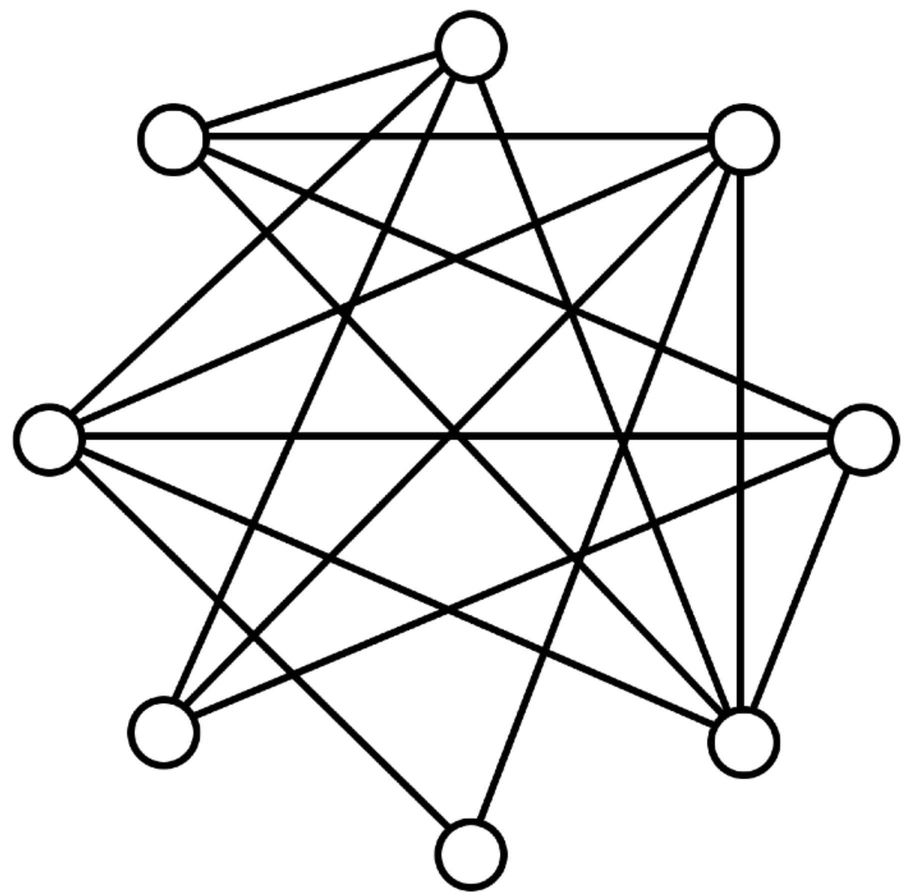
Container Lemmas and a Characterization of Testing Hereditary Graph Properties

Gaia Carenini (University of Cambridge), Cameron Seth, Yuichi Yoshida (National Institute of Informatics)

August 14 2026

3-Colorability Problem

Given a graph G on n vertices, is it 3-colorable?



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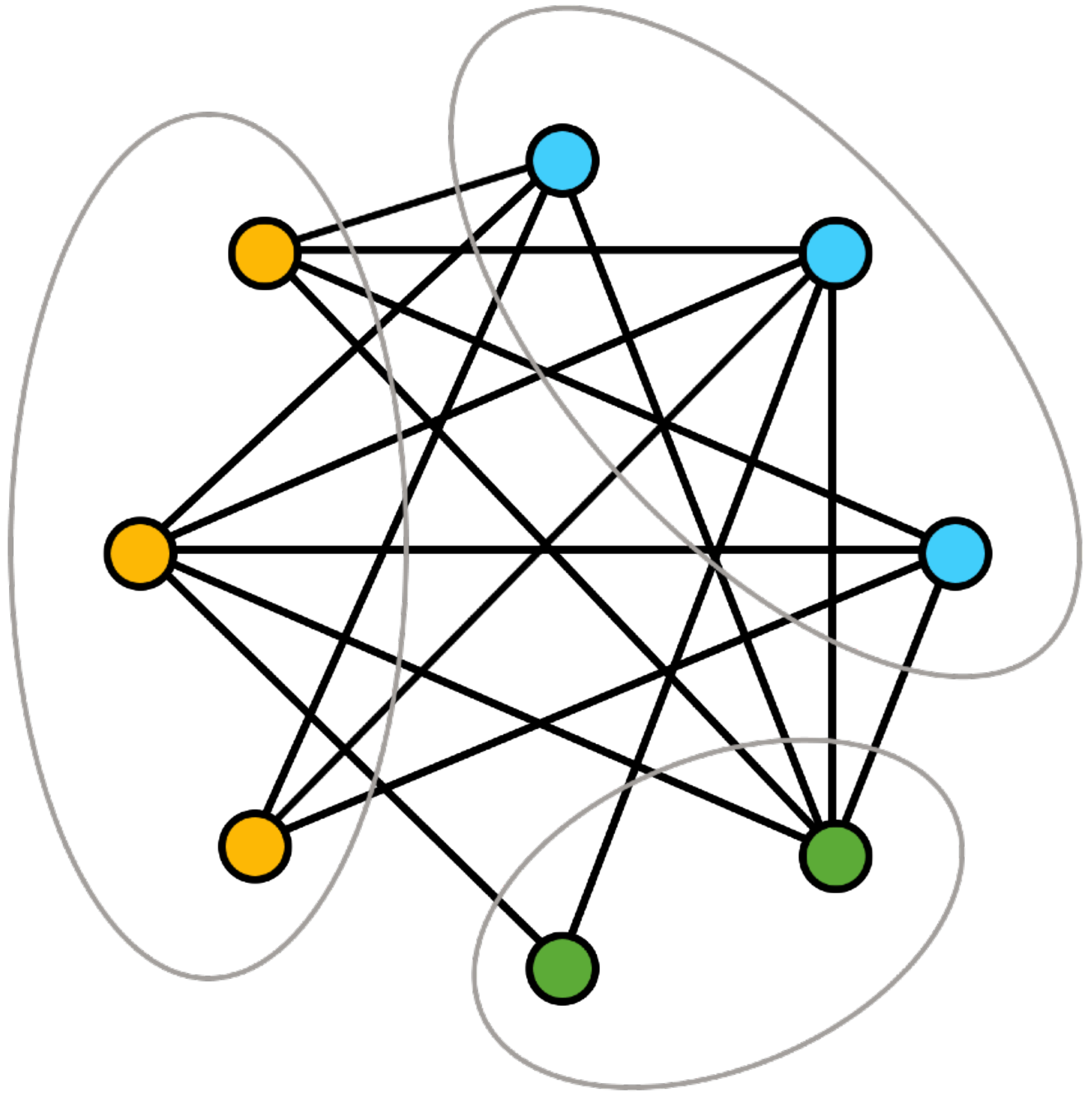

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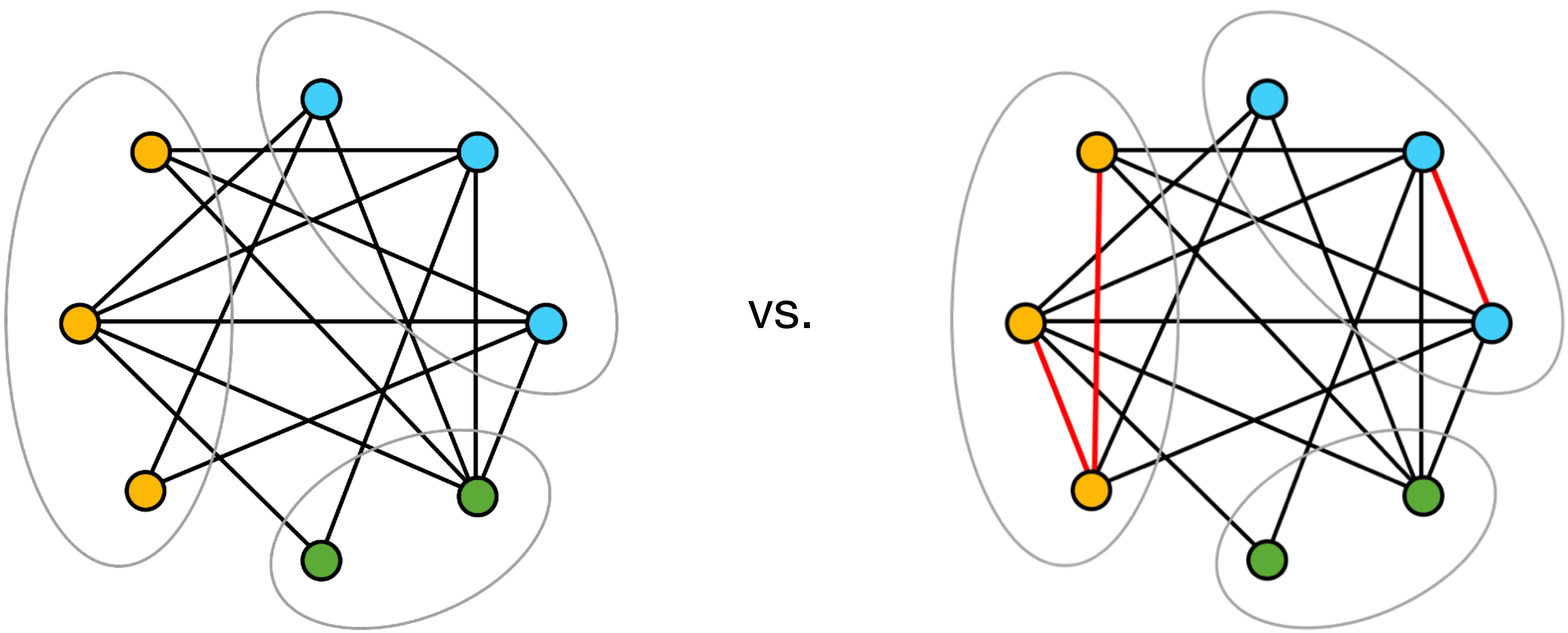

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Theorem: Inspecting a random induced subgraph on $\mathrm{poly}(1/\epsilon)$ vertices is sufficient to distinguish between case 1 and 2 (whp).

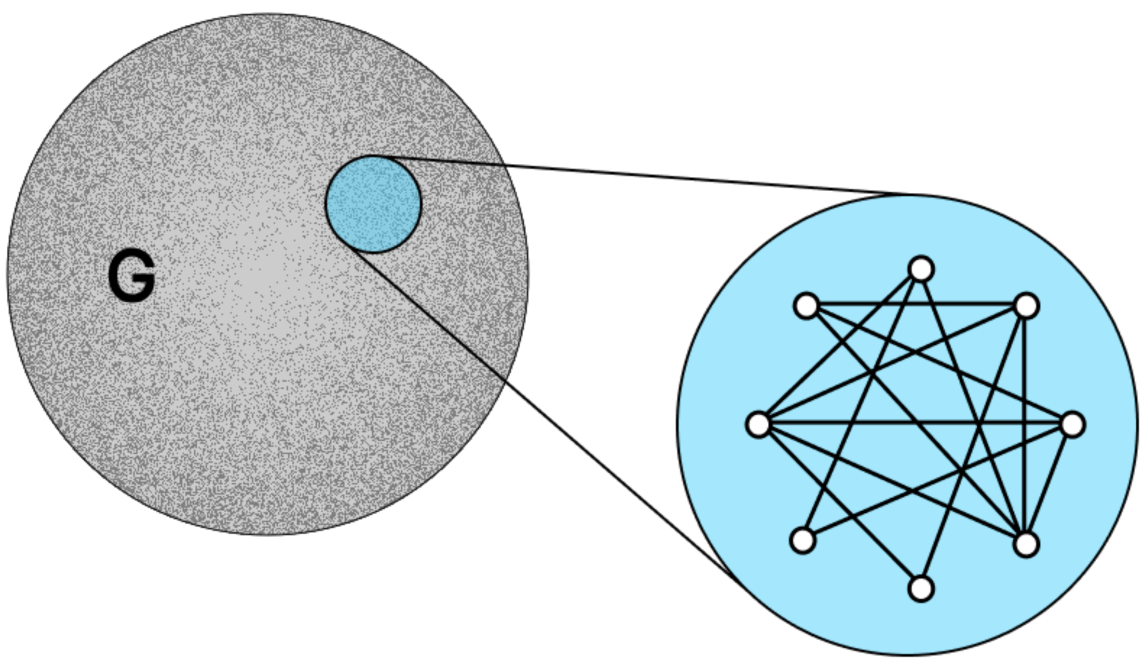
[Goldreich, Goldwasser, Ron '98]

Algorithm: On input graph $G = (V, E)$, take a random set $S \subseteq V$ of $\mathrm{poly}(1/\epsilon)$ vertices, accept iff $G[S]$ is 3-colorable.

Definitions

An ϵ-tester for Π is an algorithm that samples a set S of s random vertices, examines the induced subgraph $G[S]$, and distinguishes between the cases (with probability at least $2/3$):

1. G has the property Π, and
2. At least ϵn^2 edges of G must be modified to make G have Π (ϵ-far)



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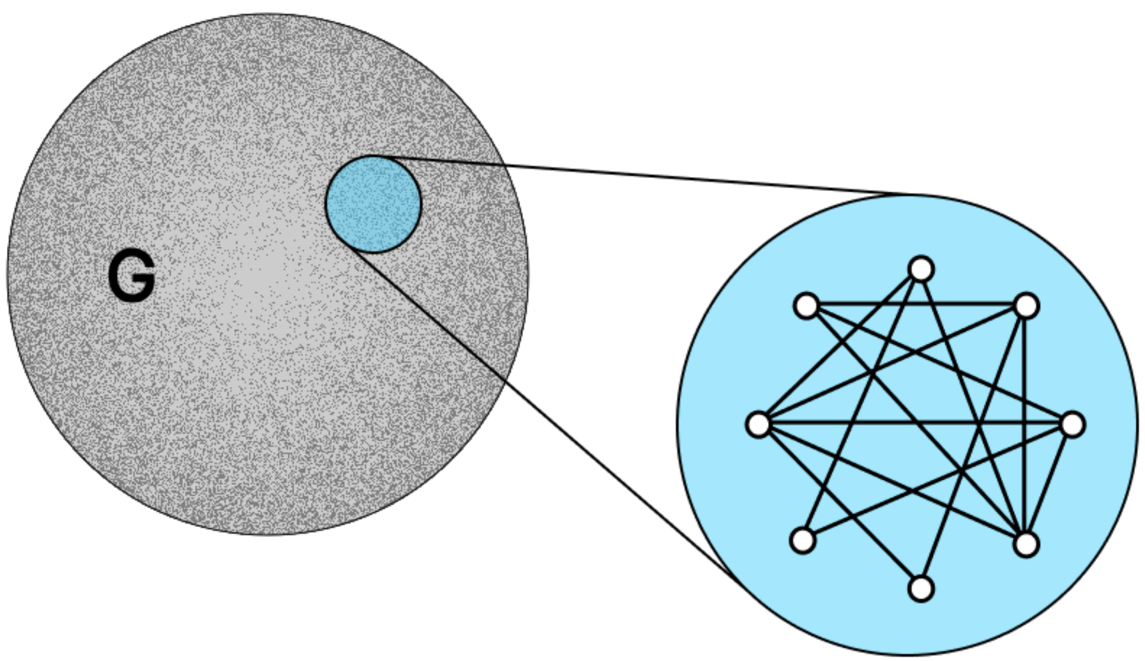

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Remark: An alternate measure is the edge query complexity, but the two measures are related by at most a quadratic factor.

[Goldreich, Trevisan '03]

First Results in Graph Property Testing

Theorem: There are ϵ-testers for k-colorability, ρ-clique, ρ-cut, and many other partition properties with sample complexity $\mathrm{poly}(1/\epsilon)$.

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The Characterization Question: What graph properties can be ϵ-tested with a sample complexity that only depends on ϵ (and not n)?

Answer to the Characterization Question

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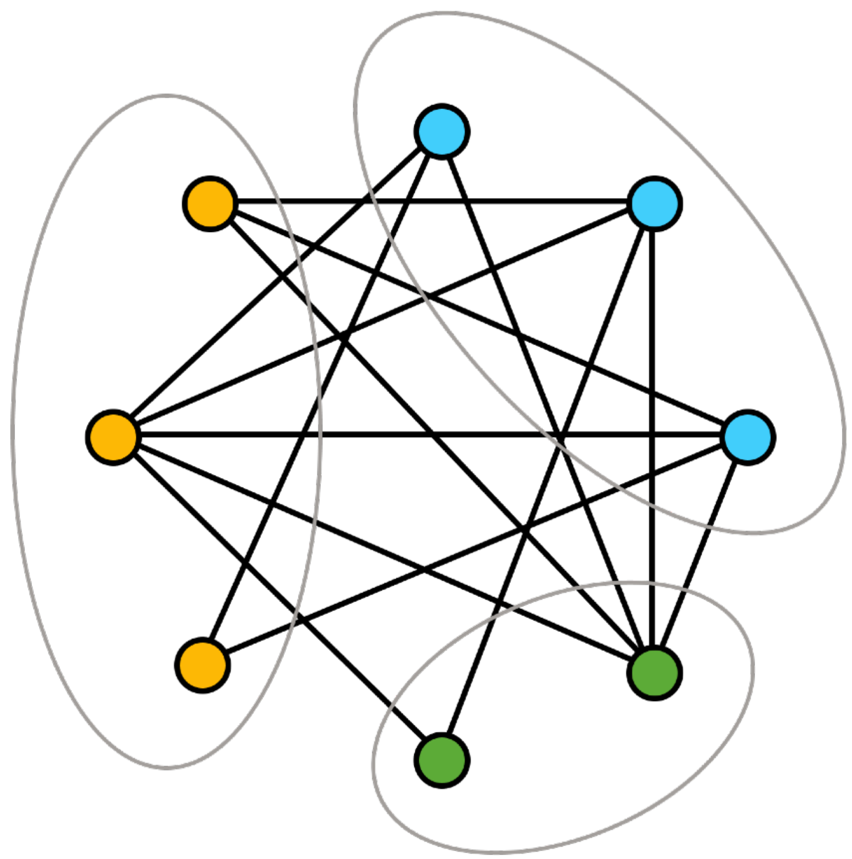
Examples: k-colorability (every induced subgraph of a k-colorable graph is also k-colorable)



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Question: Is there a more quantitative answer? For example, what is the class of graph properties that can be ϵ-tested with one sided error and sample complexity that is $\mathrm{poly}(1/\epsilon)$?

Recent Container Method based Results

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- Tight bounds on testing the ρ-independent set property

  [Blais, **S** '23]

- Improved bounds on testing k-colorability and satisfiability

  [Blais, **S** '24]

- Tight bounds on tolerant testing the ρ-independent set property

  [**S** '25]

- Improved bounds on testing the ρ-independent set property in hypergraphs

  [Grigorescu, Nasa, **S** '26]

Main Result (Very Informal)

Theorem: Let Π be a hereditary graph property. Then

Π is ϵ-testable with one sided error and sample complexity s

$$\Updownarrow$$

Π has an "ϵ-container lemma" parameterized by $\mathrm{poly}(s)$

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Π has an "ϵ-container lemma" parameterized by $\mathrm{poly}(s)$

Remarks:

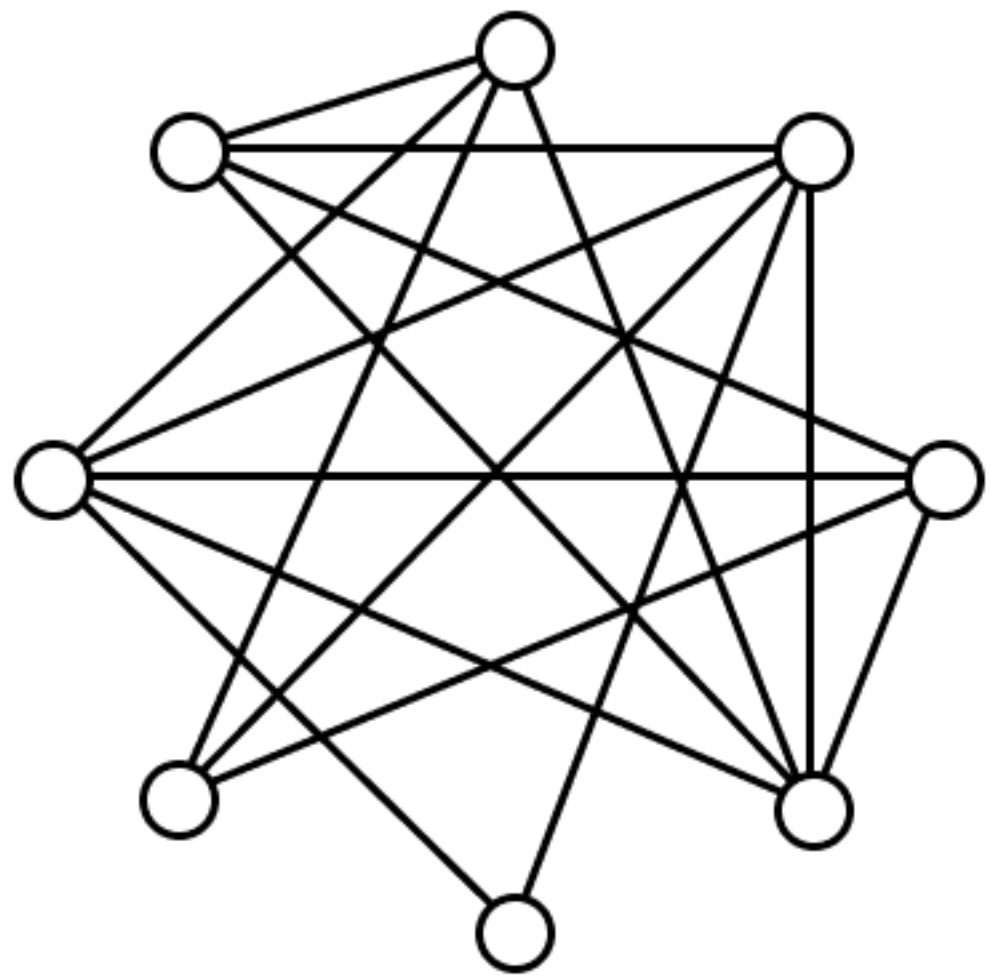
- A quantitative characterization of testability of hereditary properties
- Demonstrates that the container method captures testability of hereditary properties

Outline of Talk

- What is the container method and how can it be used for testing?
- Formally stating our theorem
- Proof sketch of our theorem
- An application to testing partitions of hereditary properties

What is the Container Method?

Answer: A tool for characterizing independent sets in some graphs.



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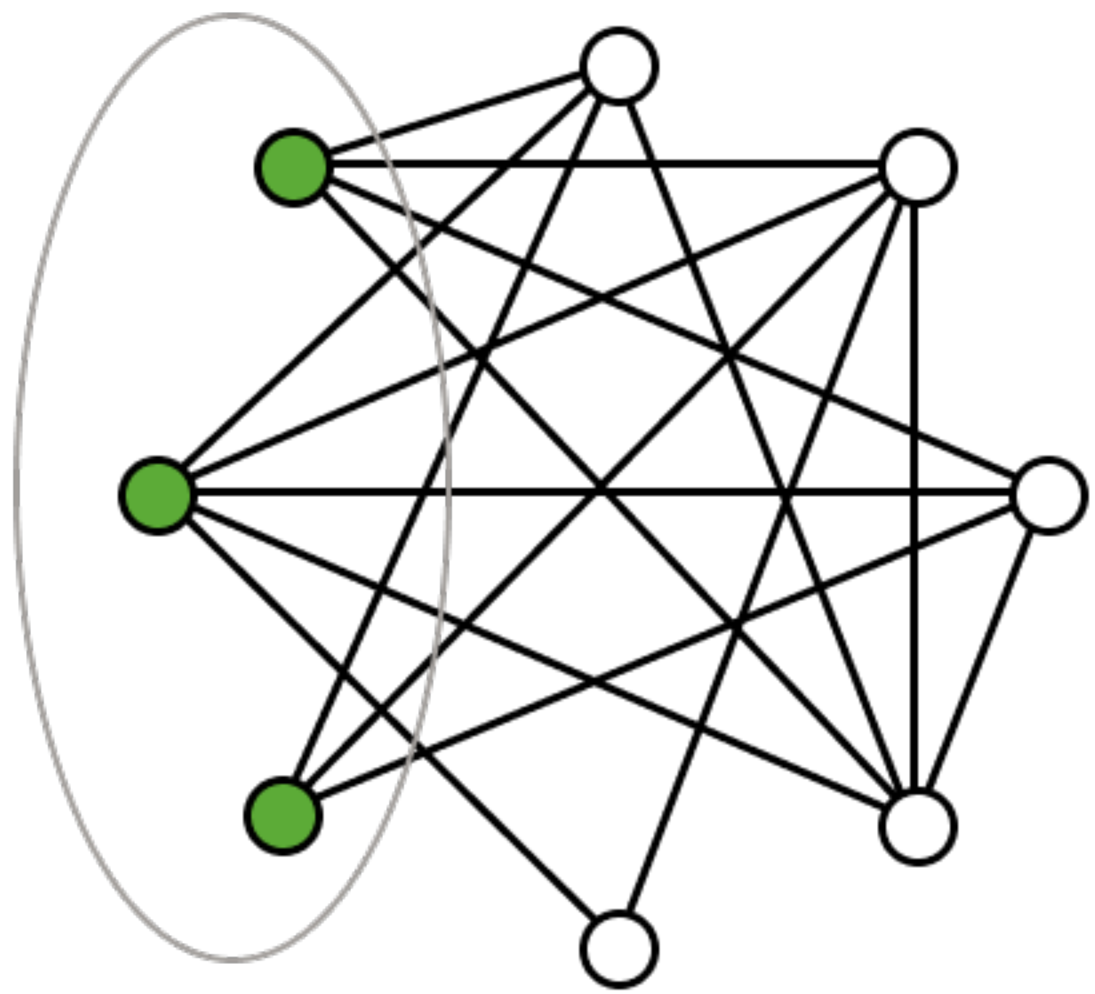

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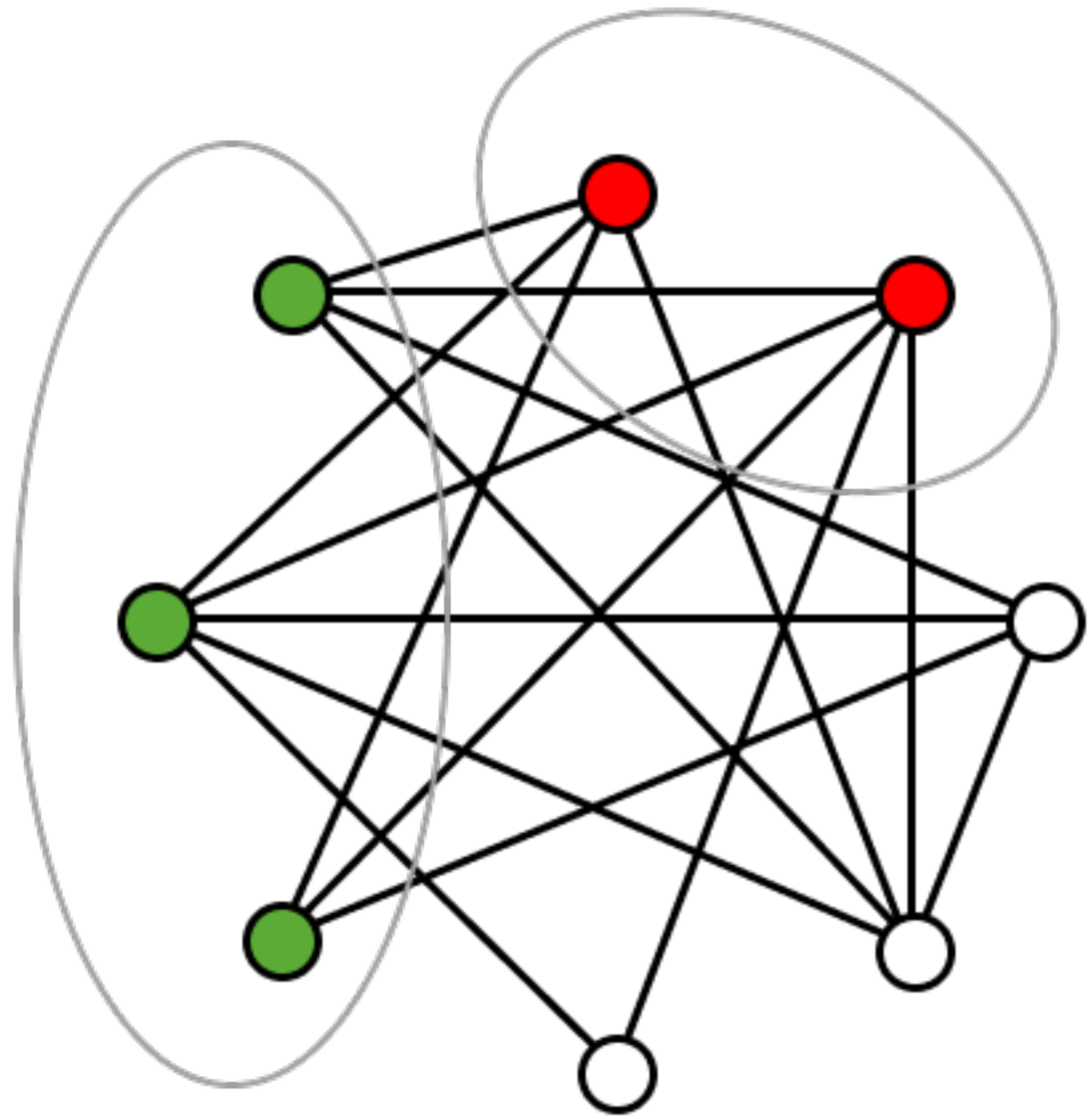

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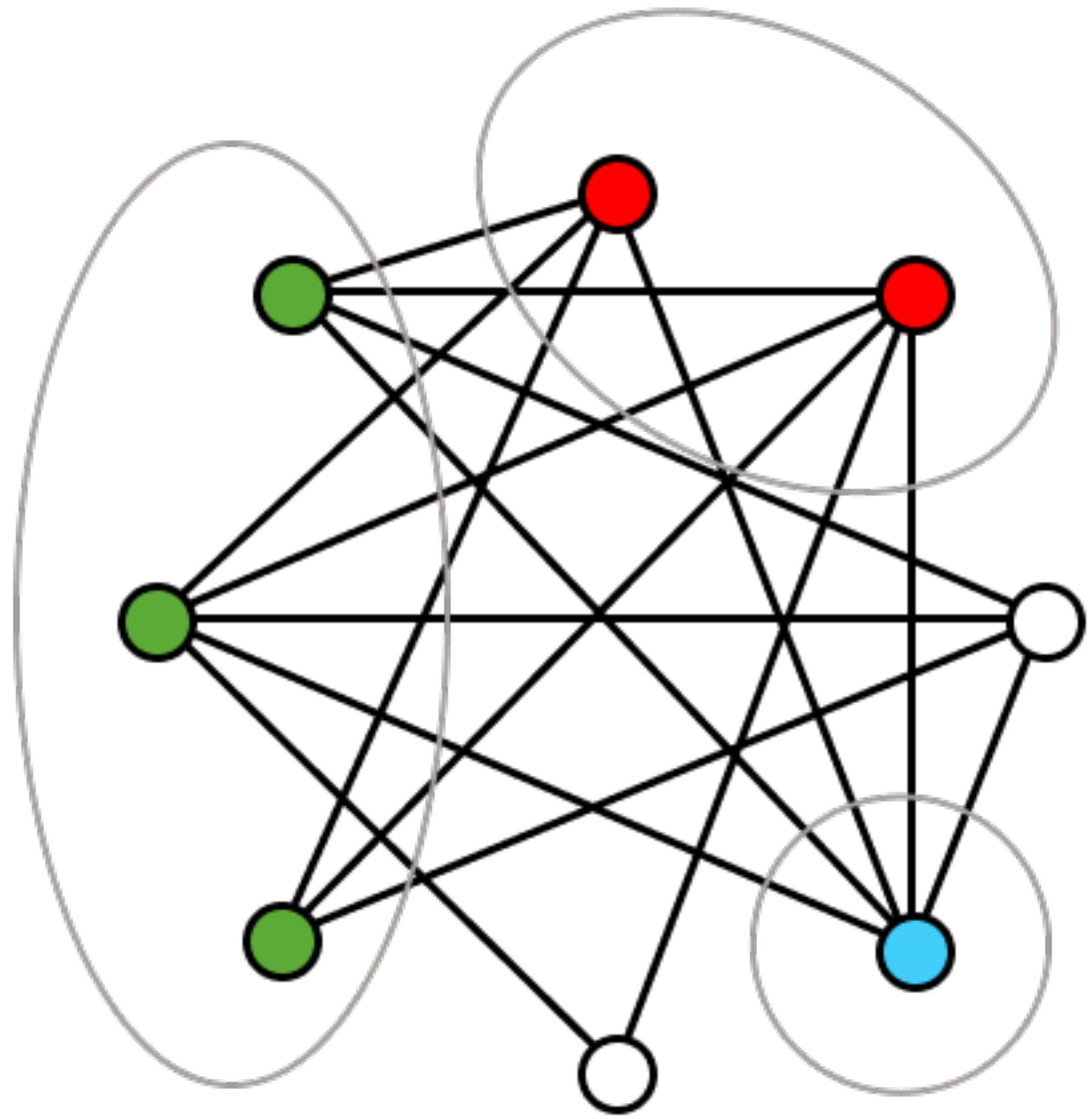

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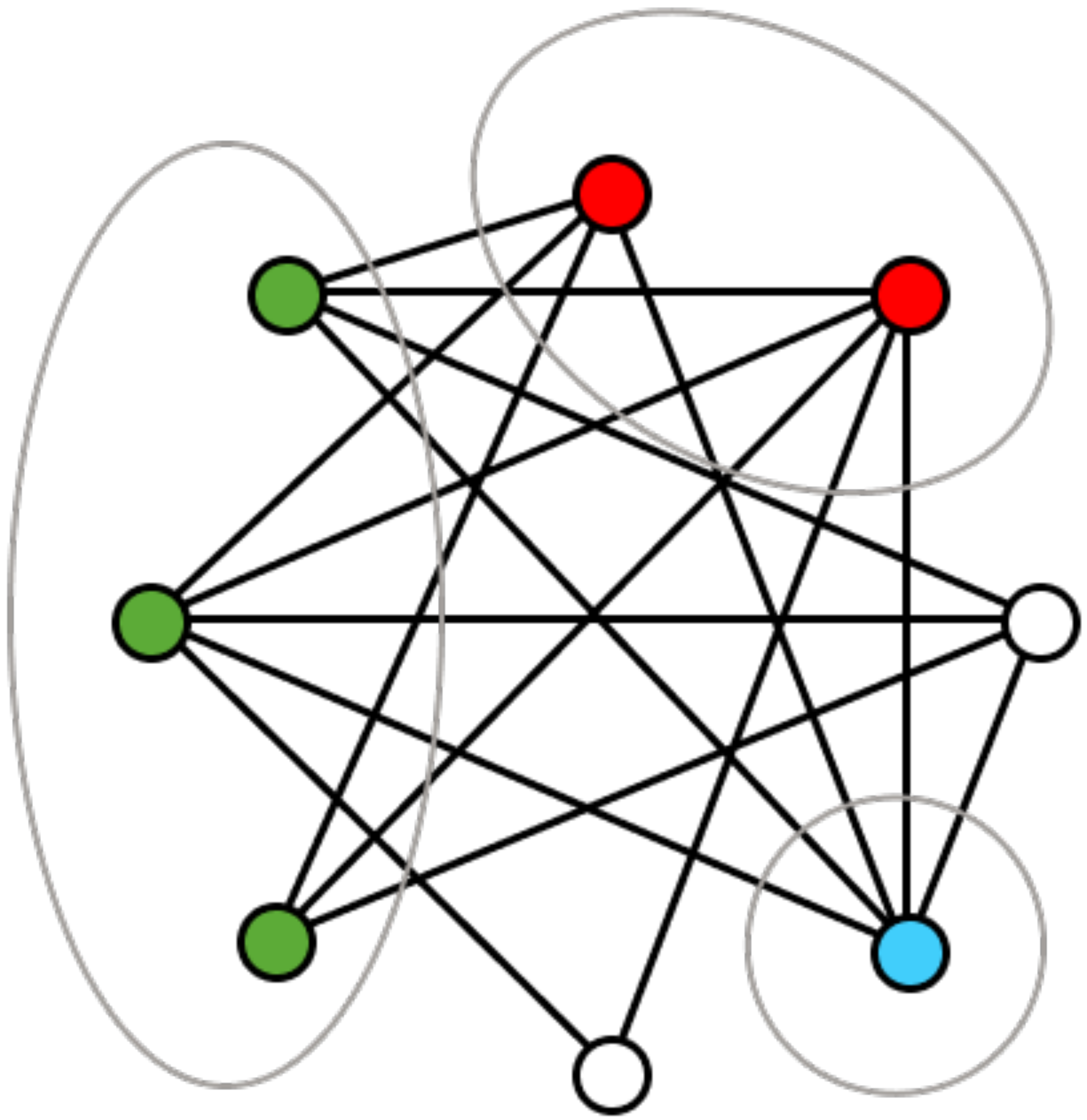

Informal Idea: For any graph satisfying some “nice” conditions, all independent sets in the graph can be covered by a small number of containers (each container is a subset of vertices).

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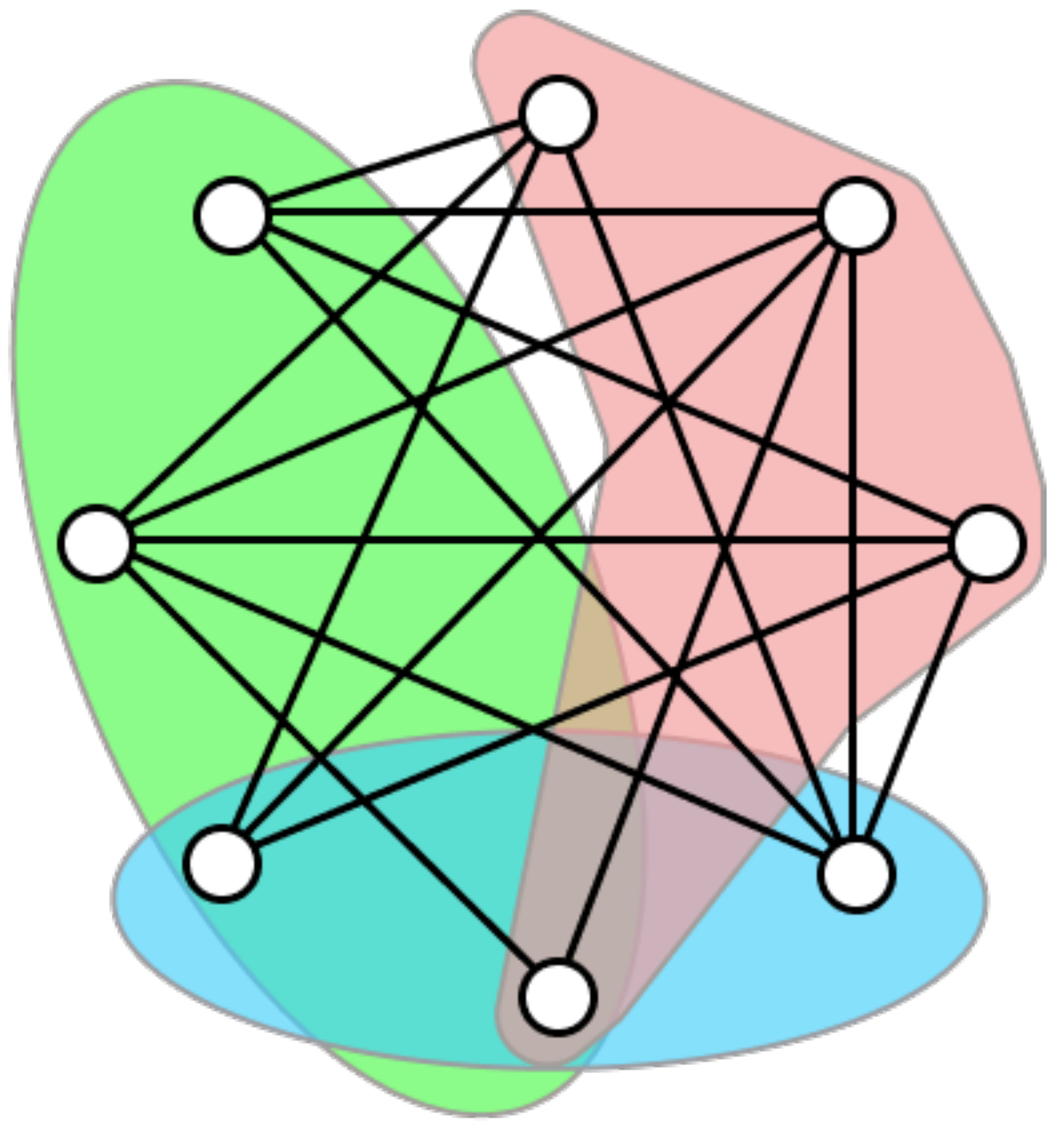

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An Initial Graph Container Lemma

Lemma: For any ϵ let $G = (V, E)$ be a graph with at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$,
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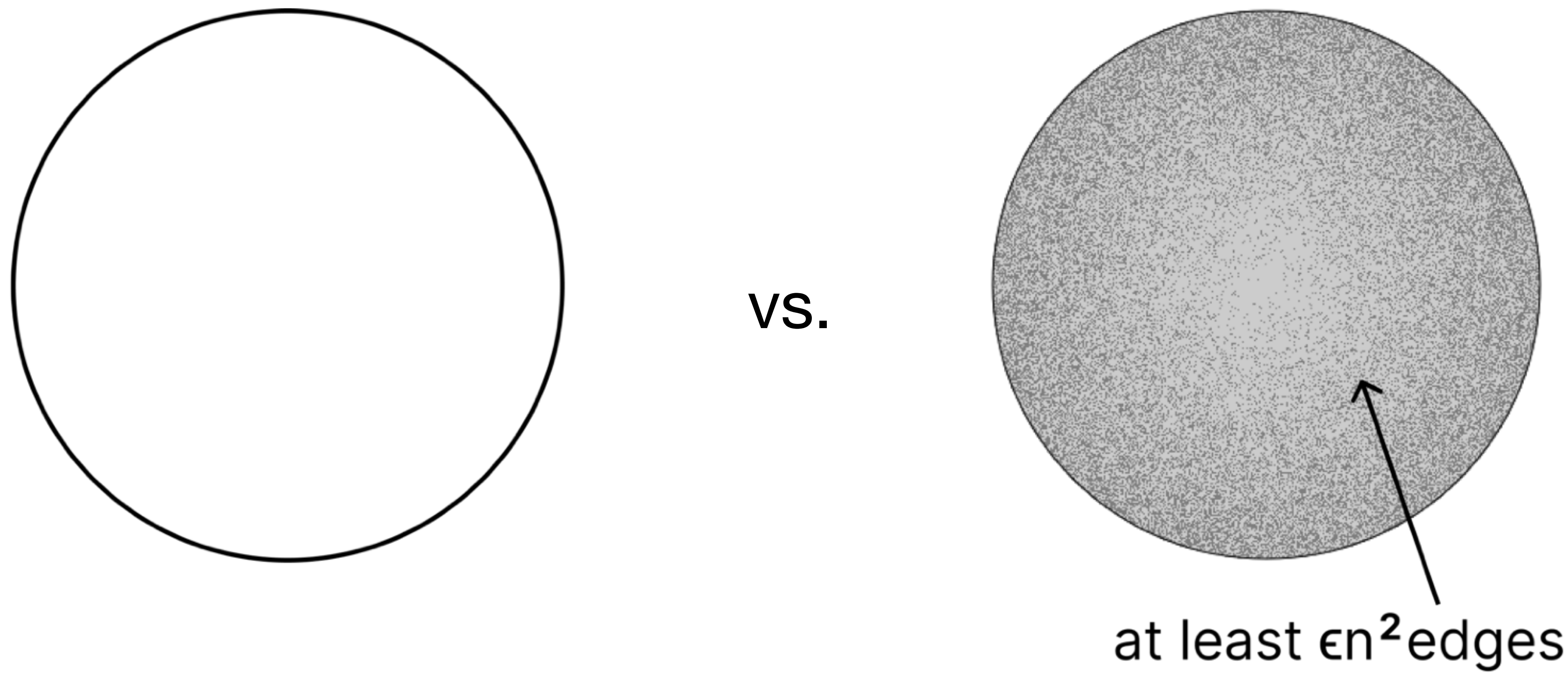
Note: for survey of combinatorial applications see "Counting Indepedent Sets in Graphs" by Samotij or "The method of hypergraph containers" by Balogh, Morris, and Samotij.

How to use Container Lemmas for Property Testing

Example: Testing Emptiness with a Container Lemma

Problem: Distinguish empty vs at least ϵn^2 edges



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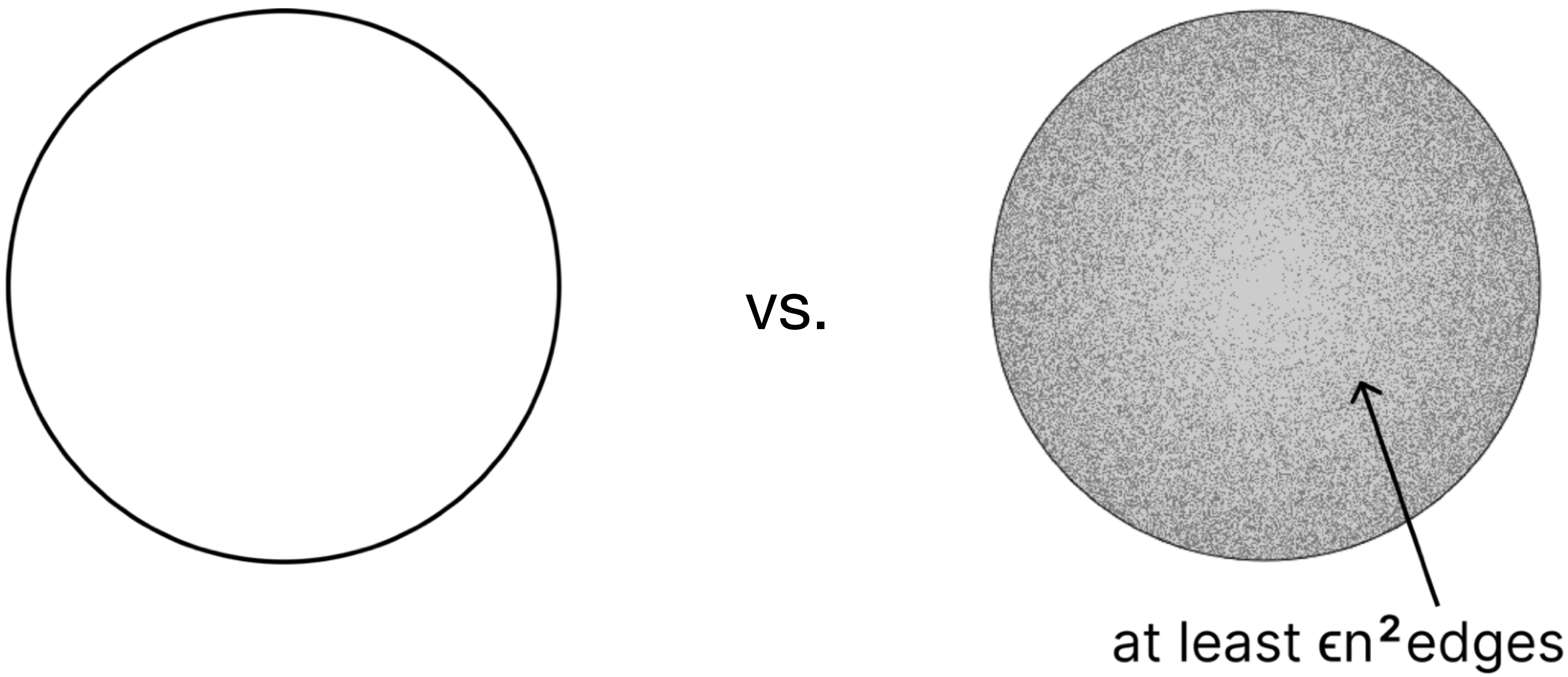

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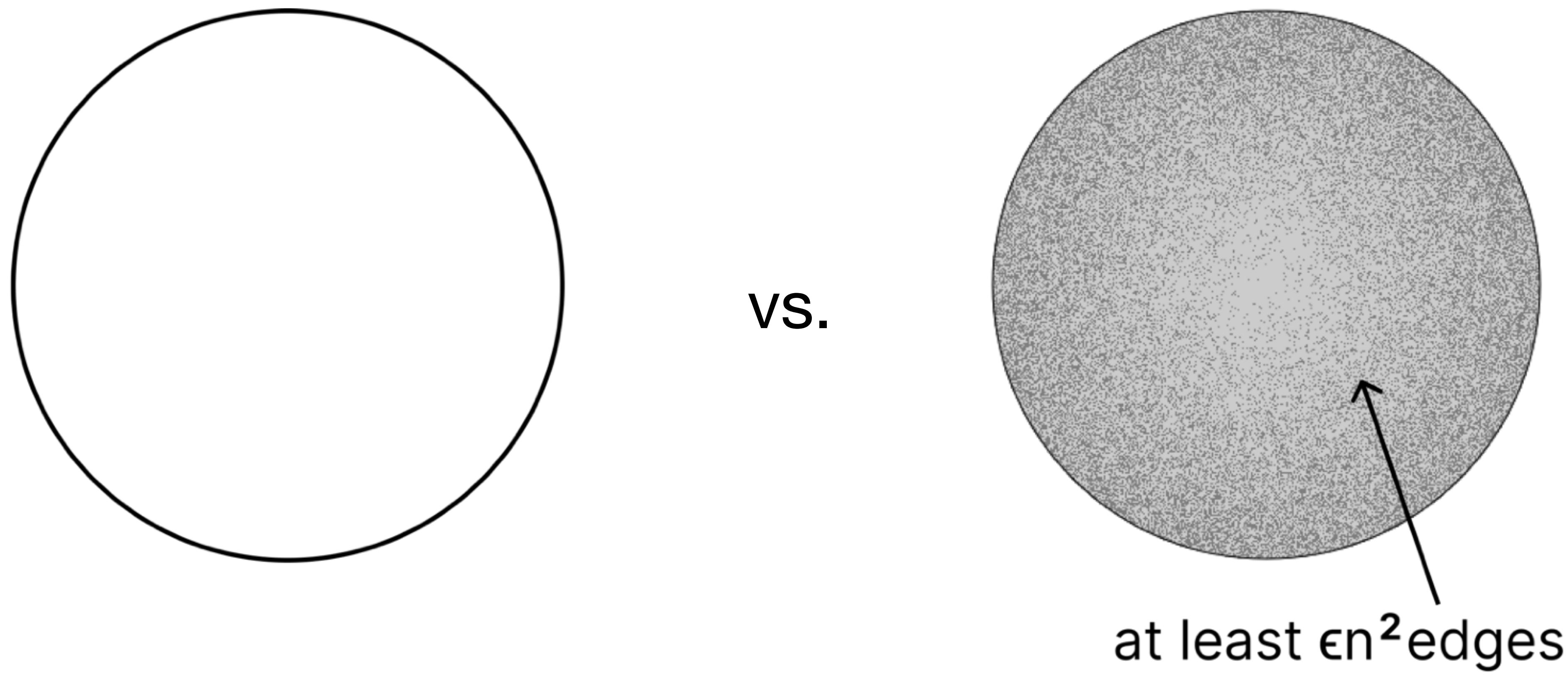

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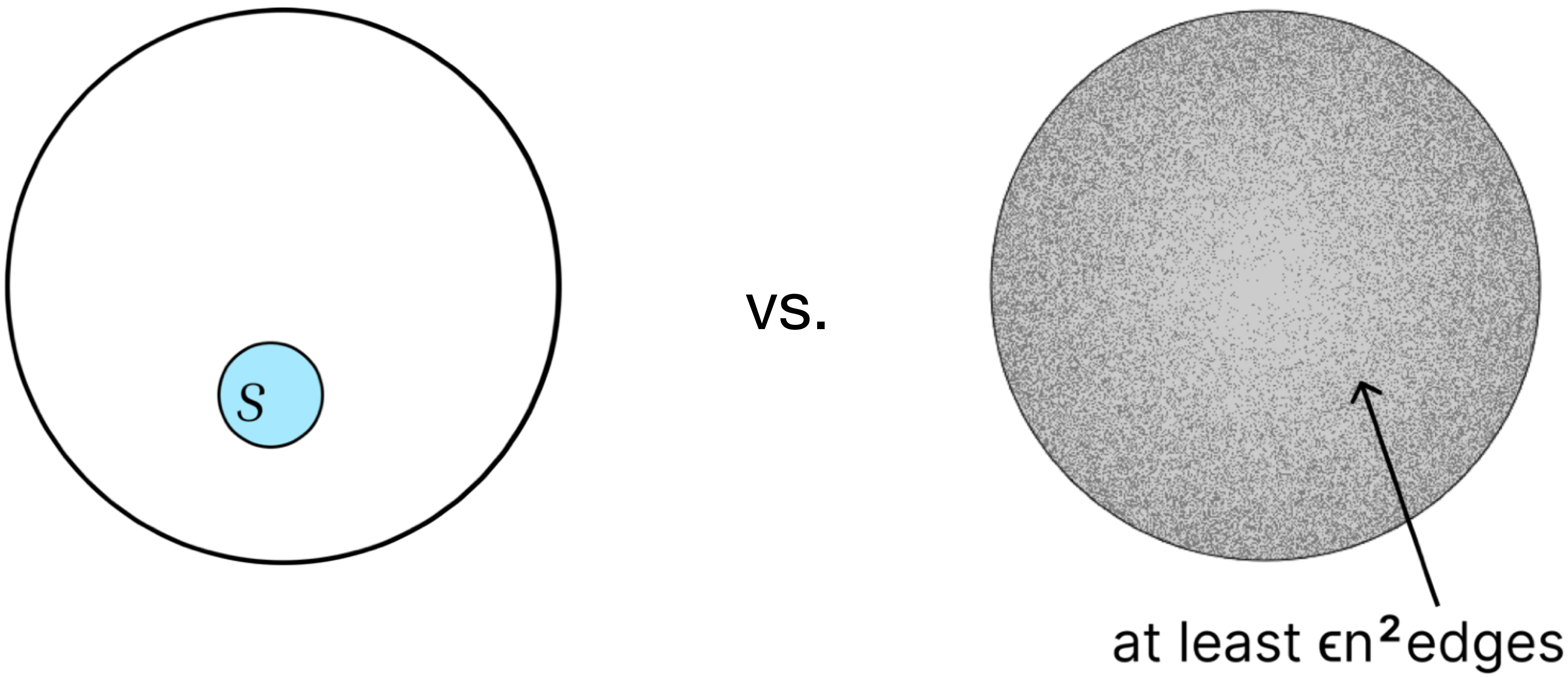

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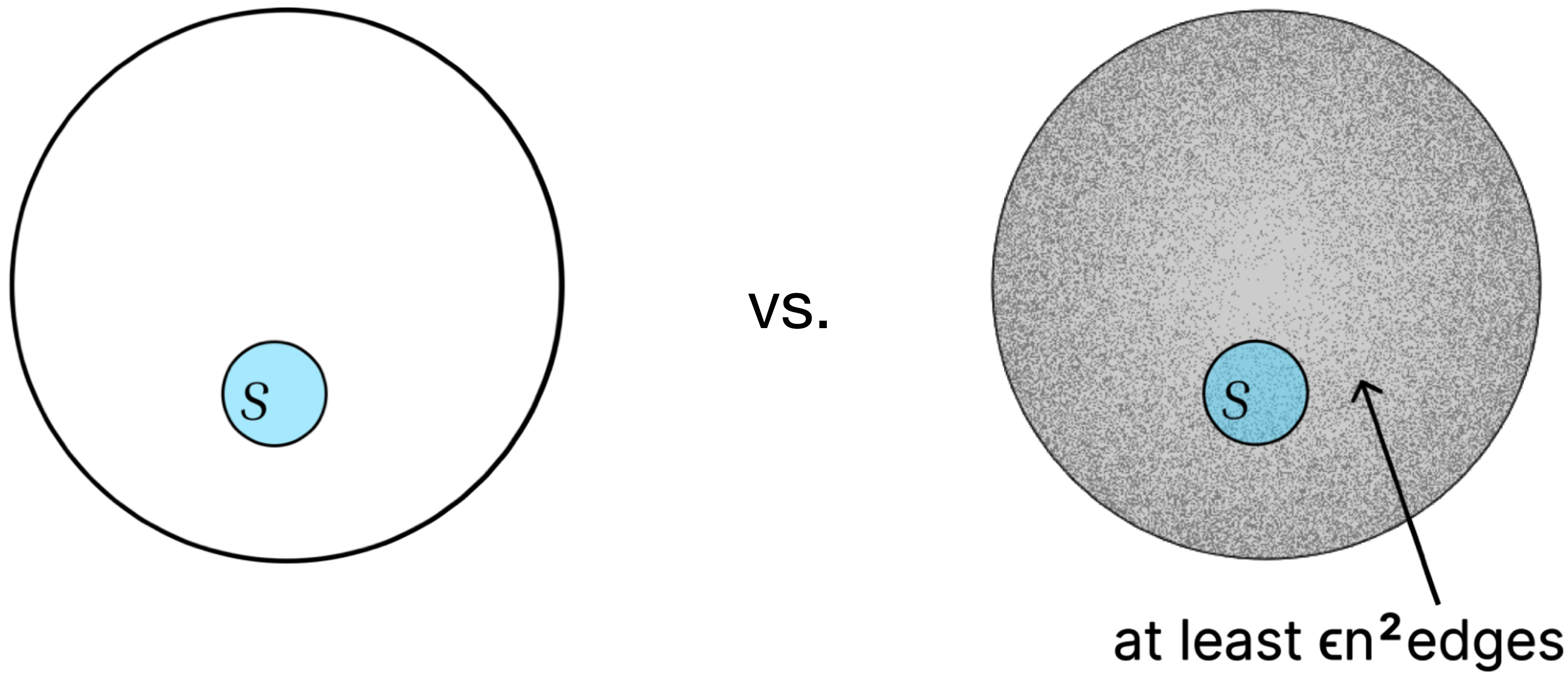

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&\leq \exp\left(\frac{1}{\epsilon}\log(n) \;-\; \frac{\epsilon\,|S|}{2}\right) \\
&\leq 1/3, \text{ as long as } s \gtrsim \frac{\log n}{\epsilon^2}.
\end{aligned}$$

How to prove a Container Lemma - An Encoding Argument

Encoding Independent Sets

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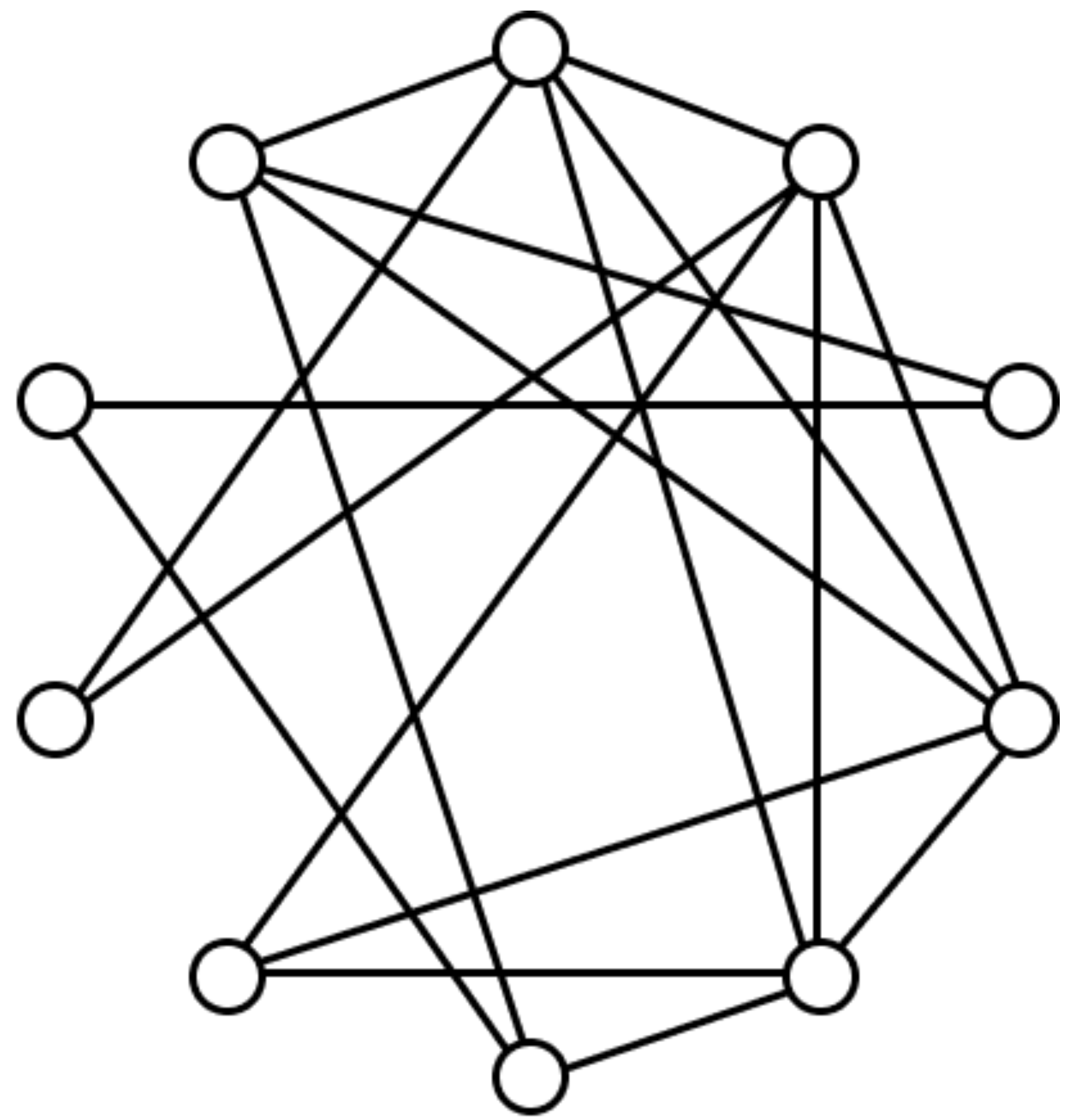

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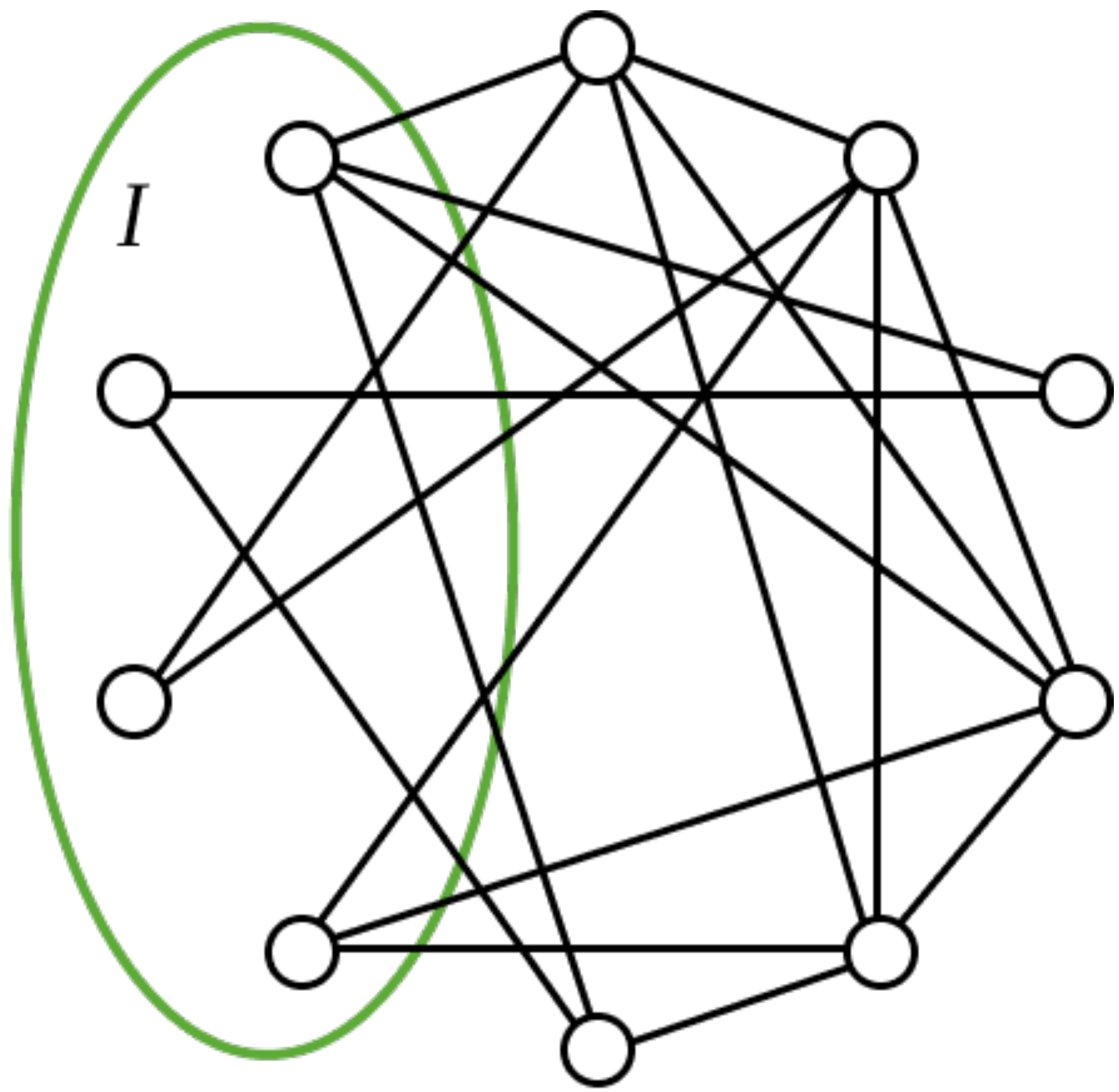

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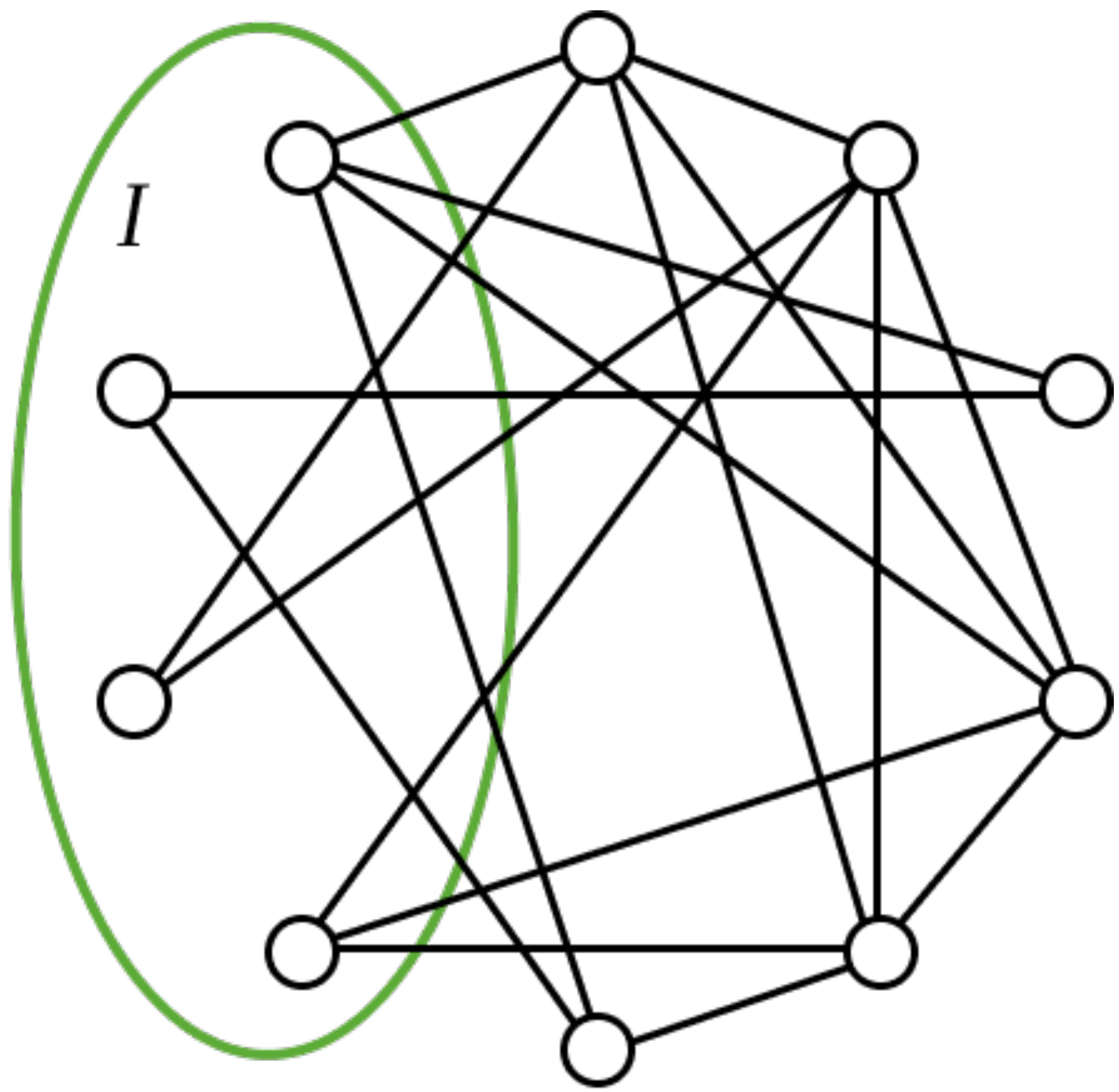

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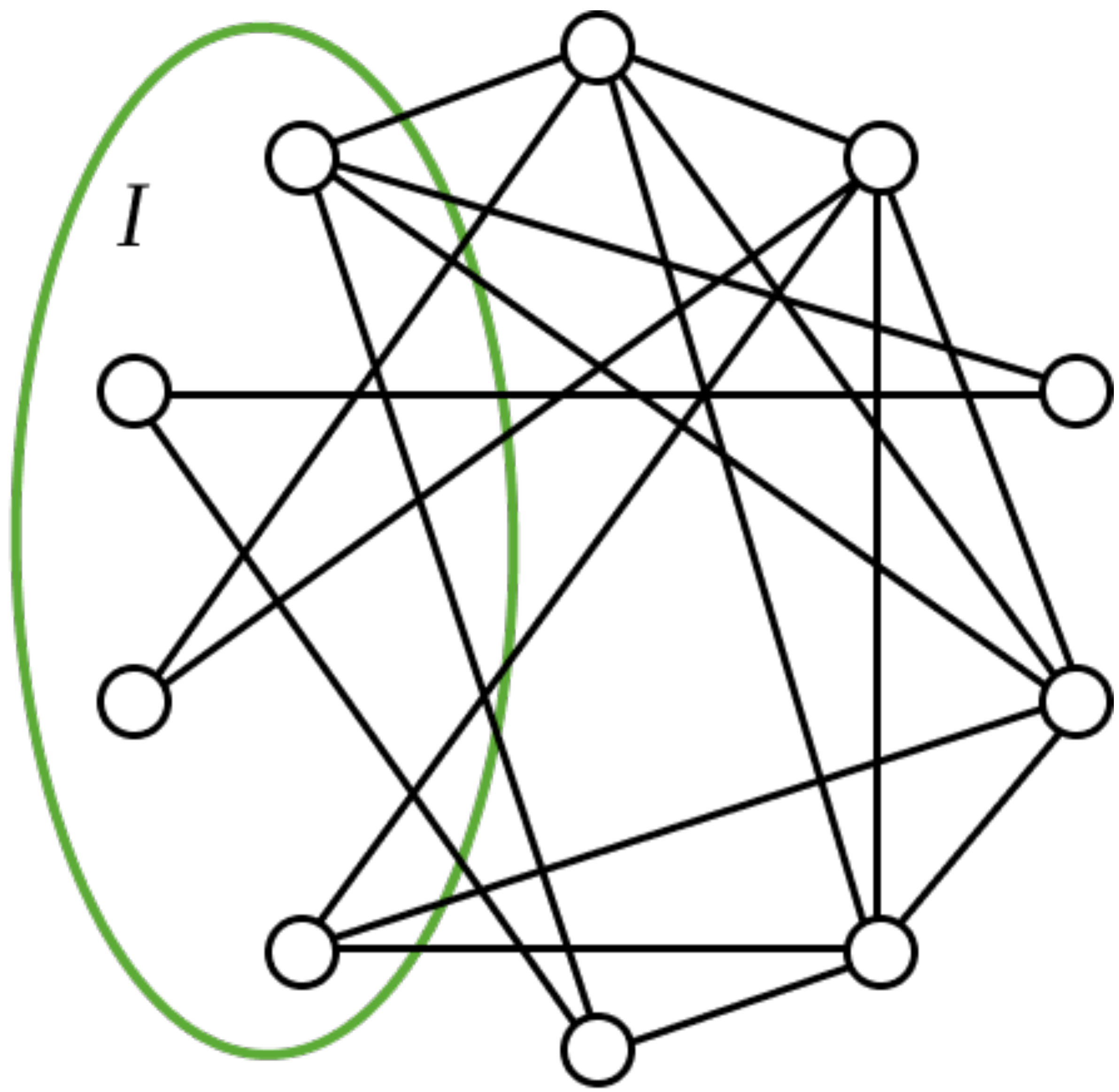

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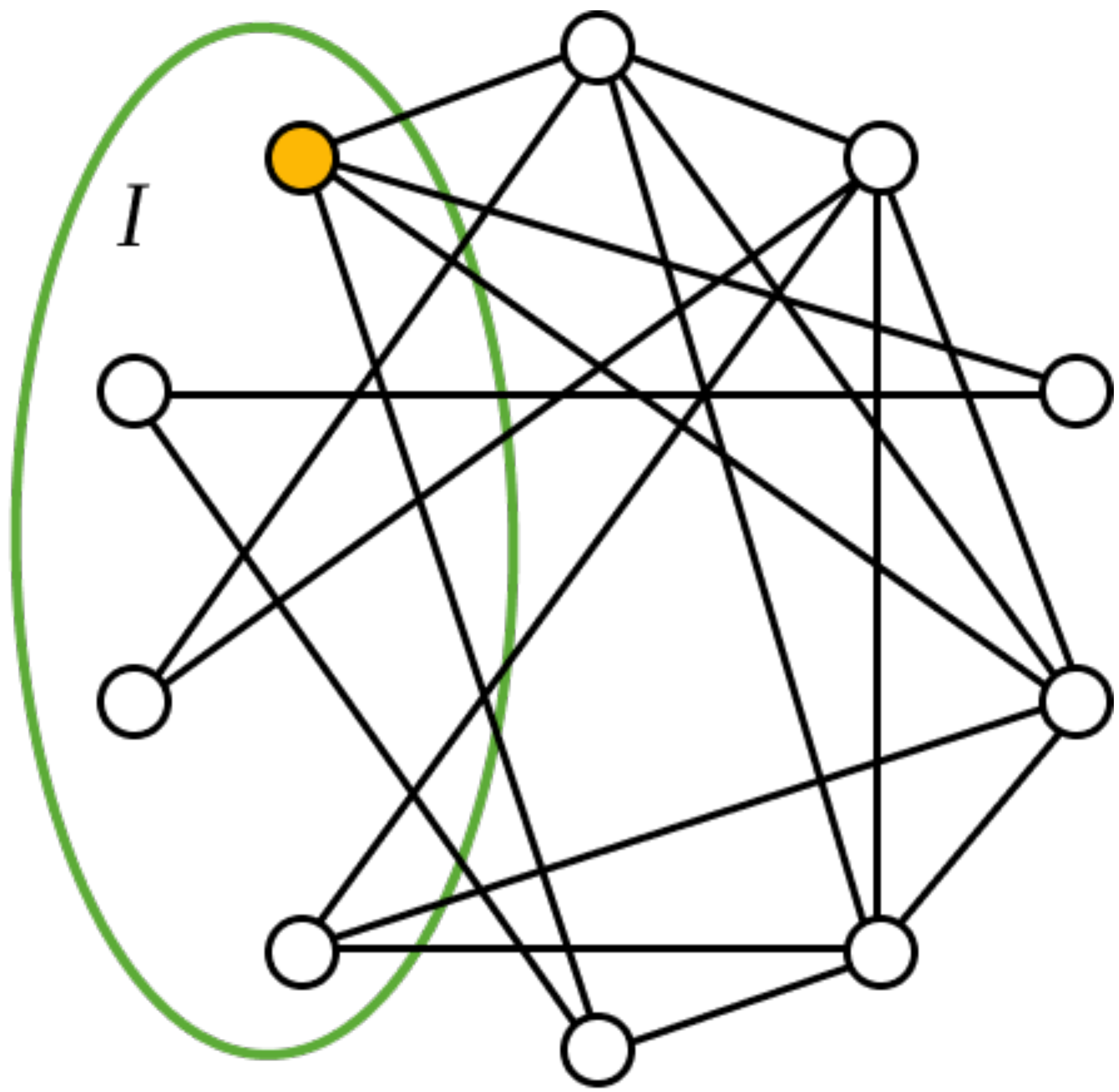

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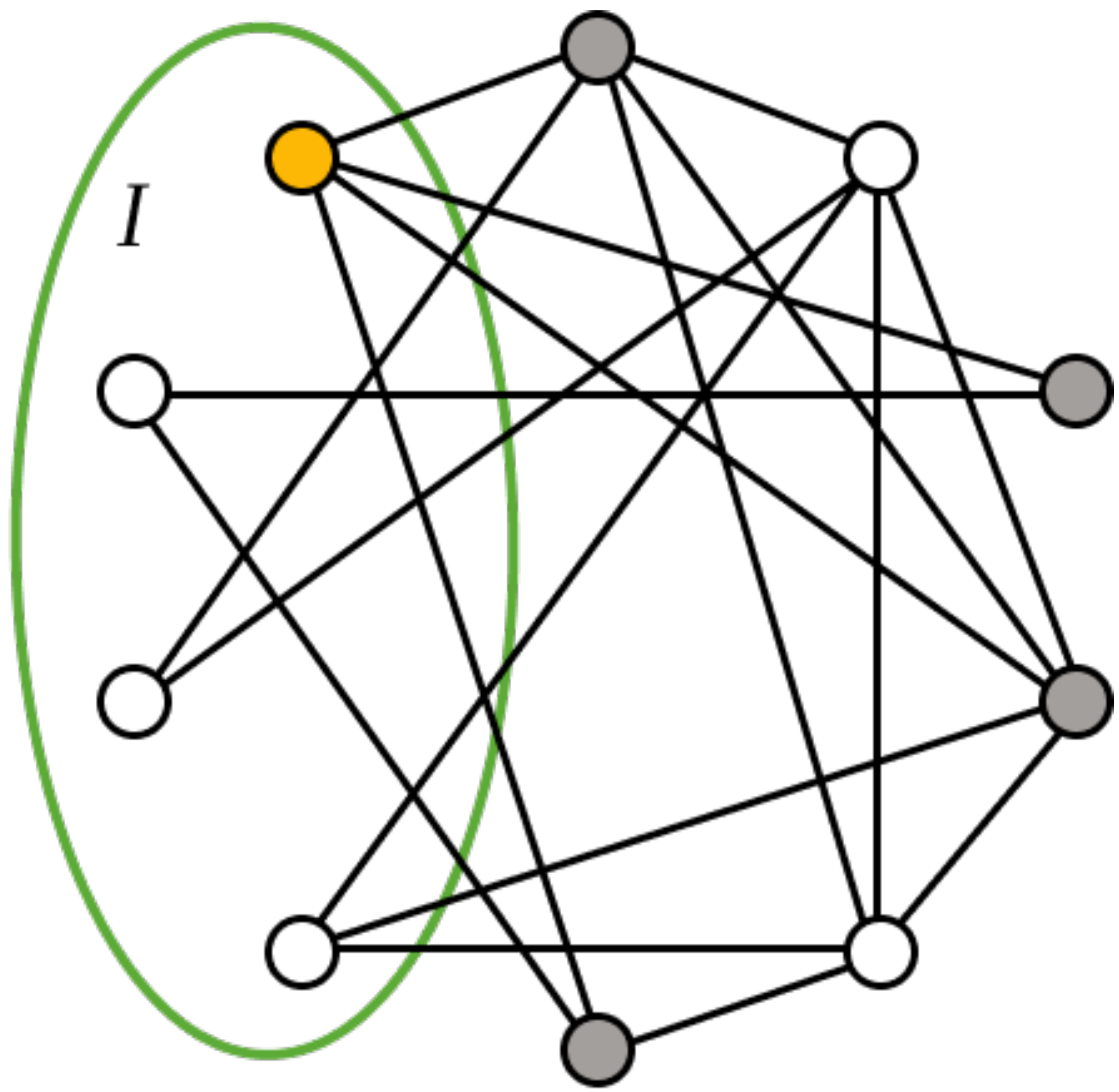

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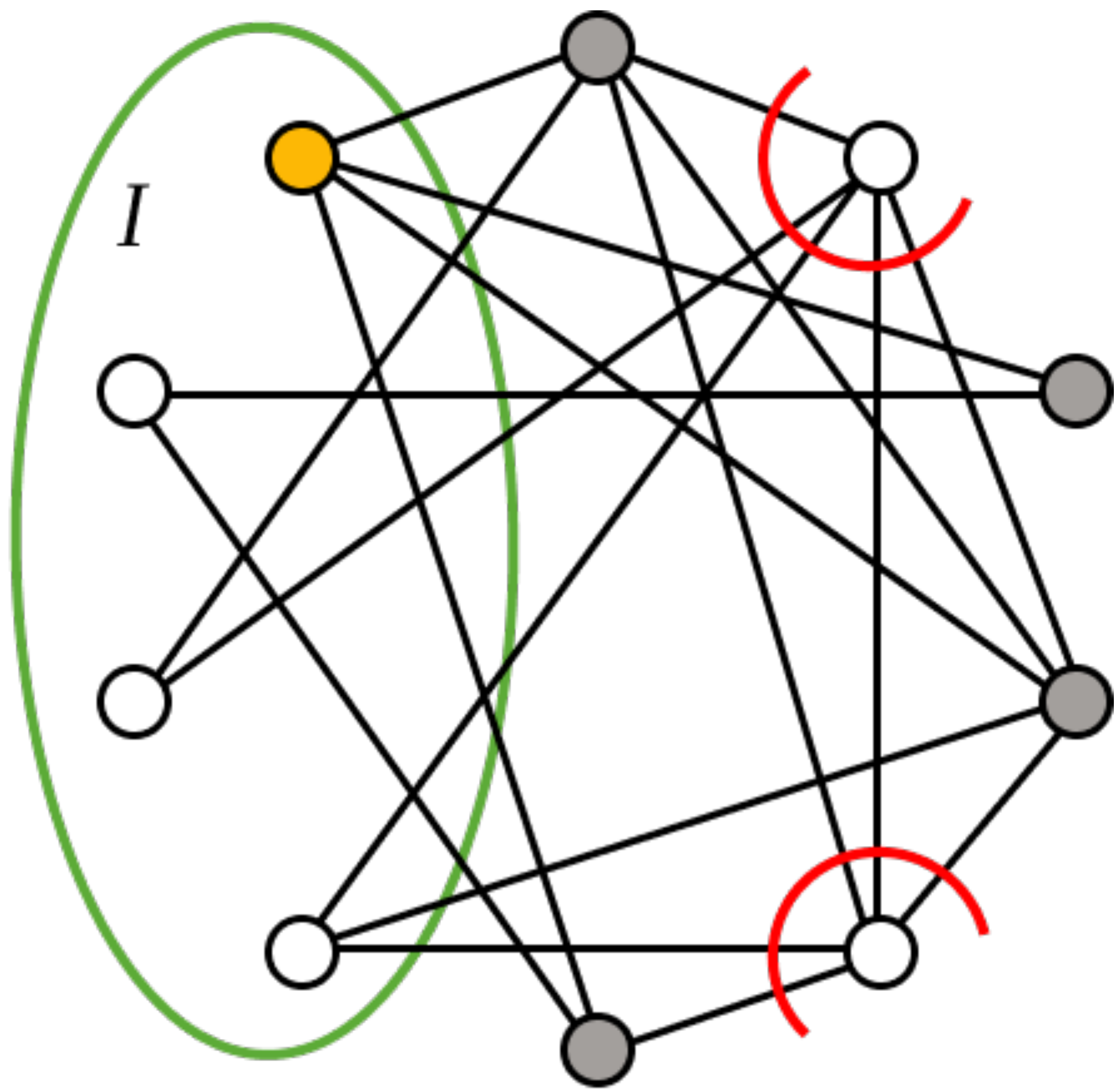

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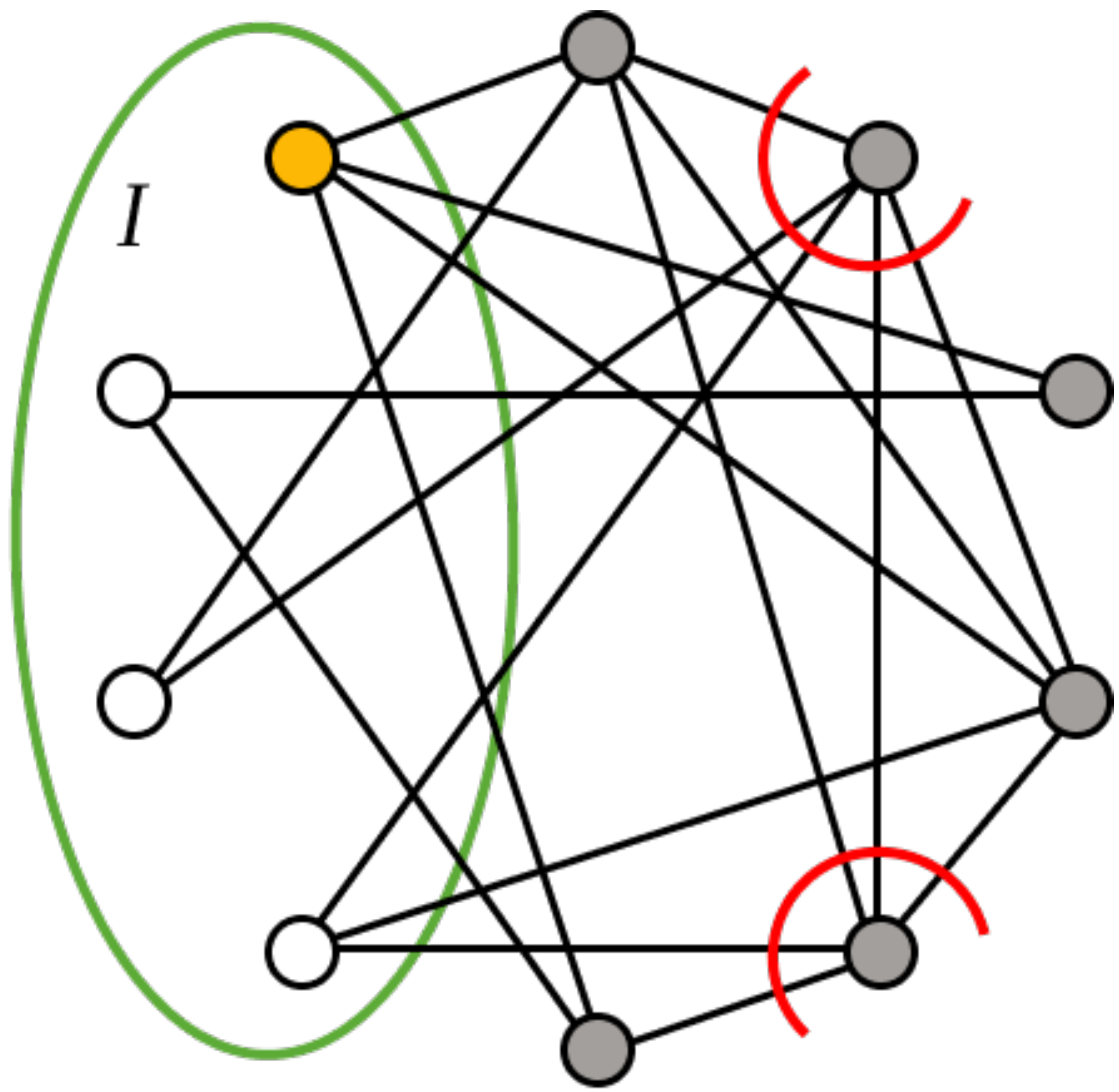

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Container Generator Algorithm

Input: Graph G and an independent set I

- Initialize fingerprint $F = \varnothing$ and container $C = V$
- Repeatedly select a vertex $v \in I$ with highest degree in $G[C]$ and add it to F. Remove $N(v)$ and all vertices with higher degree than v from C.

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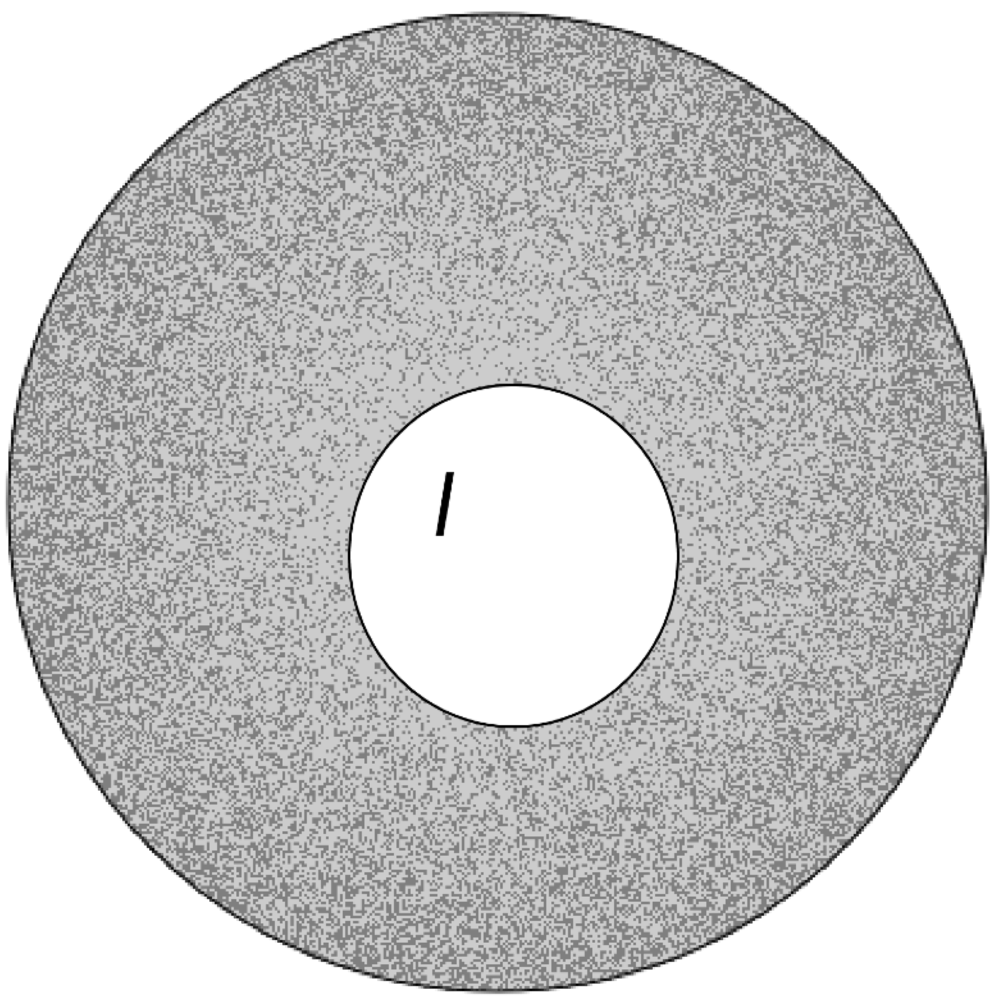

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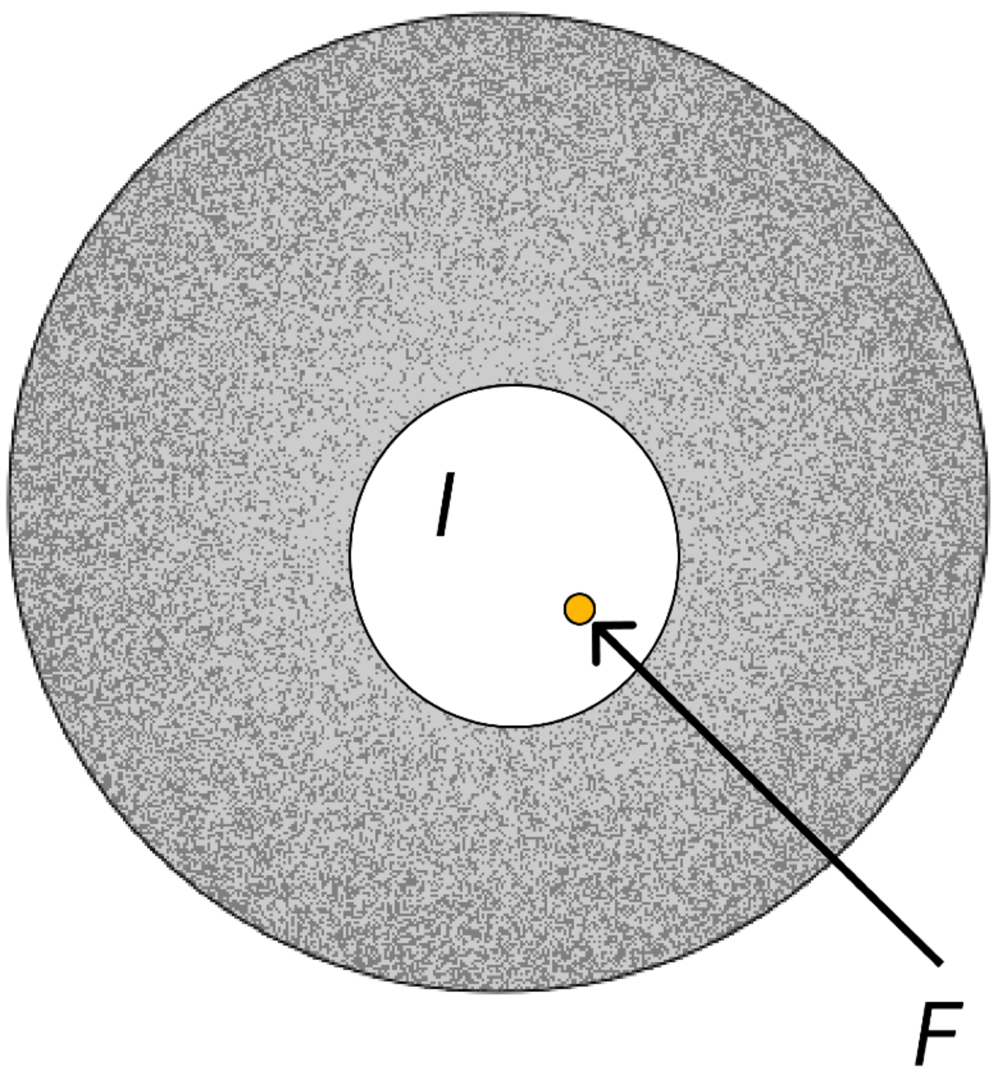

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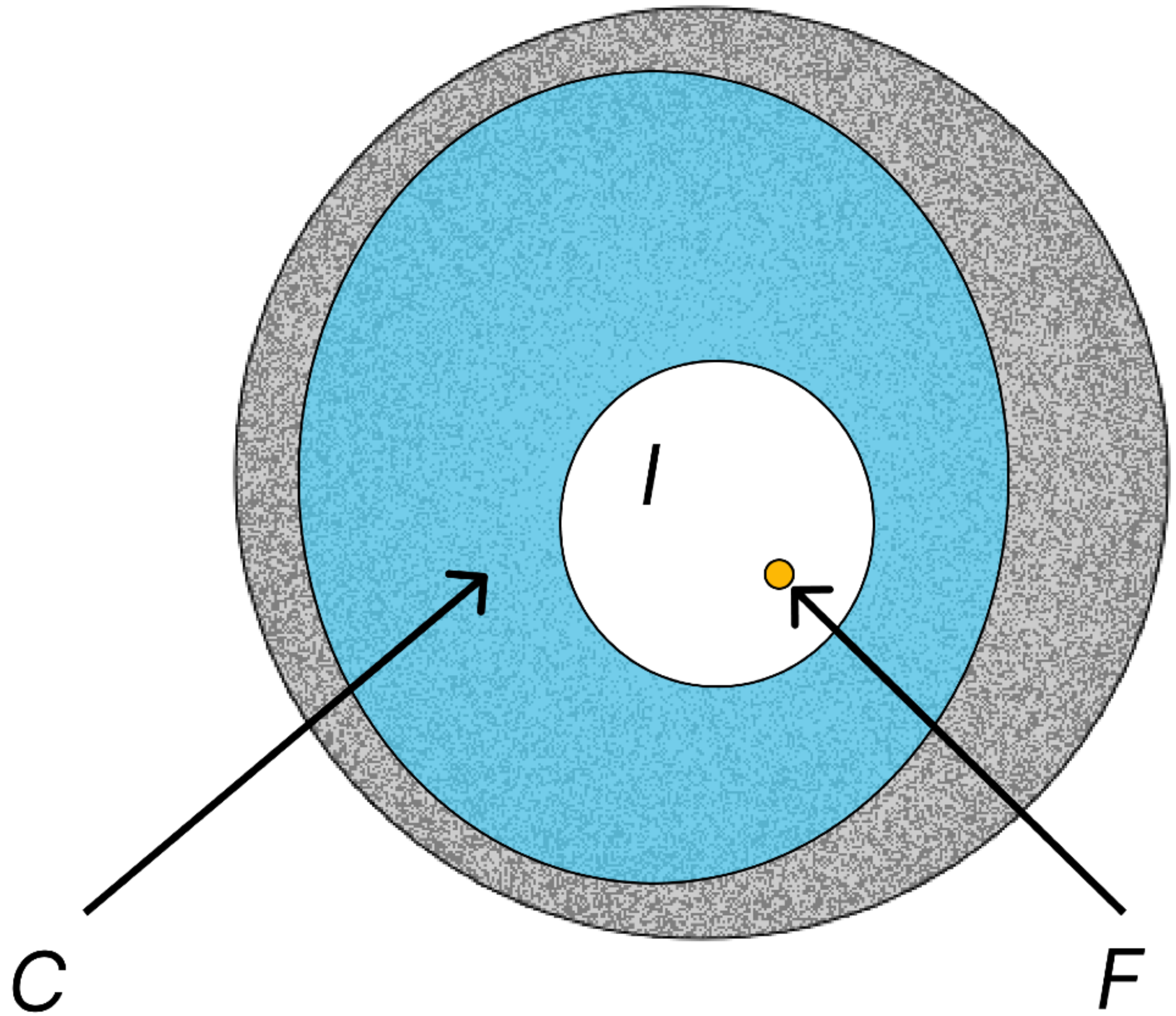

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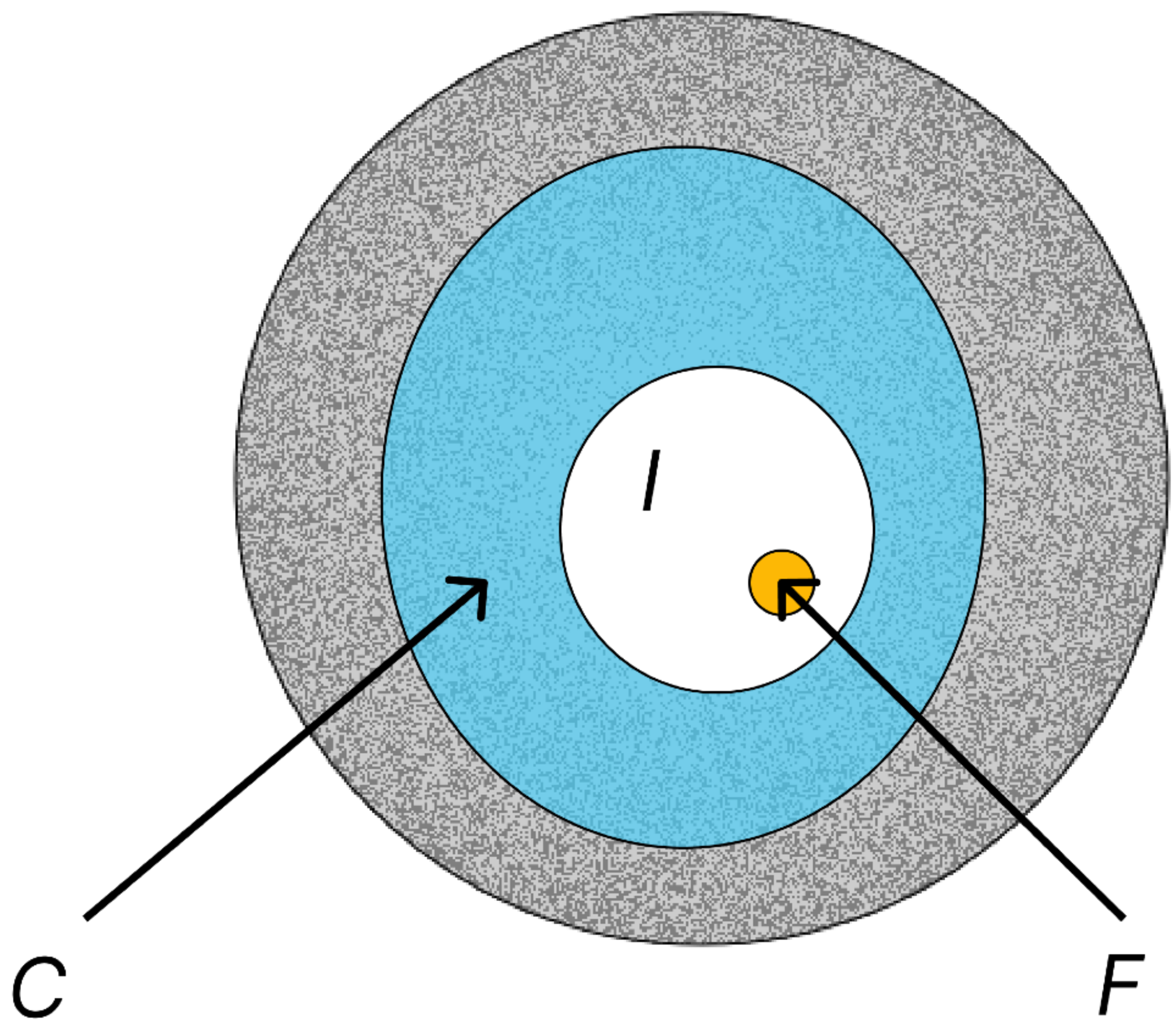

2nd Iteration

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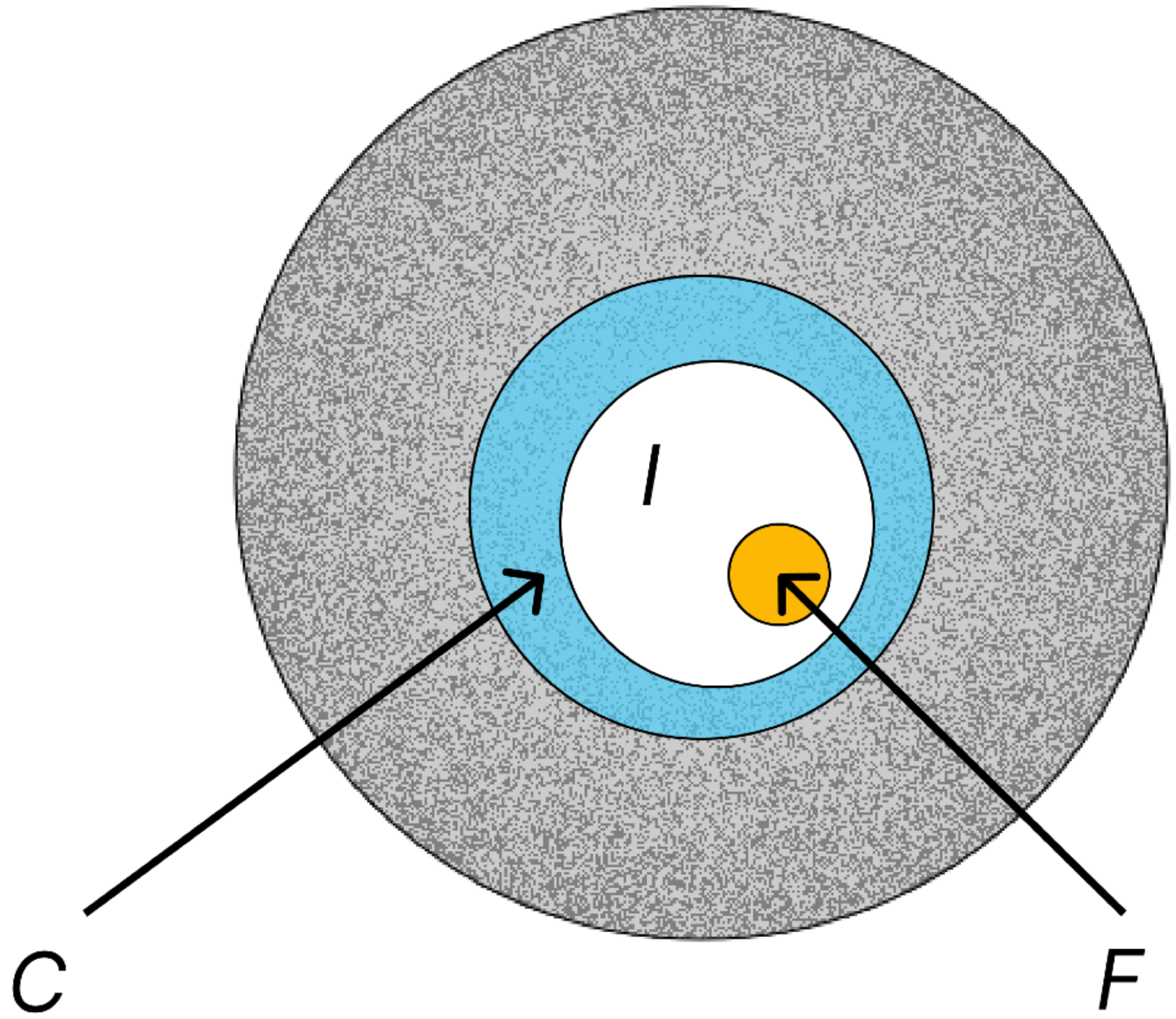

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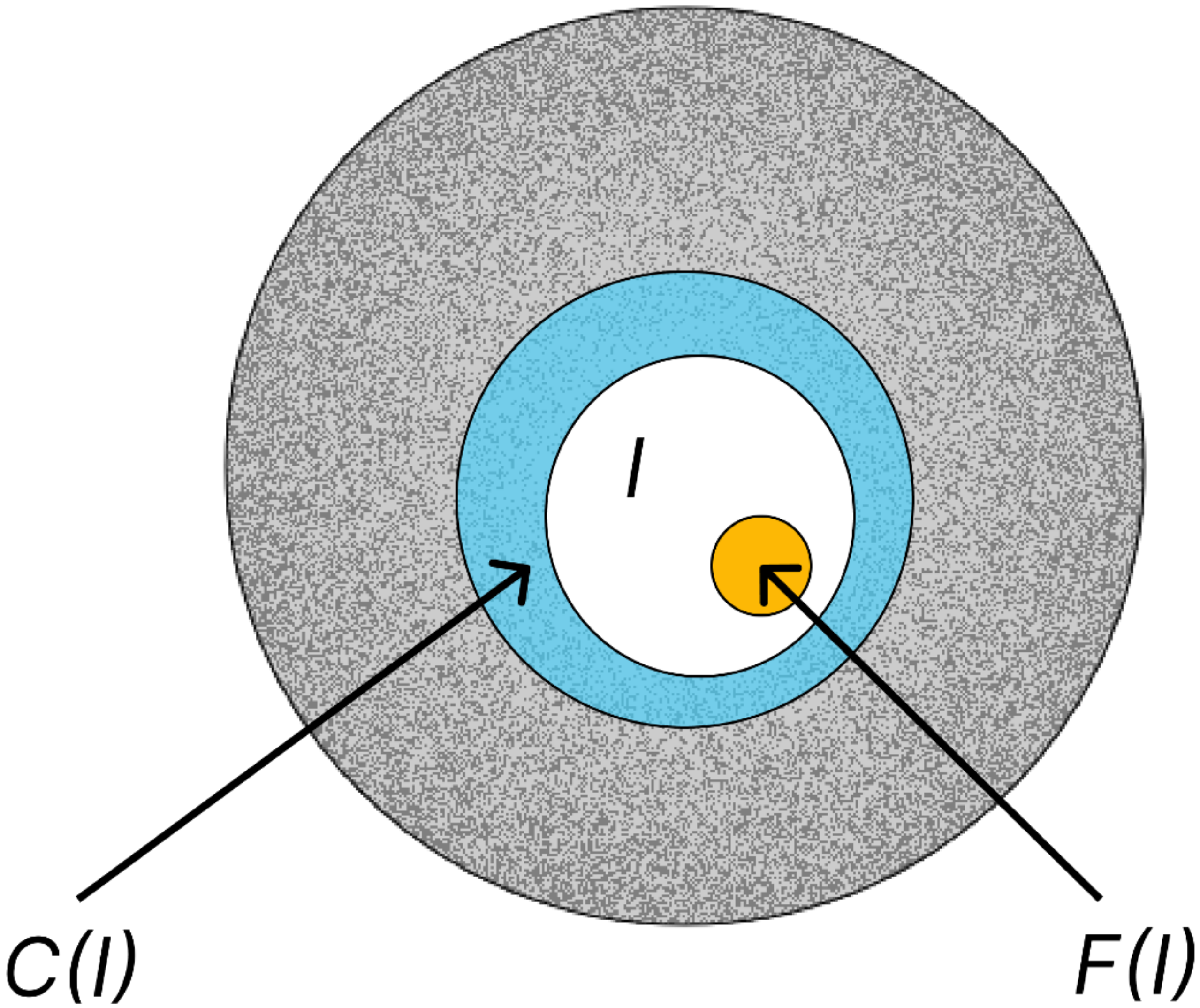

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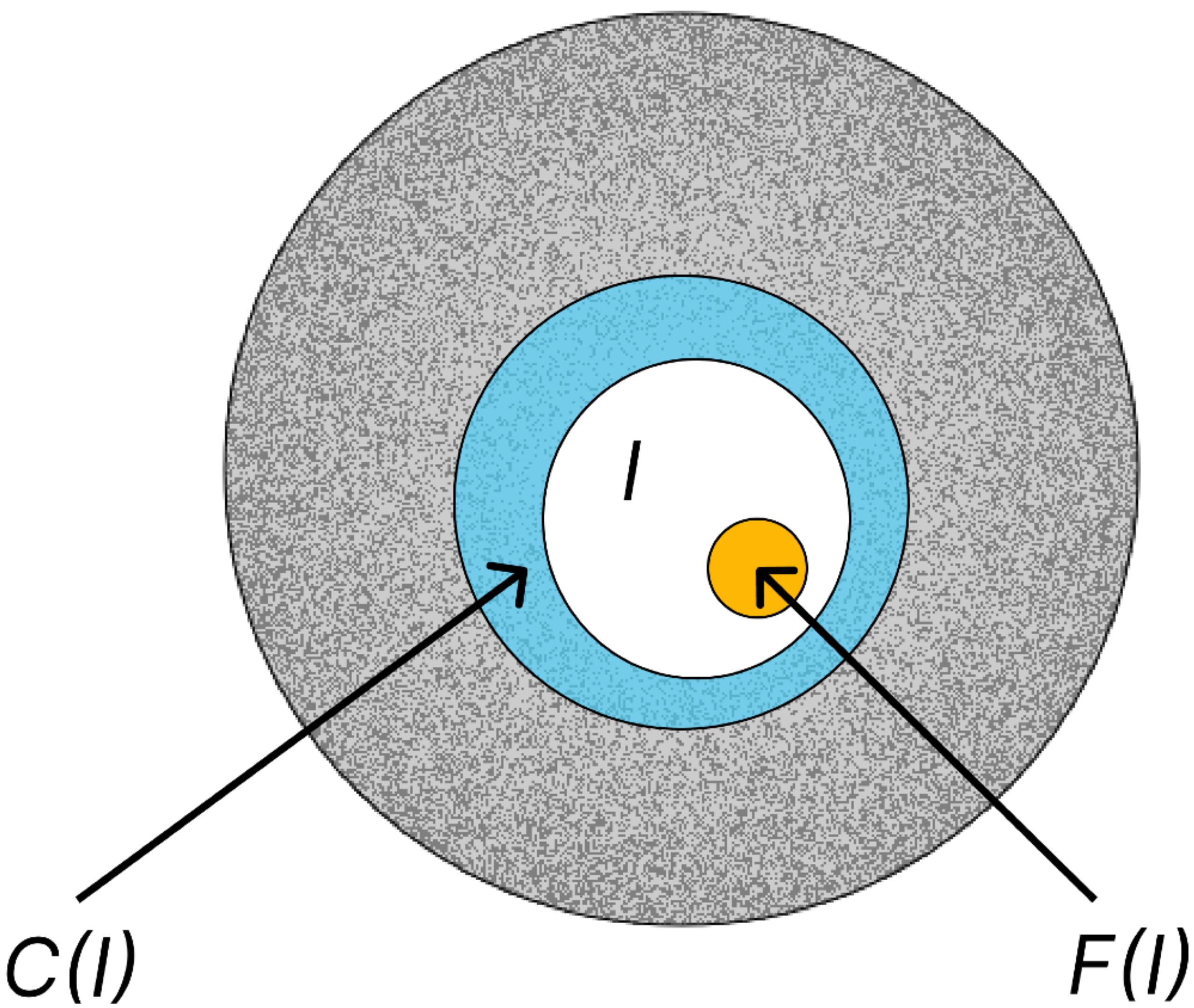

Final Iteration

Key Observations:

- $I \subseteq C(I)$
- $C(I) = C(F(I))$

Proving the Simple Graph Container Lemma

Lemma: For any ϵ let $G = (V, E)$ be a graph with at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

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Proof: Run container procedure on every independent for $4/\epsilon$ steps

Let $\mathscr{C} = \{C(I) : I \text{ is independent set in } G\}$

$= \{C(F) : F = F(I) \text{ for some independent set } I \text{ in } G\}$

Proving the Simple Graph Container Lemma

Lemma: For any ϵ let $G = (V, E)$ be a graph with at least ϵn^2 edges. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$,
2. $|\mathscr{C}| \leq \binom{n}{4/\epsilon}$,
3. for every $C \in \mathscr{C}$, $|C| \leq (1 - \epsilon/2)n$.

[Kleitman, Winston '82] - simple case

Proof: Run container procedure on every independent for $4/\epsilon$ steps

Let $\mathscr{C} = \{C(I) : I \text{ is independent set in } G\}$

$= \{C(F) : F = F(I) \text{ for some independent set } I \text{ in } G\}$

Observe that in each step, if there are more than $\epsilon n^2/2$ edges then at least $\epsilon n/4$ vertices are removed. Hence after $4/\epsilon$ steps the final container has less than $\epsilon n^2/2$ edges. And so $|C| \leq (1 - \epsilon/2)n$ $\square$

Restating our Theorem

Restating Informal Version of Theorem

Theorem: Let Π be a hereditary graph property. Then

Π is ϵ-testable with one sided error and sample complexity s

$$\Updownarrow$$

Π has an "ϵ-container lemma" parameterized by $\mathrm{poly}(s)$

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Defining ϵ-Container Lemma: An Example

Example: An ϵ-container lemma for the $\Pi =$ emptiness property

Lemma: For any ϵ let $G = (V, E)$ be ϵ-far from empty. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

1. for every independent set I, there exists $C \in \mathscr{C}$ with $I \subseteq C$,
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Defining ϵ-Container Lemma Informally

Definition (informal): An ϵ-container lemma for Π is the following:

Lemma: For any ϵ let $G = (V, E)$ be ϵ-far from Π. Then, there exists a set $\mathscr{C} \subseteq 2^V$ of containers that satisfies:

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The containers are covering Π-subgraphs instead of independent sets!

Defining ϵ-Container Lemma

Definition: Let Π be a graph property. We say Π has an

"ϵ-container lemma with fingerprint size f and container size $(1-\alpha)$"

when the following holds. For any $G = (V, E)$ that is ϵ-far from Π there exists a set $\mathcal{F} \subseteq 2^V$ of fingerprints and container function $C : \mathcal{F} \to 2^V$ that satisfies:

1. For every $S \subseteq V$ for which $G[S] \in \Pi$, there exists $F \in \mathcal{F}$ such that $F \subseteq S \subseteq C(F)$,

2. for every $F \in \mathcal{F}$, $|F| \leq f$, $|C(F)| \leq (1-\alpha)n$ and $G[C(F)]$ can be made to satisfy Π by modifying at most ϵn^2 edges.

Theorem Restated

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Theorem*: Let Π be a hereditary graph property. Then

Π is ϵ-testable with one sided error and sample complexity s

$$\Updownarrow$$

Π has an ϵ-container lemma with fingerprint size $\mathrm{poly}(s)$ and container size $\left(1 - \frac{1}{\mathrm{poly}(s)}\right)$

* Omitting some details

Proof Sketch of Theorem

Proof Sketch (testability => container lemma)

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Proof sketch with the specific example $\Pi =$ 3-colorability.

Recall that 3-colorability is testable with sample complexity $s \sim 1/\epsilon$.

[Sohler '12]

We want to prove an ϵ-container lemma like: if G is ϵ-far then there is a collection of containers that covers all the 3-colorable subgraphs.

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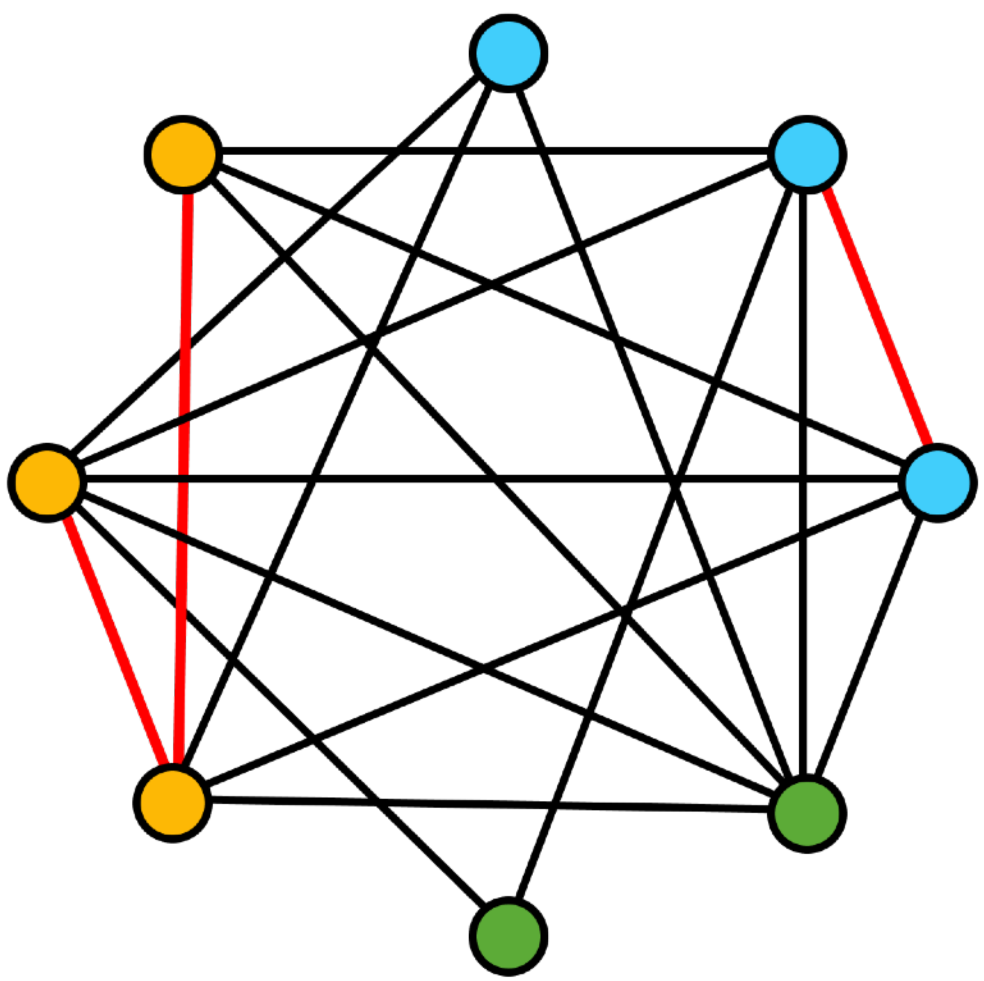

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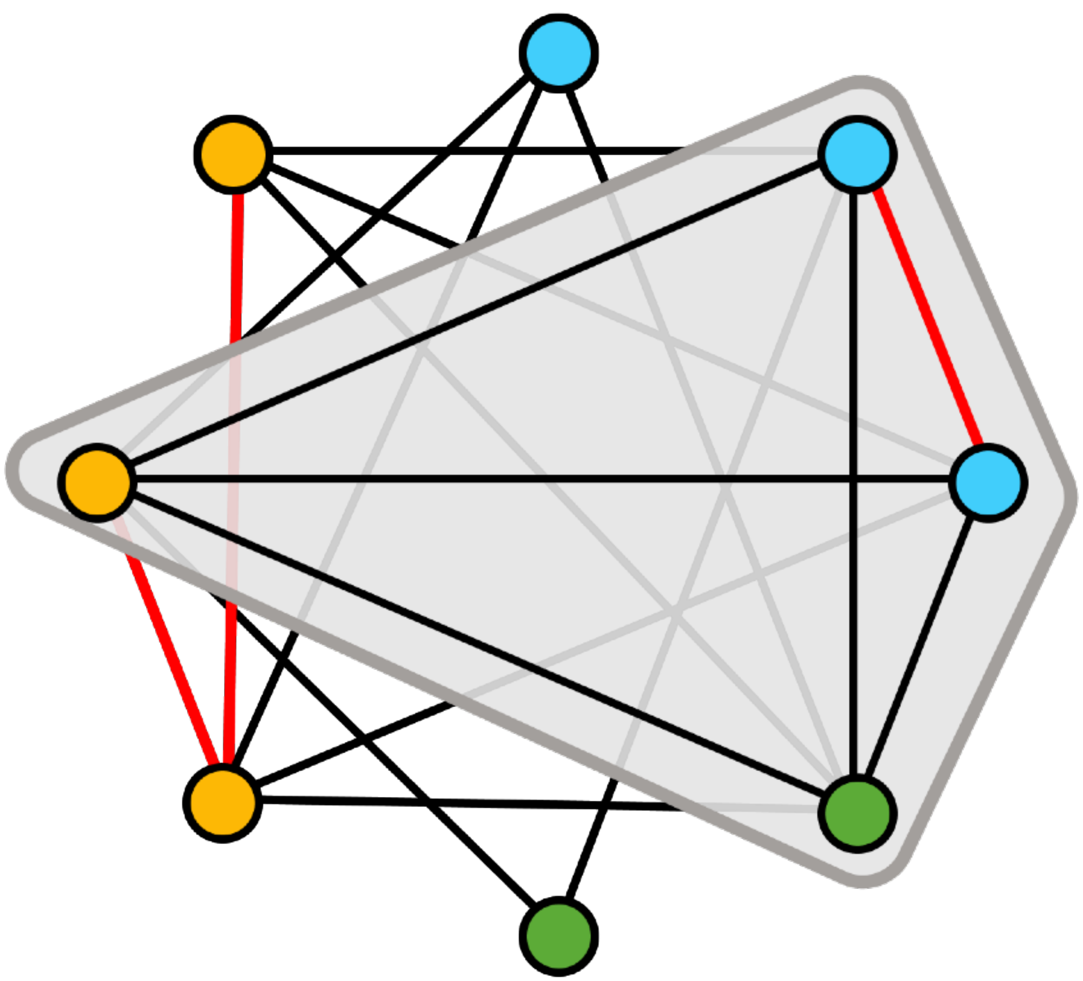

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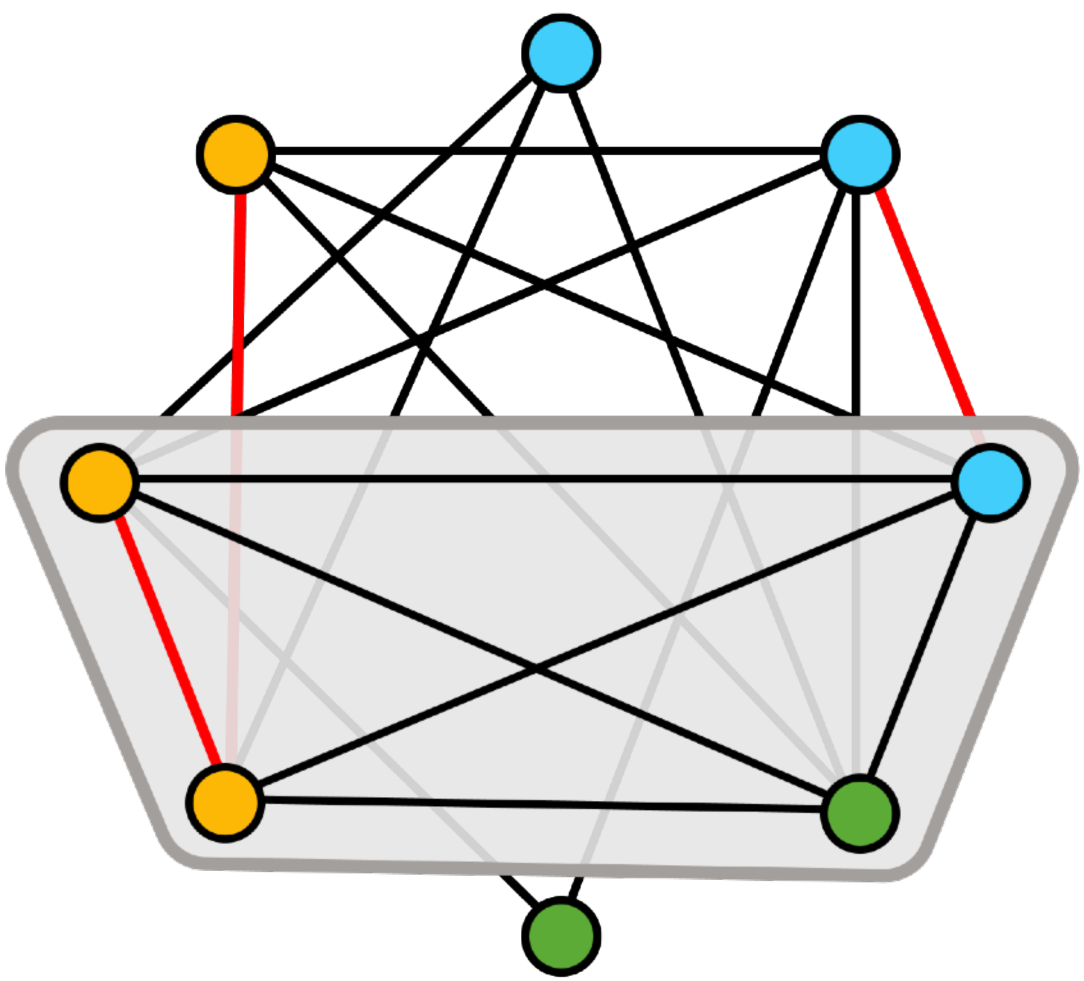

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Idea: construct s-uniform hypergraph H on vertex set V with edges corresponding to sets of s vertices that form non-3-colorable subgraphs

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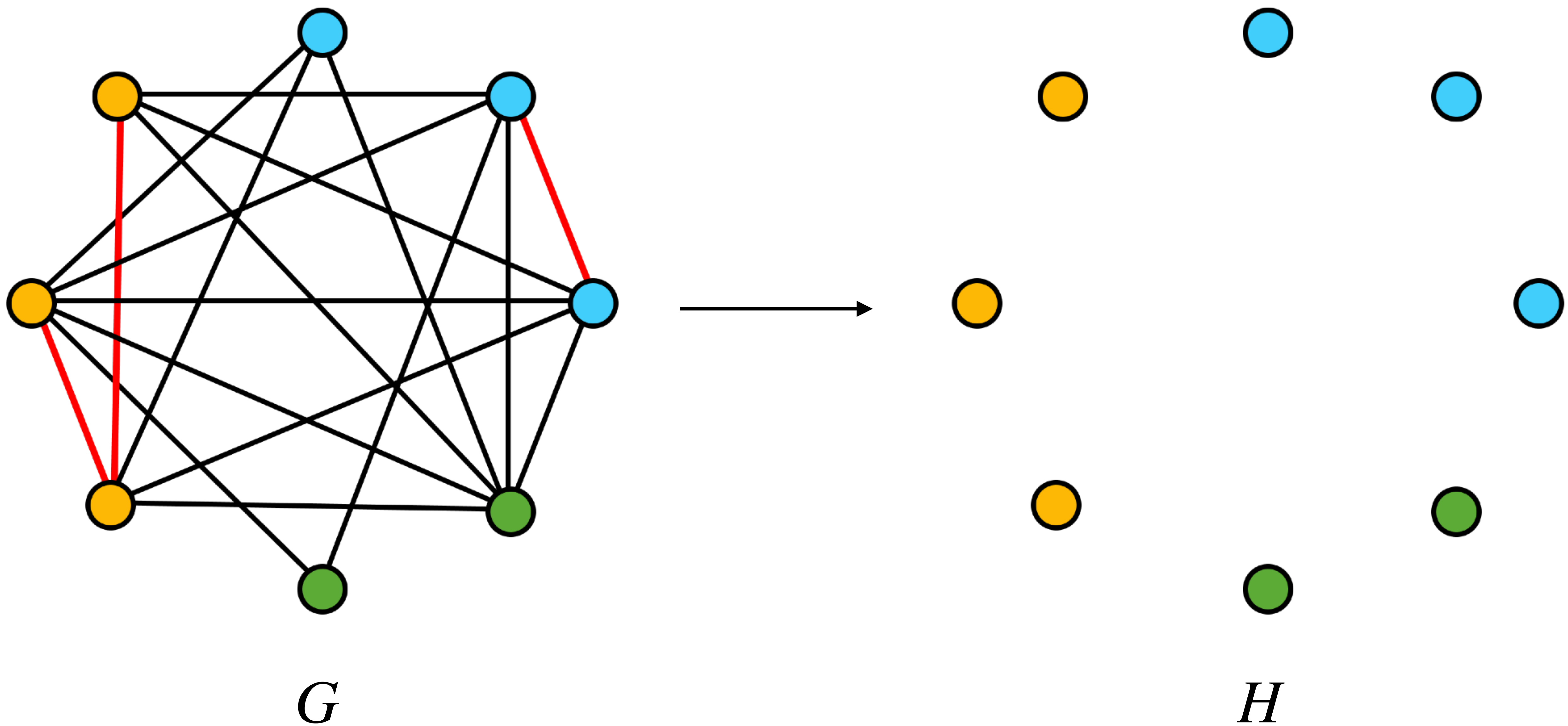

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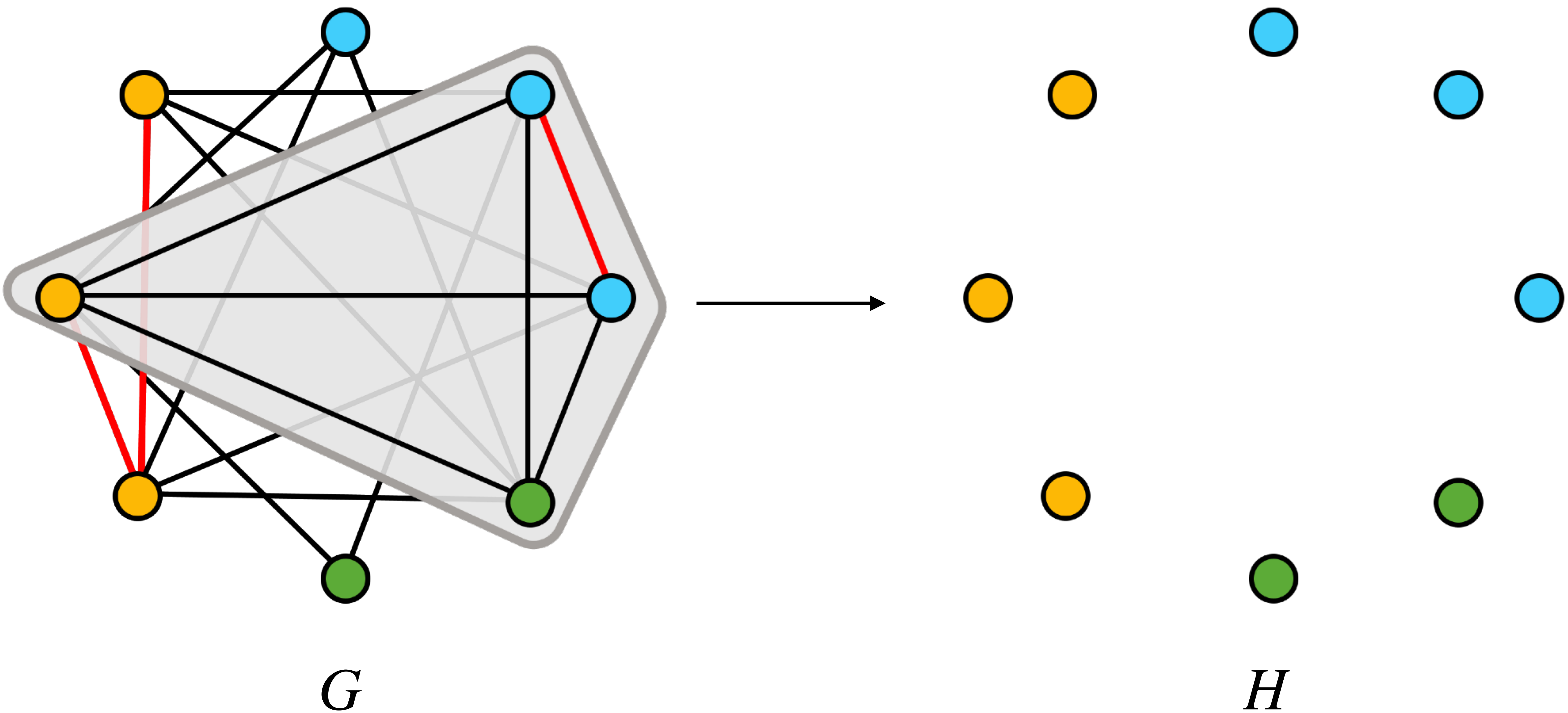

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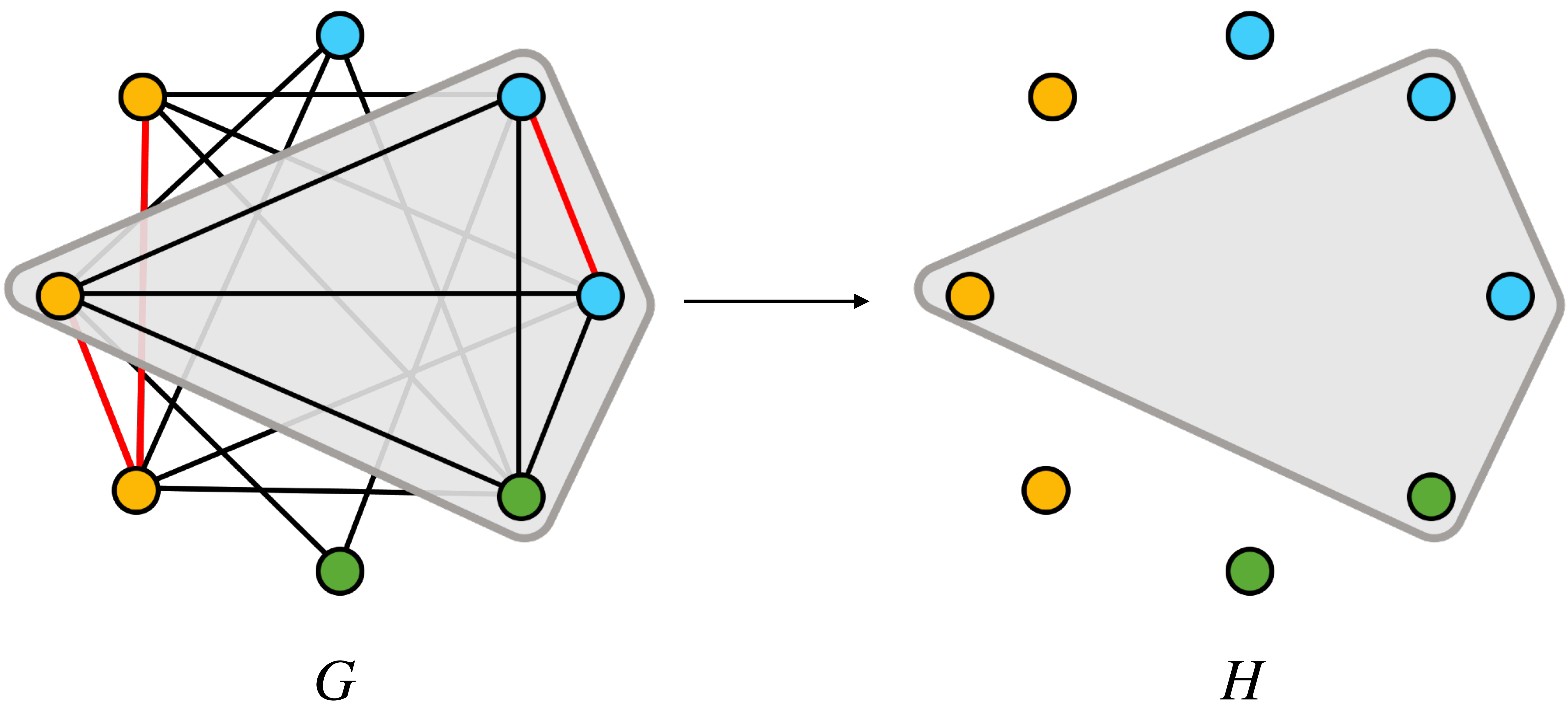

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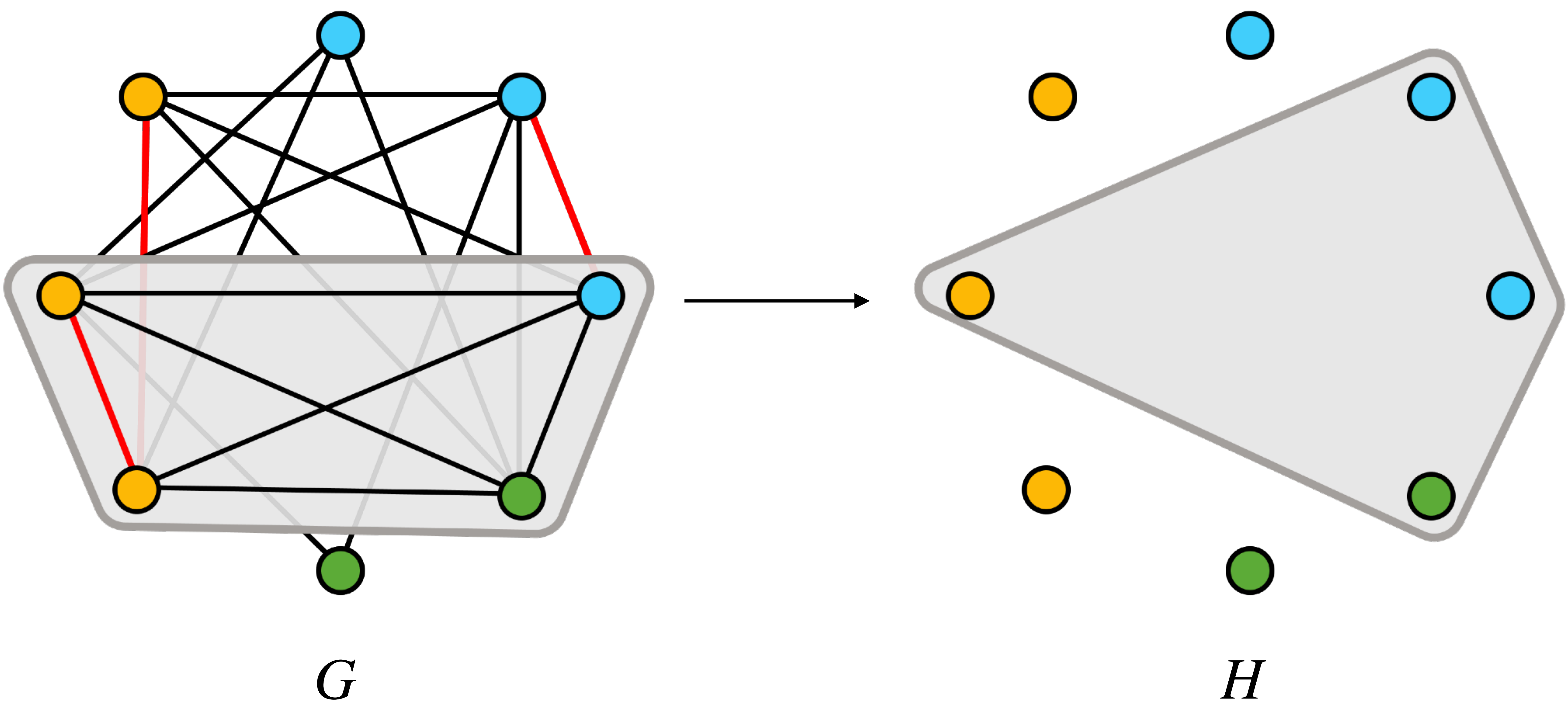

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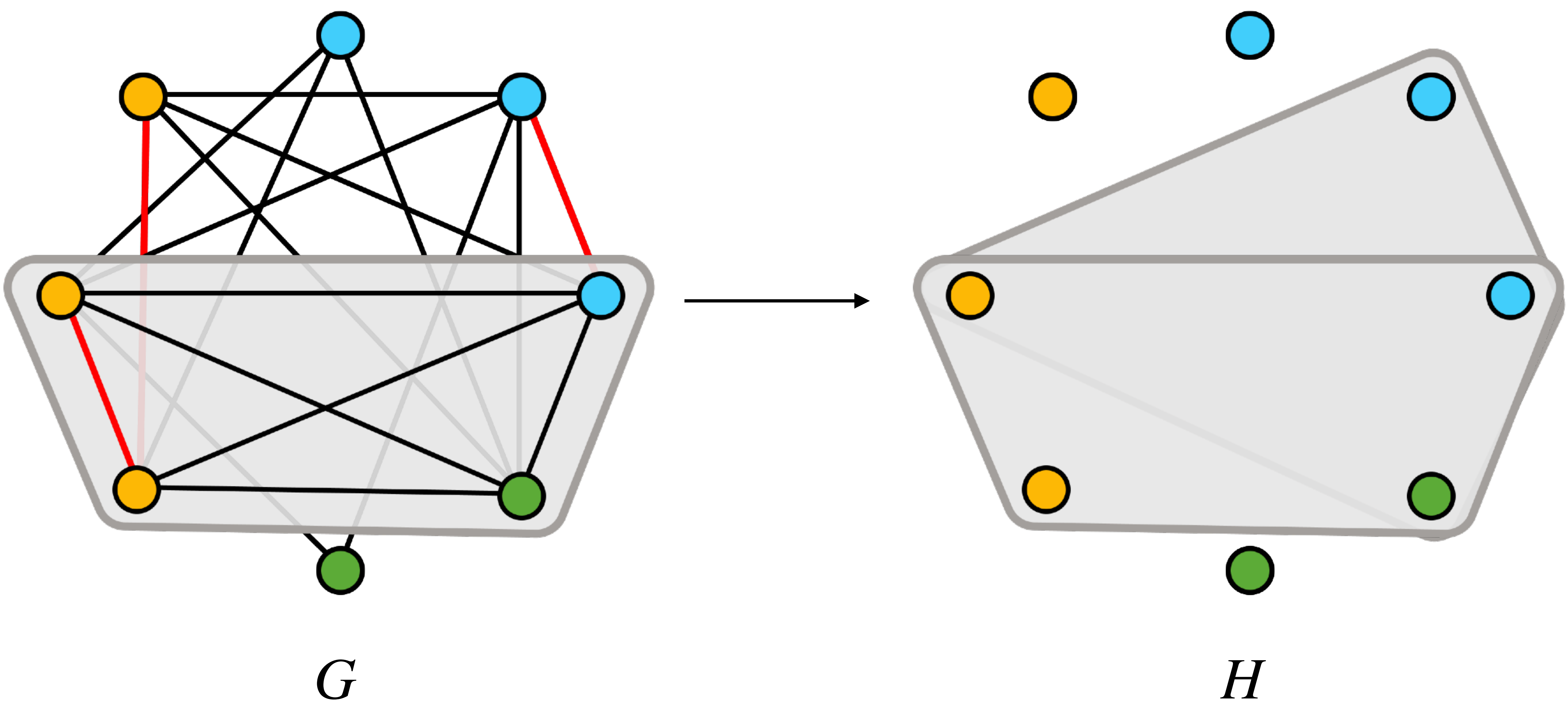

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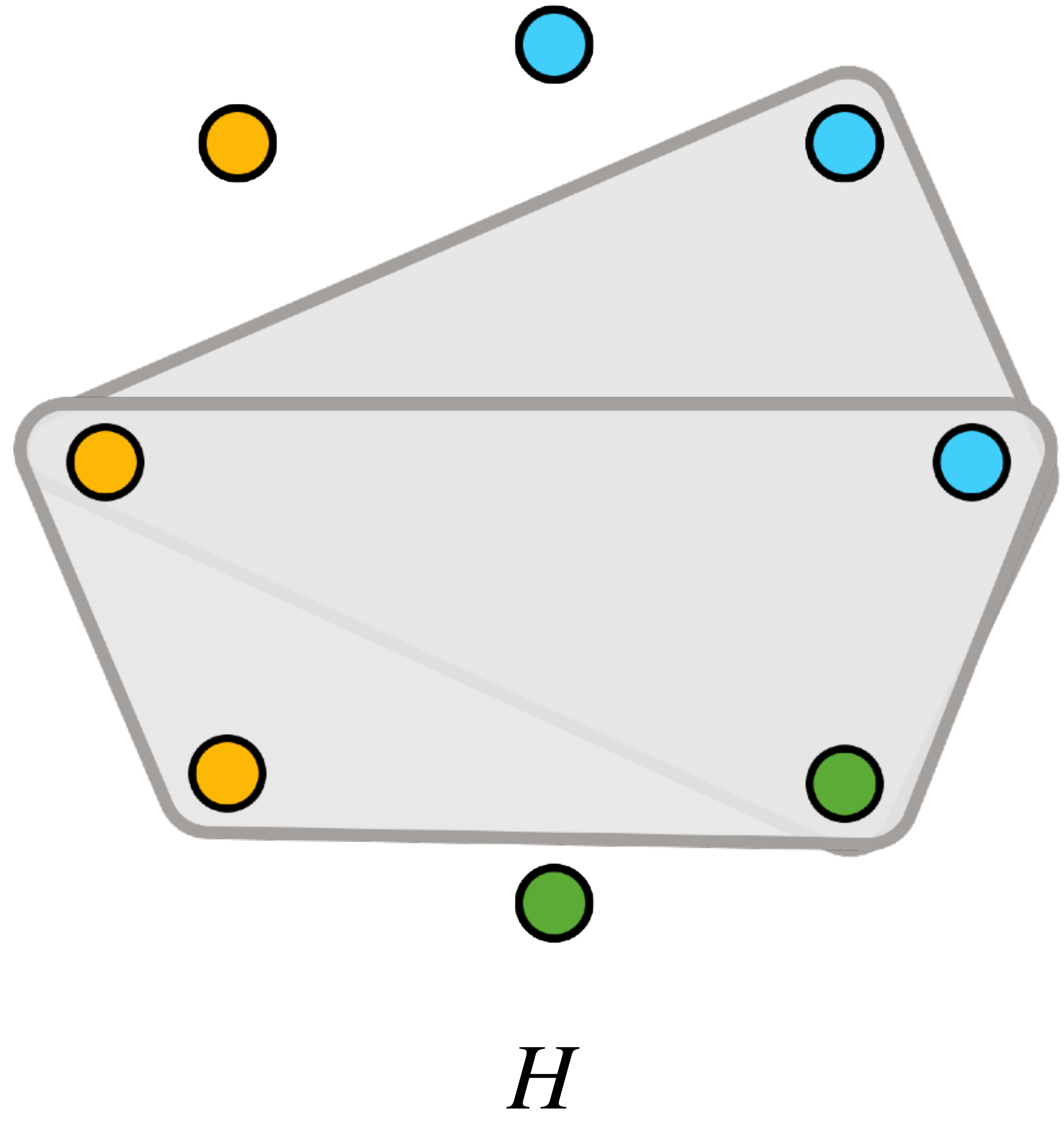

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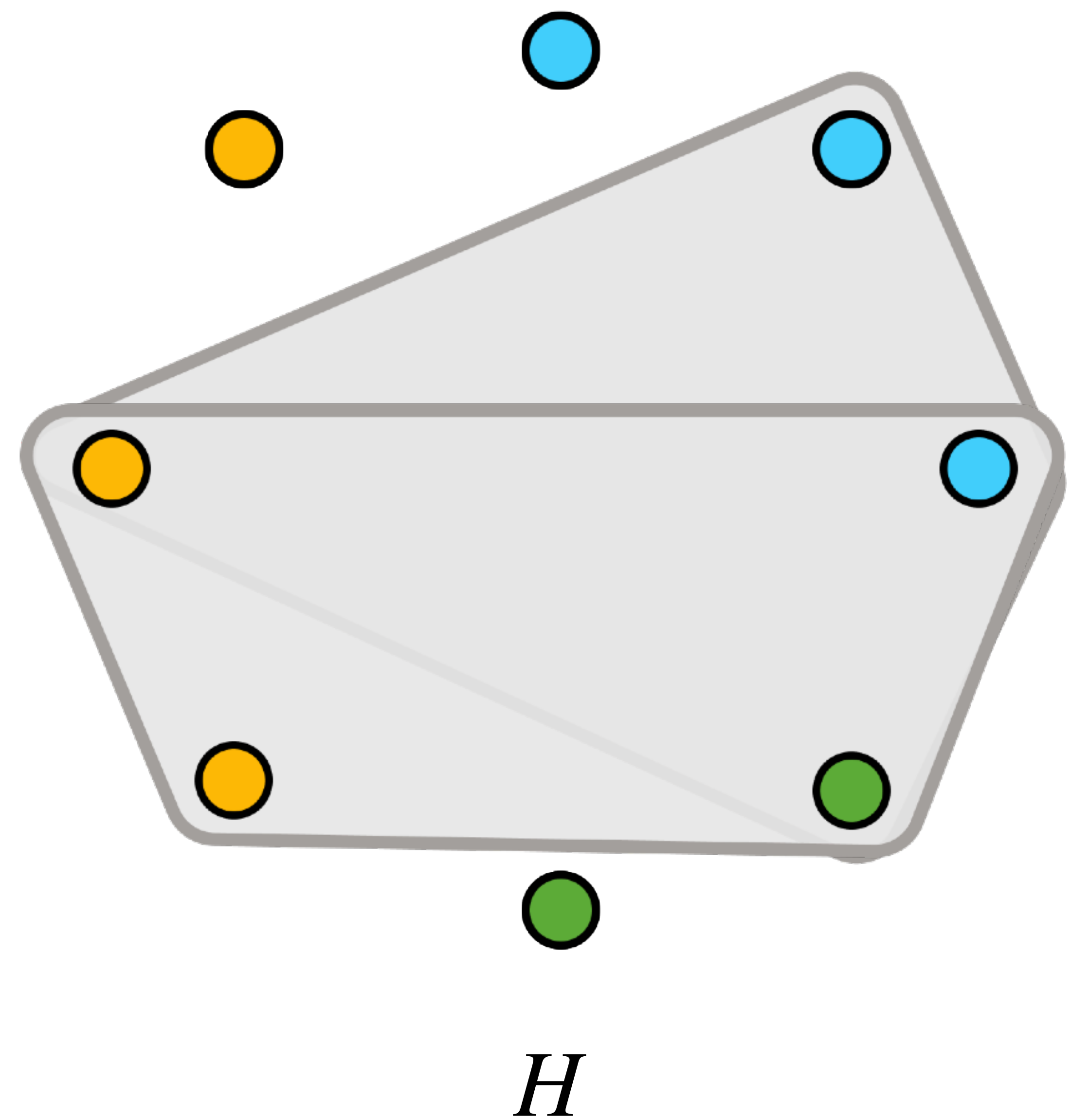
Observation: 3-colorable subgraphs in G are independent sets in H



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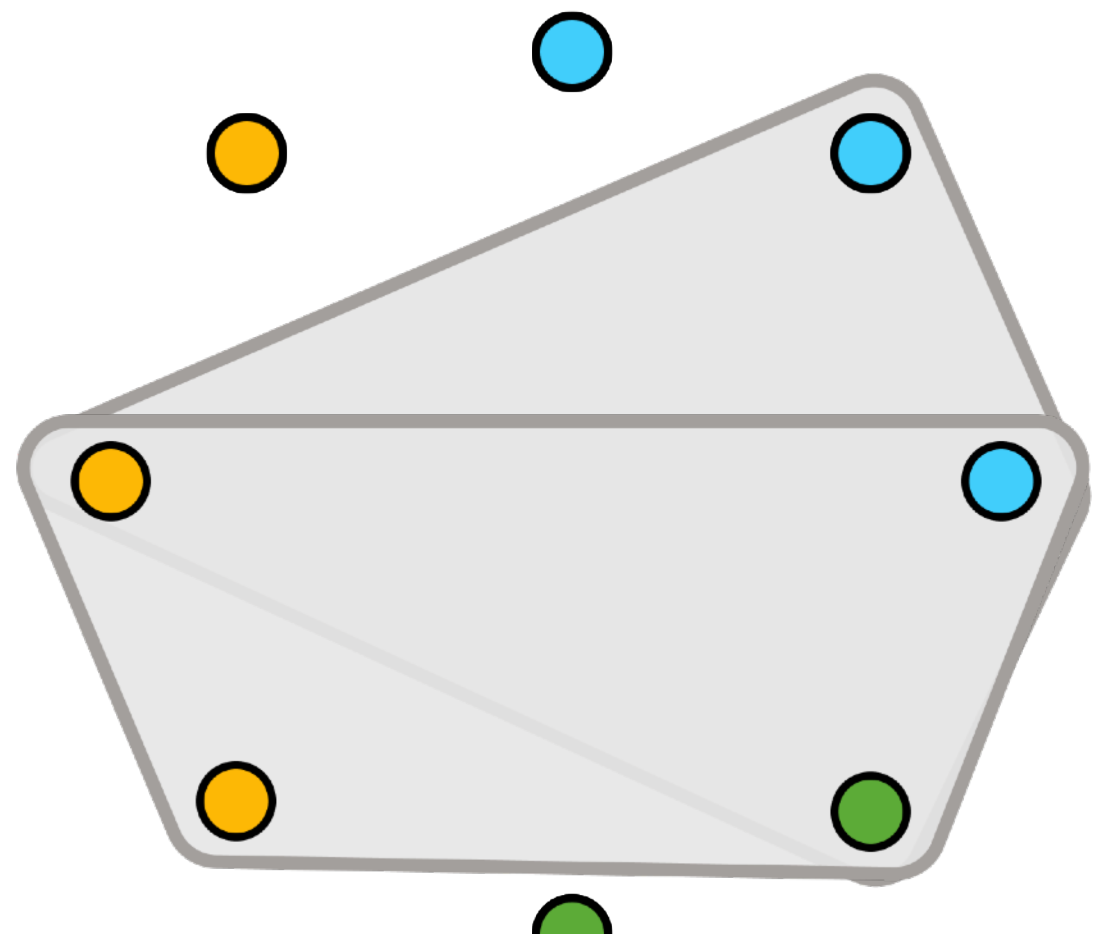
A container lemma for independent sets in H gives a container lemma for 3-colorable subgraphs in G



H

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Consider the tester that takes a random sample S of s vertices and checks if $G[S]$ satisfies Π.

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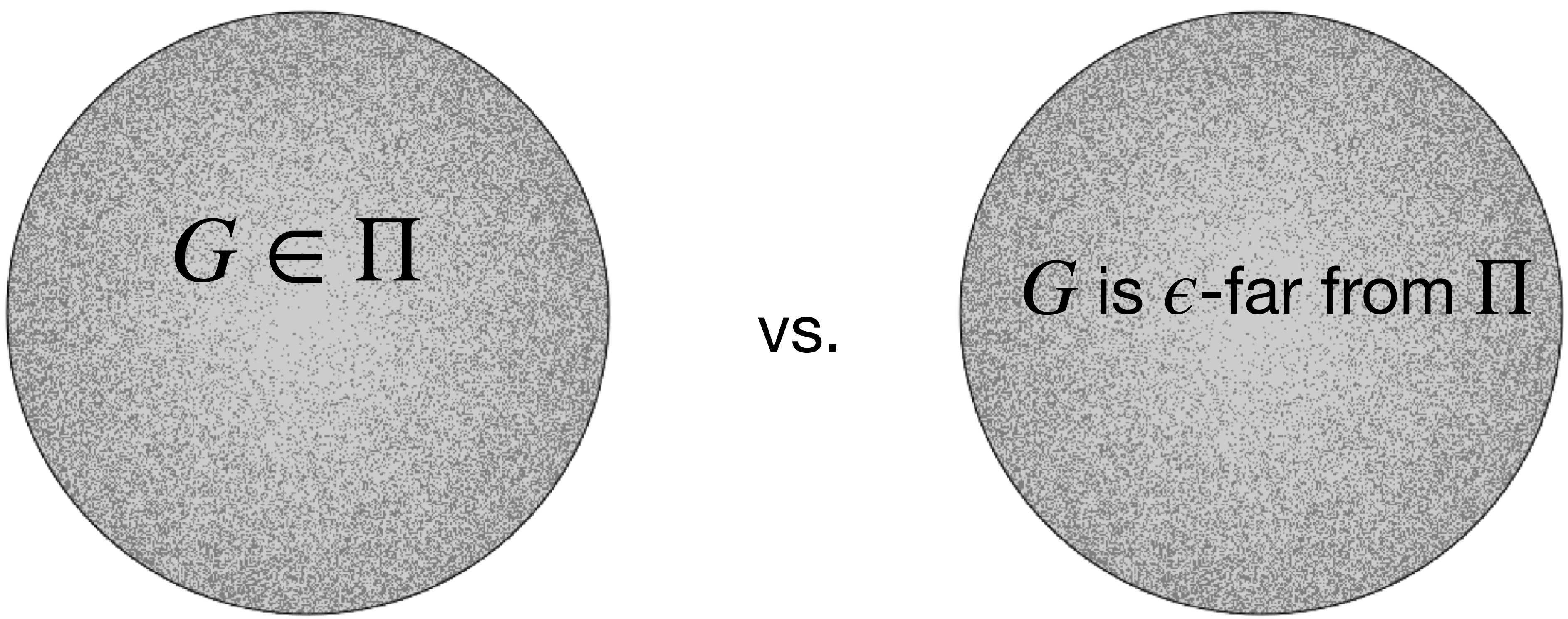

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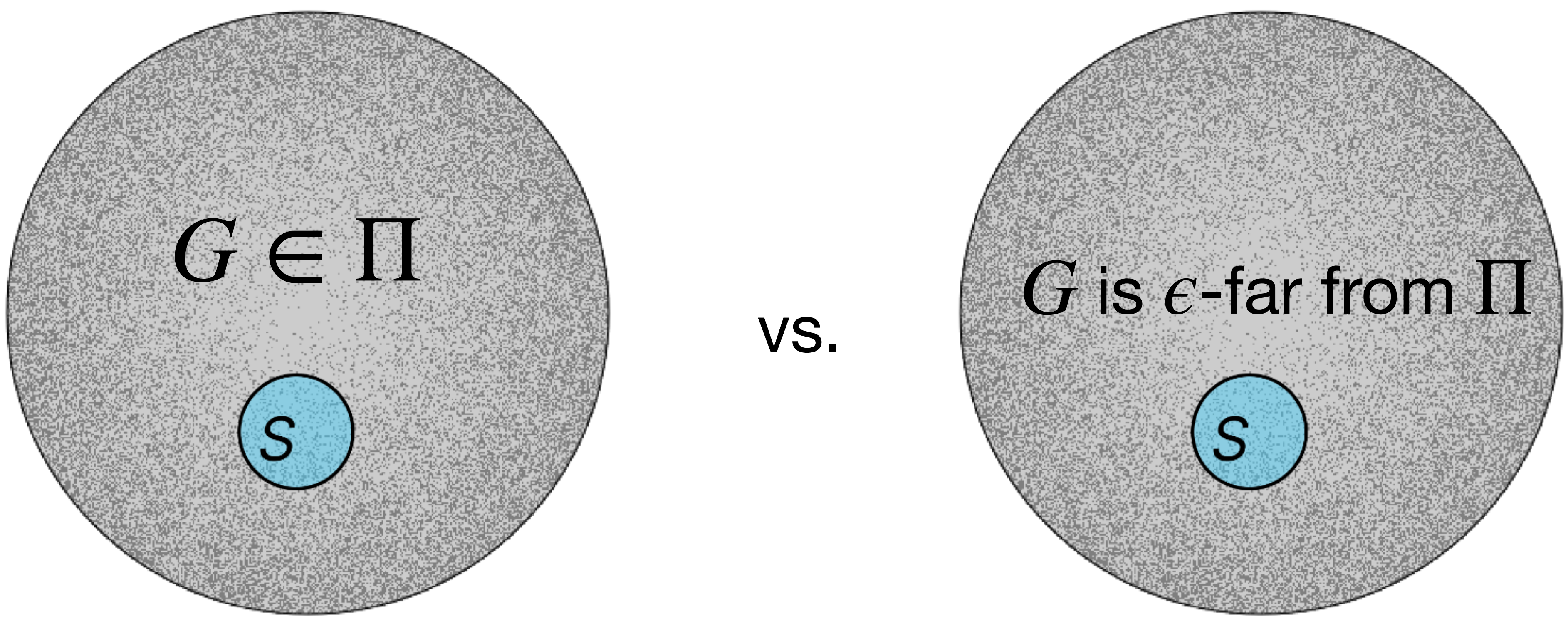

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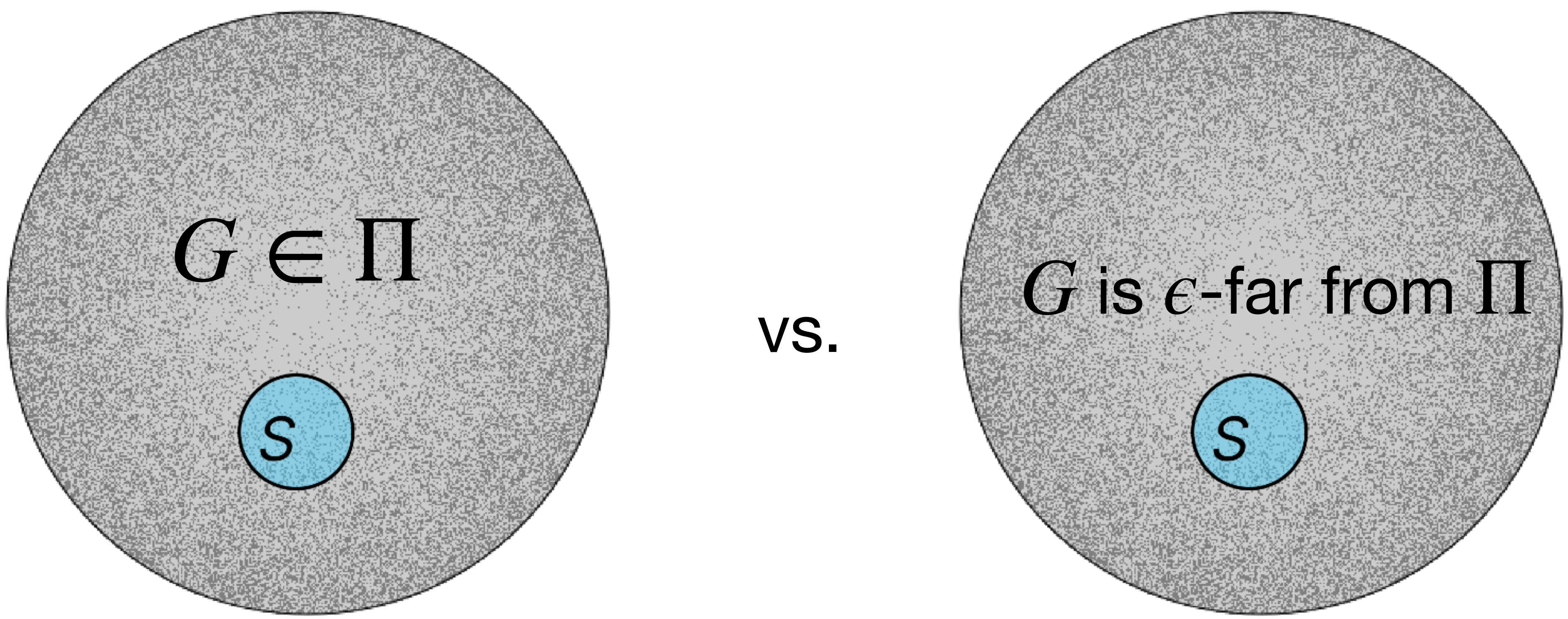

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$$\begin{aligned}\Pr[G[S] \in \Pi] &\le \sum_{F \in \mathcal{F}} \Pr[S \subseteq C(F)] \\ &\le \sum_{F \in \mathcal{F}} \left(\frac{|C(F)|}{n}\right)^{|S|} \\ &\lesssim \binom{n}{f}(1-\alpha)^{|S|}\end{aligned}$$

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&\lesssim \binom{n}{f} (1-\alpha)^{|S|} \\
&\leq \exp(f \log(n) - \alpha |S|)
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&\leq \exp(f \log(n) - \alpha |S|) \\
&= o(1) \qquad \text{as long as } |S| = \Omega(f \log(n)/\alpha)
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$$\leq \sum_{r=1}^{f} \binom{|S|}{r} (1-\alpha)^{|S|-r} \leq \sum_{r=1}^{f} \exp(r \log(|S|) - \alpha |S|) = o(1)$$

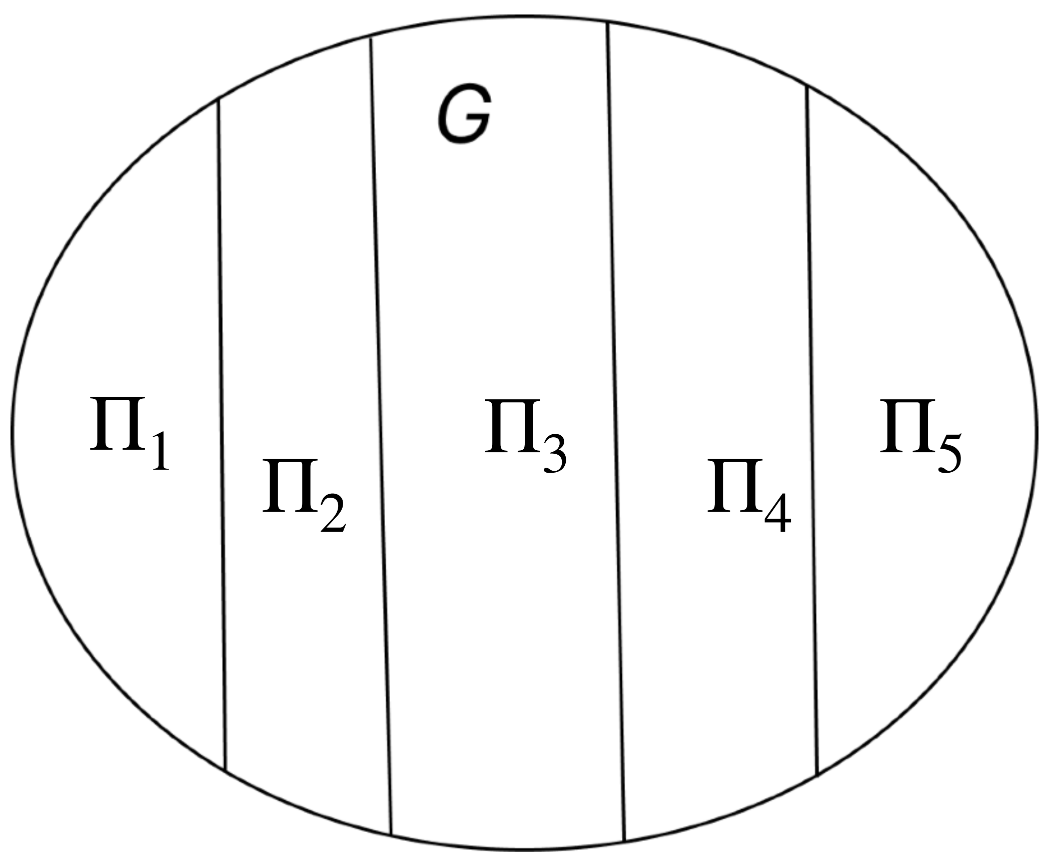
As long as $|S| = \tilde{\Omega}(f/\alpha)$

An Application of the Theorem

Application: Testing Partitions of Hereditary Properties

Corollary: Let $\Pi_1, \ldots, \Pi_k$ be hereditary graph properties that are ϵ-testable with one sided error with sample complexity $s_1(\epsilon), \ldots, s_k(\epsilon)$ respectively. Let Π be the graph property of having a partition into k parts such that the i-th part satisfies Π_i. Then Π is ϵ-testable with sample complexity roughly

$$\max(s_i(\epsilon/k))^2 \cdot k/\epsilon$$



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Example 1: If Π_i is the empty property for all $i \in [k]$ then Π is the property of being k-colorable. Corollary gives sample complexity bound of k^3/ϵ^3 since emptiness property is testable with $1/\epsilon$ samples

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Example 2: If Π_i is the property of being a DAG (directed acyclic graph) for all $i \in [k]$ then Π is the property of having dichromatic number k. Corollary gives sample complexity bound of k^3/ϵ^3 since DAG property is testable with $1/\epsilon$ samples ([Bender, Ron '02])

Application: Testing Partitions of Hereditary Properties

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Proof:

We will show that Π has an ϵ-container lemma with fingerprint size $k \max \left(s_i \left(\frac{\epsilon}{2k} \right) \right)^2$ and container size $(1 - \epsilon/2)$.

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- Each Π_i has ϵ-container lemma with fingerprint size bounded by $s_i^2(\epsilon)$ - by Theorem
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Application: Testing Partitions of Hereditary Properties

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- The fingerprints are of size $k \max\left(s_i\left(\frac{\epsilon}{2k}\right)\right)^2$.
- Again applying the Theorem, we get testability with sample complexity roughly

$$k \max\left(s_i\left(\frac{\epsilon}{2k}\right)\right)^2 / \epsilon$$

Summary and Open Questions

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Π is ϵ-testable with one sided error and sample complexity s $\quad\Longleftrightarrow\quad$ Π has an ϵ-container lemma parameterized by $\mathrm{poly}(s)$

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Thank you!