

Module 1

Overview and background

What is it all about?

CS 360: Introduction to the Theory of Computing

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1 Welcome

Topics of Module 1

- Overview of the course
- Administrivia
- Mathematical background

Ideally this is not all administrivia

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2 Overview of course

What is this course about?

Theory of computing

- What does it mean to compute?
- What kinds of machines compute?
- How much more can computers do as we give them new powers?

Who cares?

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Who cares?

We will keep coming back to that.

Here are a few reasons:

- Philosophical: It is good to have a sense of what computation means
- Mathematical: Mathematical rigour underlying computer science
- Engineering: Many practical problems are about engineering with small resources. How does this limit us?
- Scientific: Some advances will not change the power of computers; others would be enormous. Let's pursue the important ones.

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How we will study theory

We will have a focus on mathematical rigour.

Example: you have already seen a lot of the first few weeks of the course, but without full detail.

- We will prove almost everything we do
- We will work with lots of proof techniques and approaches
- We will work with functions, sets, sets of sets, *etc.*

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Course myths

This course has several:

- It is too easy
 - You have seen regular expressions in 241.
 - A lot of it sounds reasonable.
 - It is easy to convince yourself you understand the details, even if you do not!
 - This is dangerous!
- It is too hard
 - Some complicated topics in the course come back after you think you are done with them
 - The later part of the course is harder than the beginning

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Other course myths

It is irrelevant

- Problems are like puzzles, not like what happens at your co-op jobs
- Where is the connection between a finite automaton and your desktop machine?
- We are ignoring all the complexity of computation!

But these are myths.

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Only myths

Myths are not always true.

The advantages of our approach:

- We will focus on the important core of a problem, not the implementation details
- Abstraction lets us get at the essence of computing, not the wiring details

And, many CS 360 ideas are used directly in important applications.

- Research in bioinformatics, for example.
- They also come up in parsing, in data compression, and in other areas.

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The real point of 360

We will model computer science problems as mathematical questions. These problems will give rise to formal languages

- Solve a problem = test membership in a language
- Simple languages correspond with simple algorithms

Model of computers: Formal machine models

- We work from simple to complex
- It is fun to see what simple models can do, and what they cannot do

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Complexity of models

We will work from simple to complex.

- Simplest: regular languages, and finite automata
- More complex: Context-free languages, and pushdown automata
- Still more complex: recursive and recursively enumerable languages, and Turing machines

Turing machines are actually more complex than your PC (they have infinite memory).

Actually, your PC is (technically) a finite state machine. It just has *lots* of states.

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Limitations of computers

For each of these types of models, we will find some language it cannot recognize.

Finite state machines do not have a memory.

- We will find problems that need memory to solve.

A pushdown automaton cannot access its inputs in arbitrary order

- We will find problems that cannot be solved without that ability.

For Turing machines, we will identify problems that cannot be solved, even with infinite memory available.

- This is probably the most important contribution of computer science to Western thought.

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More properties and topics

Families of languages are recognized by a particular computation model

- Regular languages \leftrightarrow finite automata
- Context-free languages \leftrightarrow pushdown automata
- Recursive languages, recursively enumerable languages \leftrightarrow Turing machines

The power of additions to computation models

- Often, small changes have no effect on computing power.
- Other times, perhaps unexpectedly, they do.

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History of the field

This subject mostly dates to the middle third of the 20th century. Perhaps the most interesting person of the many interesting people we will talk about: Alan Turing (1912-1954) He was a

- Cryptographer
- Philosopher
- Mathematician
- Theoretical biologist

Read *The Enigma*, by Andrew Hodges, or *The Man Who Knew Too Much*, by David Leavitt, or see the movie *The Imitation Game* if you want to learn about him.

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End of overview

Models of computation

- Formal representations of problems
- Ever more powerful computation models
- Problems that each model cannot solve

Formal mathematical proof of CS ideas

- Lots and lots of induction proofs
- Mathematical rigour

Reduction

- Solving one problem with the answer to another problem.
- Unsolvable problems

I hope you will have fun!

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3 Administrivia

Administrivia

- Course information (assignments, lecture notes, schedules, etc.) will all be on the course web sites
- The grading scheme for the course is documented in the course outline and on the course websites
- Refer to the schedule page of the unsecured website for all due dates, except that for the final assessment (which will be scheduled by the Registrar)

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Staff

Instructor

- Collin Roberts, cd2rober@uwaterloo.ca
- Office hours: TBD (See LEARN)

The course staff will also be using Piazza to answer student questions.

We are here to help.

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Cheating

Please do not cheat.

- There are very clear expectations on the course outline.
- Do not work in groups.
- I hate undergraduate cheating. Please do not make me have to deal with it.

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4 Mathematical Background

Basic mathematical details

You have seen these before.

CS 360 primarily is concerned with finite-length strings over a finite alphabet.

- Alphabet: Finite set of symbols, usually denoted Σ

Examples:

- Binary alphabet: $\Sigma = \{0, 1\}$
- Latin alphabet: $\Sigma = \{a, b, \dots, z\}$
- Unary alphabet: $\Sigma = \{1\}$.

- A *string* is an ordered sequence of alphabet symbols. Strings are also called *words*.

Examples of words: rover, 1010110

- Denote the length of the string x by $|x|$.

In CS 360, all of our strings all have *finite length*.

- Empty string: ε

Length of the empty string: $|\varepsilon| = 0$.

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Manipulating strings

- Prefixes: The first k characters of a string v , for some $k \geq 0$
- Suffixes: The last k characters of a string v , for some $k \geq 0$
- Substrings: A string z that is found as consecutive characters in a string v . A substring can be fully described by its starting position and length, or by its starting and ending positions.

Examples: Suppose $v = \text{temptation}$

- Prefixes: t , tempt , ε , *etc.*
- Suffixes: n , ation , temptation , ε , *etc.*
- Substrings: All of these, and also tat , mptat , *etc.*

Question: How many non-empty substrings does a string v of length k have? (N.B. we are only interested in counting the possible choices, not enforcing that the substrings obtained be distinct from one another.) **Answer:** $\frac{k(k+1)}{2}$. Can you derive this expression for yourself?

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Other string operations

Concatenation:

- One string after another
- Example: if $x = car$, $y = rot$, then $xy = carrot$.
- Not surprising: if $x = \varepsilon$, then $xy = y$.
- Formal definition of prefix:
 x is a prefix of y if $y = xz$, for some string z .
- Length of a concatenation equals sum of lengths: $|xy| = |x| + |y|$.

Power operation:

- x^k : k copies of string x , concatenated.
- Example: if $x = 010$, then $x^3 = 010010010$.
- Convention: $x^0 = \varepsilon$, for ease in recursive definitions.

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More string operators

- Counting members:
 $n_b(x) = \#$ of times alphabet symbol b is found in x .
Example: $n_b(banana) = 1$, $n_a(banana) = 3$.
Question: Is $n_\varepsilon(x)$ well-defined, for some string x ? **Answer:** No. Formally, ε is never included in any alphabet. Practically, there is no way to make this expression unambiguous. Asking for $n_\varepsilon(x)$ is analogous to asking how many factors of 1 can be pulled from some integer z .
- Reversing strings:
 $x^R = x$, written in reverse order. Example: if $x = stressed$, then $x^R = desserts$. Not x to the power of R !
- Palindrome: x a *palindrome* if $x = x^R$.

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5 Languages

Language

Σ^* : All finite strings over alphabet Σ

Language: a subset of Σ^* .

- That is, a set of strings over Σ .
- Some languages are finite; others are not.

Examples:

- $\{1, 01, 001, 001, 0001, \dots\}$
- $\{cat, dog, hamster\}$
- $\{aa, abab, abbabb, abbbabbb, \dots\}$
- $\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \dots\}$
- \emptyset

The second last of these is just $\{0, 1\}^*$.

We can describe languages by rules, or by listing all their elements if they are finite.

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Classes of languages

Class of languages: A set of languages.

- Note: a set of sets of words.

Example:

$$\mathcal{C} = \{ \{1, 11, 111, 1111, 11111, \dots\}, \\ \{0, 00, 000, 0000, 00000, \dots\}, \\ \{0, 1, 00, 11, 000, 111, \dots\}, \\ \{\varepsilon\} \}$$

We will care about classes with something in common. (Example: $\mathcal{C} = \{\text{all languages with ten words}\}$.)

(This would be hard to argue about the class \mathcal{C} that contains the four languages above!)

Remark: Languages in a class need not be disjoint. The above example demonstrates this.

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Important details about classes of languages

Important: A string is not a member of a class of languages.

- Strings are members of languages.
- Languages are members of classes.
- The only class that is also a language: \emptyset

Also, note:

- ε is a word.
- $L = \{\varepsilon\}$ is a language.
- The language L can be part of a class of languages (it is part of \mathcal{C}).
- ε cannot be part of a class of languages. (It is a word.)
- Letting $L_1 = \{\varepsilon\}$ and $L_2 = \emptyset$, we have that $L_1 \neq L_2$, as $|L_1| = 1 \neq 0 = |L_2|$.

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Language operations

Language: set of strings

- We can take unions and intersections of languages.
- Also: *complement*: $L' = \Sigma^* \setminus L$.
 - L' depends on what Σ is!
 - Recall, $L \subseteq \Sigma^*$ by definition.
 - Hence $L' = \emptyset$ if and only if $L = \Sigma^*$.
- *Concatenation*: If L and M are languages, then $LM = \{xy \mid x \in L, y \in M\}$.
 - Example: $L = \{\text{love}, \text{true}\}$, $M = \{\varepsilon, \text{ly}, \text{r}\}$ $LM = \{\text{love}, \text{true}, \text{truely}, \text{lovely}, \text{lover}, \text{truer}\}$

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Powers of languages

- *Finite powers*:
 - L^k is k copies of L , concatenated.
 - $L^0 = \{\varepsilon\}$.
 - $L^1 = L$.
 - If $L = \{\text{sorta}, \text{kinda}\}$, then $L^2 = \{\text{sortasorta}, \text{kindakinda}, \text{sortakinda}, \text{kindasorta}\}$
- *Kleene star operator* (so named after S.C. Kleene):
 - $L^* = \bigcup_{i=0}^{\infty} L^i$.
 - $L^+ = \bigcup_{i=1}^{\infty} L^i$.
 - If $L = \{\text{sorta}, \text{very}\}$, then
 - * $L^* = \{\varepsilon, \text{sorta}, \text{very}, \text{sortavery}, \text{verysorta}, \dots\}$, and
 - * $L^+ = \{\text{sorta}, \text{very}, \text{sortavery}, \text{verysorta}, \dots\}$.
 - If $M = \{1\}$, then
 - * $M^* = \{\varepsilon, 1, 11, 111, 1111, \dots\}$, and
 - * $M^+ = \{1, 11, 111, 1111, \dots\}$.
 - If $N = \{\varepsilon, 1\}$, then
 - * $N^* = \{\varepsilon, 1, 11, 111, 1111, \dots\}$, and
 - * $N^+ = \{\varepsilon, 1, 11, 111, 1111, \dots\}$.
 - * Note, $N^* = N^+$ for this choice of N .
 - It is generally true that $L^+ = LL^*$.
 - The proof is left as an exercise.

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How to define languages

Can describe languages using these sorts of operations:

Example:

- $M = (L \cap N)^*OP$ is a language.
- Each word in M consists of:
 - A concatenation of a finite number of strings from $L \cap N$,
 - followed by a string from O ,
 - followed by a string from P .

Very important: Keep your set notation clear, so that you know when you are dealing with sets, strings or symbols.

- This is especially a pain for ε and $\{\varepsilon\}$, and \emptyset .

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Proofs about languages

Three important proof ideas:

- To prove set A equals set B , show $A \subseteq B$ and $A \supseteq B$.
- To prove a statement is false, give a counterexample.
- To prove a statement is true for all members of a set A , use induction. Start with simple objects of A , move to complicated ones.

Quote from a TA: “CS 360 is just one long induction proof.” (This will not be true later in the course.)

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Technique 1: equality by subsets

Given: languages L_1, L_2 and L_3 .

Is $L_1(L_2 \cup L_3) = L_1L_2 \cup L_1L_3$ true for all choices of L_1, L_2 and L_3 ?

Yes.

Proof: We show each language is a subset of the other, so they must be equal.

- Let x be a word in $L_1(L_2 \cup L_3)$.
- We show it is also in $L_1L_2 \cup L_1L_3$; thus, $L_1(L_2 \cup L_3) \subseteq L_1L_2 \cup L_1L_3$.
- Since $x \in L_1(L_2 \cup L_3)$, we know $x = yz$, for some $y \in L_1$ and $z \in L_2 \cup L_3$.
- If $z \in L_2$, then $x = yz \in L_1L_2$, while if $z \in L_3$, then $x = yz \in L_1L_3$.
- In both cases, $x \in L_1L_2 \cup L_1L_3$.

(The other direction is similarly easy and is left as an exercise.)

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Technique 2: Proof by counterexample

Given: Languages L_1, L_2, L_3 . Is $L_1(L_2 \cap L_3) = L_1L_2 \cap L_1L_3$ always true for all choices of L_1, L_2, L_3 ?

Answer: No.

- Counterexample:
 - $L_1 = \{a, ab\}, L_2 = \{b\}, L_3 = \{\varepsilon\}$.
 - Since $L_2 \cap L_3 = \emptyset, L_1(L_2 \cap L_3) = \emptyset$.
 - (Note: $L\{\varepsilon\}$ is different from $L\emptyset$!)
 - But $L_1L_2 = \{ab, abb\}$, and $L_1L_3 = \{a, ab\}$, so $L_1L_2 \cap L_1L_3 = \{ab\}$.
 - And $L_1(L_2 \cap L_3) \neq L_1L_2 \cap L_1L_3$
- It is not true for all L_1, L_2 , and L_3 .
- It is not a theorem, period.

Remember: A *single case* that is not true proves a conjecture false.

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Technique 3: induction

Three kinds of induction:

- Weak induction If hypothesis is true for k , and if being true for ℓ implies being true for $\ell + 1$, then it is true for all $\ell > k$.
- Strong induction If hypothesis is true for k , and if being true for all m between k and ℓ implies being true for $\ell + 1$, then it is true for all $\ell > k$
- Structural If hypothesis is true for simple objects of a given type, and it is true for how we combine them to make new objects of the type, then it is true for all objects of a given type.

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Induction examples

We will use induction over and over. Here are some first theorems.

Theorem: If $L^2 \subseteq L$, then $L^+ \subseteq L$. (Remember: $L^+ = LL^*$.)

Proof:

- We want to show that $LL^i \subseteq L$ for all $i \geq 0$.
- The proof is by induction on i .
- Base cases:
 - $i = 0$. This is just $L = L$.
 - $i = 1$. That is the theorem's hypothesis, that $L^2 \subseteq L$.
- Induction ($i > 1$):
- Induction hypothesis: $L^2 \subseteq L$ implies $LL^k \subseteq L$, for $k = i - 1$
- Need to show: If the induction hypothesis is true, then $LL^i \subseteq L$.

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First induction example, continued

- Induction hypothesis: $L^2 \subseteq L$ implies $LL^{i-1} \subseteq L$.
- Need to show: If induction hypothesis is true, then $L^2 \subseteq L$ implies $LL^i \subseteq L$.
- Let $x \in LL^i$ be arbitrary.
 - Then $x = yz$, where $y \in L$ and $z \in L^i = LL^{i-1}$.
 - By the induction hypothesis, $z \in LL^{i-1} \subseteq L$, in other words $z \in L$.
 - So $yz \in L^2$.
 - But the hypothesis of the Theorem is that $L^2 \subseteq L$.
 - So $x = yz \in L$, and this proves that $LL^i \subseteq L$.
- Remember: we have to formally state what we are trying to prove, and how.

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Another example with reversals

Recall: x^R is x , written backwards.

Theorem: $(xy)^R = y^R x^R$.

Proof: By induction on $|y|$:

- Base case ($|y| = 0$): Then $y = \varepsilon$, so that

$$(xy)^R = (x\varepsilon)^R = x^R = \varepsilon x^R = \varepsilon^R x^R = y^R x^R.$$

- Induction step ($|y| = k > 0$): The induction hypothesis is that $(xz)^R = z^R x^R$ holds for all z of length $< k$. We need to argue that this implies $(xy)^R = y^R x^R$.
- Write $y = za$, for a single letter a . (We can do this, because $|y| > 0$.)
Then $xy = xza$, and

$$(xy)^R = (xza)^R \underbrace{=}_{\text{reversal}} a(xz)^R \underbrace{=}_{\text{I.H. for } z} a(z^R x^R) \underbrace{=}_{\text{reversal}} (za)^R x^R = y^R x^R.$$

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Recursive structure, induction

Many objects are defined recursively.

Examples:

- Fibonacci numbers
- Sequences of letters (either empty, or a sequence of letters followed by another letter)
- Binary trees (either empty, a single leaf, or the result of combining two binary trees with a new root)
- And so on. (Regular languages, *etc.*)

Structural induction: Show a theorem true for simple objects, and true for however objects are combined.

You have seen a lot of this in CS 135/136/245/...

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Example of structural induction

(We will actually come back to structural induction later.)

Let L be the language over $\Sigma = \{+, a, (\,)\}$ that constructed according to these rules:

- $a \in L$
- If w_1 and w_2 are in L , so is $(w_1 + w_2)$ (including the parentheses).

Examples of words in L : $a, (a + a), (a + (a + a)), ((a + a) + (a + a)), \dots$

Theorem: No string in L contains $)$. Proof: by structural induction. Let $w \in L$ be arbitrary.

- Base case: $w = a$ does not contain $)$, so the base case holds.
- Induction step: As we are no longer in the base case, we have that $w = (w_1 + w_2)$ for some $w_1, w_2 \in L$, each constructed in strictly fewer steps than w .
- The induction hypothesis applied to both of w_1, w_2 then says that neither w_1 nor w_2 contains $)$.
- Now we must use this assumption to argue that $(w_1 + w_2)$ contains no $)$.

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Proof by structural induction

Proof of induction:

- To show: no $)$ in $(w_1 + w_2)$ comes right before a $($.
- Certainly, the last $)$ cannot; it is the last letter.
- And no $)$ in w_2 comes directly before a $($, by the induction hypothesis.
- This is also true of all the $)$'s in w_1 , by the induction hypothesis. The last $)$ in w_1 now may come before a $+$, but not before a $($.
- And that is all of the $)$'s that can occur in $(w_1 + w_2)$.
- This completes the induction step.
- So no member of L contains $)$.

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Structural induction: main idea

- Show that a theorem is true for simple objects, and that complicated objects are just simple objects put together.
- It is sometimes easy to forget: We do *not* need to count steps in a recursive definition; instead just put "simpler" objects together.

We will use structural induction over and over in the first two thirds of the course:

- Regular languages are defined recursively!
- Context-free grammars are inherently recursive.

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Wrap-up

Module 1 goals:

- Overview of the basic goals of the course
- Boring administrivia
- Background: alphabets, strings, languages, proof techniques

Next module:

- Finite automata, regular languages and regular expressions

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