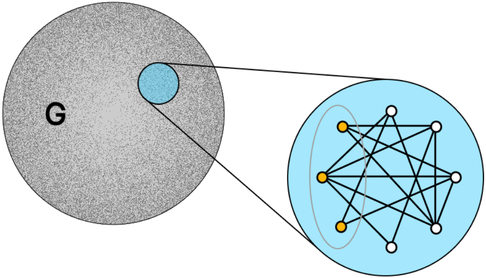


by inspecting a small fraction of G.

[Goldreich, Goldwasser, Ron '98]

A Simple Tester for Independent Set Property

Testing Algorithm: Take a random sample S of s vertices, check if the induced subgraph $G[S]$ has a ρs independent set (for some small s, which we call the sample complexity).



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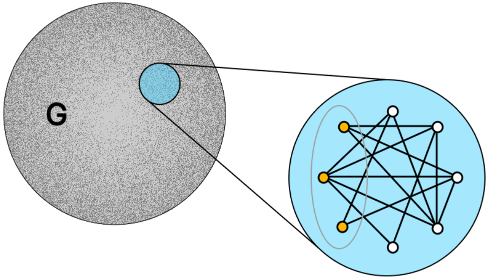

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- Since each container is small, can upper bound the probability of getting ρs vertices from a specific container.
- Take union bound over all containers.

New Container Lemmas and Property Testing Results

Prior Testing Results using the Container Method

Theorem:
There is an ϵ-tester for ρ-IndepSet that only inspects a random induced subgraph on $\widetilde{O}(\rho^3/\epsilon^2)$ vertices.

There is an ϵ-tester for k-Color that only inspects a random induced subgraph on $\widetilde{O}(k/\epsilon)$ vertices.

[Blais, Seth FOCS '23]

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Question: What else can the container method give in testing?

Result 1: Satisfiability Testing

Definition: An instance ϕ of (q, k)-$\textsc{Sat}$ is a constraint satisfaction problem with q variables per constraint and alphabet size k.

ϕ is ϵ-far from satisfiable if at least $\epsilon\binom{n}{q}$ constraints must be removed to make ϕ satisfiable.

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Theorem 1: There is an ϵ-tester for (q, k)-$\textsc{Sat}$ with sample complexity $\widetilde{O}(kq^3/\epsilon)$.

- Improving on prior bounds of $2^{k^{2q}}/\epsilon^2$ [Alon, Shapira '03] and k^{3q}/ϵ [Sohler '12].
- Via new **hypergraph** container lemma.

Corollaries of Result 1

Theorem 1: There is an ϵ-tester for (q, k)-$\textsc{Sat}$ with sample complexity $\widetilde{O}(kq^3/\epsilon)$.

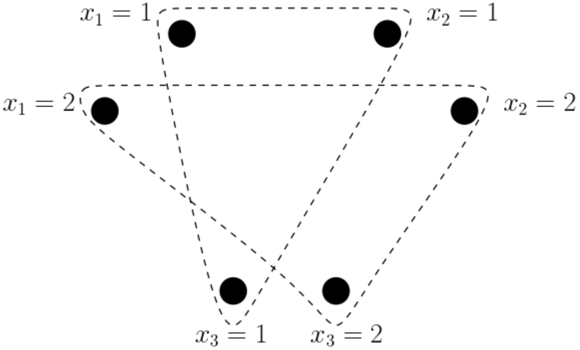
Corollaries:

- new bound for ϵ-testing k-colorability on q-uniform hypergraphs.
- new bound for ϵ-testing any $0 - 1$ graph partition property.

Hypergraph Encoding for Satisfiability

Idea: Encode (q, k)-SAT instance ϕ as a hypergraph H_ϕ. The vertex set is all possible single variable assignments. Edges correspond to falsified constraints.

Example: Two hyperedges formed by the constraint that variables x_1, x_2, x_3 must not all be equal.



New Hypergraph Container Lemma for Satisfiability

Lemma (informal). Let ϕ be an instance of (q, k)-$\textsc{Sat}$. If ϕ is ϵ-far from satisfiable then there exists a collection of containers $\mathscr{C} \subseteq P(V(H_\phi))$ that satisfies:

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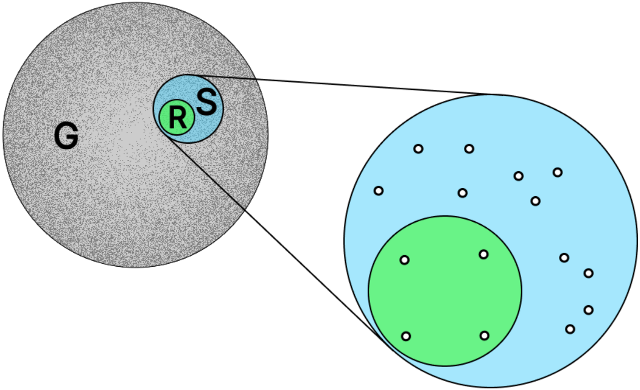
Result 2: Query Complexity of Independent Set Testing

Theorem 2: There is an ϵ-tester for ρ-IndepSet with query complexity $\widetilde{O}(\rho^5/\epsilon^{7/2})$.

- Demonstrates gap between query complexity of optimal sample based tester of $\widetilde{O}(\rho^6/\epsilon^4)$ by Blais, Seth ('23).
- Via new graph container lemma for an object we call independent set stars.

The New Tester



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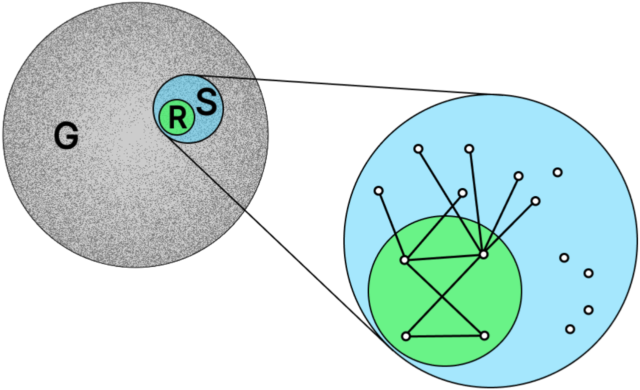

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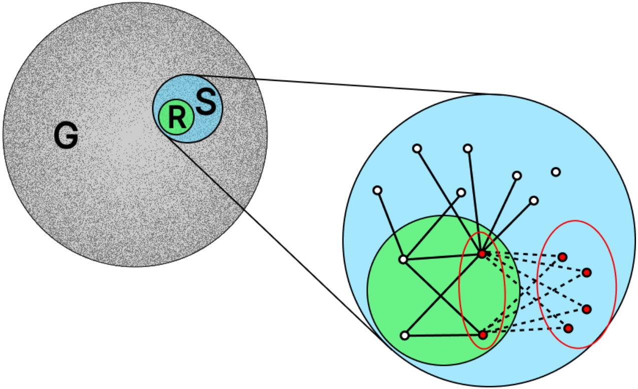
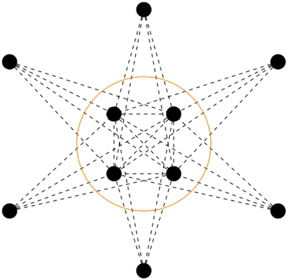


- Take vertex sample S and smaller sample $R \subset S$.
- Query all edges in R, and all edges between R and S.
- Accept if and only if R contains a $\rho|R|$ independent set I and S contains a $\rho|S|$ set J such that $E(I, J) = 0$.

Independent Set Stars and New Container Lemma

Definition: Let $G = (V, E)$ be a graph. A pair of subsets (I, J) of V form an independent set star if I is an independent set and $|E(I, J)| = 0$.



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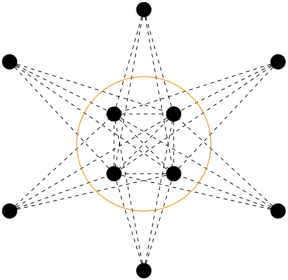

Lemma (informal). Let G be ϵ-far from having a ρn independent set. Then, there exists a collection of containers that cover all independent set stars.

Summary: Property Testing and the Container Method

New graph and hypergraph container lemmas can be used to show:

- ϵ-tester for ρ-IndepSet with sample complexity $\widetilde{\Theta}(\rho^3/\epsilon^2)$
- ϵ-tester for k-Color with sample complexity $\widetilde{O}(k/\epsilon)$

[Blais, Seth '23]

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