CS 360
Lecture Notes
Winter 2022

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1 Lecture 01

Outline

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1.1 Introduction to CS 360

1. Refer to the Course Outline and M1 slides 1-17 for the details.
2. I will lecture on the document camera, with the text and old lecture slides as resources.
3. I will announce pre-reading for all the future lectures by email.

1.2 Terminology

Definition 1.2.1. *An **alphabet** is a non-empty finite set of symbols, usually denoted* Σ.

Examples:

1. Binary alphabet: $\Sigma = \{0, 1\}$ Note, a surprising amount of work can be done over this small alphabet.
2. Latin alphabet: $\Sigma = \{a, b, \ldots, z\}$
3. Unary alphabet: $\Sigma = \{1\}$

Definition 1.2.2. *A **string** (a.k.a. **word**) is an ordered sequence of alphabet symbols.*

Examples:

1. 1010110 (Binary)
2. *rover* (Latin)

Notation / Conventions:

1. Denote the **length** of the string x (i.e. the number of alphabet symbols composing x) by $|x|$.
2. Denote the **empty string** (i.e. the string of no symbols) by ε.

Remarks:

1. In CS 360, all strings have **finite lengths**.
2. By definition, $|\varepsilon| = 0$.

Manipulating Strings

1. **Concatenation:** One string after another. (e.g. if $x = car$, $y = rot$, then $xy = carrot$.)

2. **Powers:** Let $k \in \mathbb{N}$. Let x be a string. Then x^k is k copies of x, concatenated together. (Convention: $x^0 = \varepsilon$, for any x.)
3. **Prefixes:** x is a **prefix** of y if $y = xz$, for some z. In other words a prefix of y is the first k characters of y, for some $k \geq 0$. **Remark:** ε is a prefix of every string.
4. **Suffixes:** x is a **suffix** of y if $y = zx$, for some z. In other words a suffix of y is the last k characters of y, for some $k \geq 0$. **Remark:** ε is a suffix of every string.
5. **Substrings:** x is a substring of y if $y = zxw$, for some z and w. A substring can be fully described
   (a) by its starting position and length, or
   (b) by its starting and ending positions.

   Remark: Every string x is a prefix, suffix and substring of itself.
6. **Counting characters:** $n_b(x) = \#$ of times the alphabet symbol b is found in x.

   Example: $n_b(banana) = 1$, $n_a(banana) = 3$.
   (a) **Q:** Is $n_\varepsilon(x)$ well-defined, for some (any) string x?
   A: No. Formally, ε is never included in any alphabet. Practically, there is no way to make this expression unambiguous. Asking for $n_\varepsilon(x)$ is analogous to asking how many factors of 1 can be pulled from some integer z.
   (b) **Q:** Can we extend this definition to $n_w(x)$, where w is a word?
   A: No. There are many problems associated with attempting to do this, e.g.
      i. Should $n_{aba}(ababa)$ be 1, or 2? Do occurrences of the word have to be non-overlapping?
      ii. We must at least enforce that the desired word be non-empty, for the above reason.
7. **Reversing a string:** $x^R = x$ written in reverse order.
   Example: if $x = stressed$, then $x^R = desserts$. **Not** x to the power of R!
8. **Palindromes:** x a **palindrome** if $x = x^R$, e.g. $x = radar$.

Remarks:

1. The length of a concatenation equals the sum of the lengths of its substrings: $|xy| = |x| + |y|$.

Questions and Answers:

1. Is ε an alphabet symbol?
   Answer: No. Every alphabet symbol must have length 1, but $|\varepsilon| = 0$.

2. Does every alphabet contain ε?
   Answer: No, as above.

1.3 Languages

Definition 1.3.1. *Let Σ be an alphabet. A **language over** Σ is any set of words constructed using symbols from Σ.*

Notation: Denote by Σ^* the set of **all** words constructed using symbols from Σ.

Examples of Languages:

1. The following sets are languages over $\Sigma = \{0, 1\}$:
   (a) $\{0, 1, 010, 001001, 0101\}$
   (b) $\{\varepsilon, 0, 00, 000, \ldots\}$
2. The following sets are languages over the Latin alphabet:
   (a) $\{car, rot, carrot, fish\}$

Questions and Answers:

1. Does every language contain ε?
   Answer: No. For example, two of the above three examples of languages do not contain ε.
2. Does a prefix of a string have to be over the same alphabet as the string itself?
   Answer: Yes.
3. Does a prefix of a string have to be a member of the same language as the string itself?
   Answer: No. The notions of prefix/suffix/substring of a given string, are completely separate from the question of to what language the given string might belong. For example, consider the abvove language $L = \{car, rot, carrot, fish\}$. Then $fish \in L$, however fi is a prefix of $fish$, and clearly $fi \notin L$.
4. Can two languages be equal?
   Answer: Yes. In fact, much of our work in CS 360 will be to prove that two languages, described in seemingly different ways, are equal (equal as sets, i.e. they contain exactly the same words).
5. Does the finiteness of alphabets and strings imply that all languages must be finite?
   Answer: No. For example, the language $\{\varepsilon, 0, 00, 000, \ldots\}$ is infinite.

Remarks:

1. **Why we insist that Σ is non-empty:** If $\Sigma = \emptyset$, then the only possible languages over Σ are $\emptyset$ and $\{\varepsilon\}$. There is not much to study here! **Q:** is $\emptyset = \{\varepsilon\}$? **A:** No. These sets have different cardinalities.
2. Since Σ is non-empty, therefore Σ^* is always infinite. If $a \in \Sigma$, then Σ^* contains the infinite subset $\{\varepsilon, a, aa, aaa, \ldots\}$, hence it must be infinite.
3. By definition, any language L over Σ is a subset of Σ^*.
4. A language can be finite or infinite.
5. We can describe a language by listing its elements if it is finite, or in any other unambiguous way if it is infinite.

Definition 1.3.2. *A **class** of languages is a set of languages.*

Questions and Answers:

1. Does a class have to contain only distinct languages?
   Answer: Yes. A class is a set after all. When we define a set, we are only interested in which elements belong to that set, regardless of order or possible repetitions.

Examples:

1.

$$\begin{aligned} \mathcal{C} \;=\; & \{\{1, 11, 111, 1111, 11111, \ldots\}, \\ & \{0, 00, 000, 0000, 00000, \ldots\}, \\ & \{0, 1, 00, 11, 000, 111, \ldots\}, \\ & \{\varepsilon\}\} \end{aligned}$$

Remarks:

1. We will always fix our alphabet Σ, before doing anything else. We can make Σ as large as we need in any particular example.
2. Be careful with your set theory notation. Make it clear to your reader what is what. See the lecture slides for common pitfalls.
   (a) Make sure you understand the difference between $\emptyset$ and $\{\varepsilon\}$!
   (b) A string is **not** a member of a class of languages.
      i. Strings are members of languages.
      ii. Languages are members of classes.
      iii. The only class that is also a language: $\emptyset$
   (c) $L = \{\varepsilon\}$ is a language.
   (d) The language L can be part of a class of languages (it is part of $\mathcal{C}$).

(e) ε cannot be part of a class of languages. (It is a word.)

3. Since languages are just sets, we can take unions, intersections and complements of languages.
4. **Concatenations:** If L and M are languages, then

$$LM = \{xy | x \in L, y \in M\}.$$

Examples:

(a) $L = \{love, true\}$, $M = \{\varepsilon, ly, r\}$
$LM = \{love, true, truely, lovely, lover, truer\}$

(b) Let $\Sigma = \{0, 1\}$. Let $L = \{0, 00, 000, \ldots\}$ and $M = \emptyset$. I claim that $LM = \emptyset$. Why? Recall that $LM = \{xy | x \in L, y \in M\}$. Since $M = \emptyset$, it is not possible to find a $y \in M$. Thus we cannot construct any concatenation xy of the required form to lie in LM.

(c) Let $\Sigma = \{0, 1\}$. Let $L = \{0, 00, 000, \ldots\}$ and $M = \{\varepsilon\}$. **Q:** What is LM this time? **A:** $LM = L$ (proved soon)

5. **Powers:** Let $k \in \mathbb{N}$. L^k is k copies of L concatenated together, as in 4 above.

Remarks: For any L,

(a) $L^0 = \{\varepsilon\}$.
(b) $L^1 = L$.

Examples:

(a) If $L = \{sorta, kinda\}$, then
$L^2 = \{sortasorta, kindakinda, sortakinda, kindasorta\}$

6. **Kleene star operator** (so named after S.C. Kleene):

(a) $L^* = \bigcup_{i=0}^{\infty} L^i = L^0 \cup L^1 \cup L^2 \cup \cdots$.
(b) $L^+ = \bigcup_{i=1}^{\infty} L^i = L^1 \cup L^2 \cup L^3 \cup \cdots$.

Examples:

(a) If $L = \{sorta, very\}$, then
- $L^* = \{\varepsilon, sorta, very, sortavery, verysorta, \ldots\}$, and
- $L^+ = \{sorta, very, sortavery, verysorta, \ldots\}$.

Remarks:

(a) In general, $L^+ = LL^*$.
(b) The proof is left as an exercise.

Examples of Recursively Defined Objects

1. the natural numbers, $\mathbb{N}$ (base: $n = 0$, otherwise $n = s(m)$, for some natural number, m, where s is the successor function). Inducting on this recursive structure leads to POMI.

2. in CS 245, syntactically correct propositional formulas. Inducting on this structure requires POSI.
3. a word w over some alphabet Σ (base $w = \varepsilon$, otherwise $w = xa$, for some alphabet symbol a and some word x). We'll need this setup soon, when we define the **extended transition function** for a **deterministic finite automaton (DFA)**.

Questions and Answers

1. Why is this base case allowed, when ε is never part of any alphabet?
   A: ε is a word over any alphabet. It is simply the word containting no letters from the chosen alphabet.

1.4 Techniques of Proof

Three important proof ideas:

1. In CS 360, we often need to prove that two languages, L_1 and L_2, are equal. Since languages are just sets, this means proving that the sets are equal. To prove that language L_1 equals language L_2, show $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$.
   Example: Prove that $LM = L$ in the earlier example (where $M = \{\varepsilon\}$).
   Recall we must prove $LM \subseteq L$ and $L \subseteq LM$.
   Proof that $LM \subseteq L$:
   - Let $xy \in LM$ be arbitrary.
   - In other words, let $x \in L$ and $y \in M$ be arbitrary.
   - Because $M = \{\varepsilon\}$, therefore $y = \varepsilon$.
   - Then $xy = x\varepsilon = x \in L$.

   Proof that $L \subseteq LM$:
   - Let $x \in L$ be arbitrary.
   - Then $x = x\varepsilon \in LM$.
2. To prove a statement is false, one counterexample is enough.
   Example: Is it true that every natural number which is congruent to 0 modulo 3 is congruent to 0 modulo 6?
   Answer: No.
   Counterexample: The natural number 9 is congruent to 0 modulo 3, but is not congruent to 0 modulo 6. Remember: A **single case** that is not true proves a $\forall$-conjecture false.
3. To prove some statement about all words in some language L (over some Σ) is true, induct on the structure of w. This can be

(a) induction on $|w|$, if we have nothing else to go on, or
(b) induction on the structure of $w \in L$, depending on how L is defined.

2 Lecture 02

Outline

1. Deterministic Finite Automata - M2 1-13
2. Nondeterministic Finite Automata - M2 14-30

2.1 Deterministic Finite Automata

Definition 2.1.1. *A **deterministic finite automaton (DFA)** is a model of a computer consisting of five ingredients:*

1. *Σ: an alphabet (for input words)*
2. *Q: a finite set of computation states*
3. *$\delta : Q \times \Sigma \to Q$: a transition function*
   - **Important:** *In a DFA, there must be a transition defined for **every state** and for **every alphabet symbol**.*
4. *$q_0 \in Q$: a start state*
5. *$F \subseteq Q$: a set of accept (final) states*

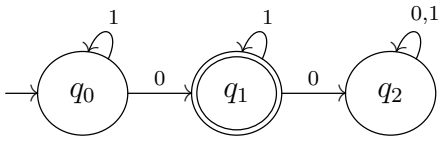
Intuition:

1. As it reads an input word, a DFA moves from one state to another as it reads each symbol.
2. At the end of the input, the machine either **accepts** or **rejects** the input, depending on whether the terminating state belongs to F, or not.

Vital constraints:

1. A DFA cannot work backward in its input; it can only go forward.
2. A DFA's only memory is its state; aside from that, it has forgotten everything.

Example:



1. $Q = \{q_0, q_1, q_2\}$,

2. $\Sigma = \{0, 1\}$,
3. δ is defined as in the picture.
   It includes $(q_0, 0) \mapsto q_1$
4. initial/starting state $= q_0$,
5. $F = \{q_1\}$.

This DFA **accepts** the **language** of words having exactly one 0 symbol (i.e. $L(1^*01^*)$).
Question: How would you prove that?

Definition 2.1.2. *Let D be a DFA and let w be a word. We say that D **accepts** w if D terminates in an accept state after processing w.*

We need the next definition to make this precise.
Intuitively, we want $\hat{\delta}(q, w)$ to be the state we terminate in if we start at state q and follow δ for each letter in the word w in turn. To make this work for all words w, we must define $\hat{\delta}$ recursively.

Definition 2.1.3. *$\hat{\delta}(q, w)$ is defined recursively, as follows:*
1. *$\hat{\delta}(q, \varepsilon) = q$, for all states q.*
2. *Otherwise, if $|w| > 0$, then we can write $w = xa$, for some alphabet symbol, a, and some word x over Σ. Then define $\hat{\delta}(q, w) = \delta(\hat{\delta}(q, x), a)$.*

Then we can make Definition 2.1.2 above precise by saying that D **accepts** w if $\hat{\delta}(q_0, w) \in F$.

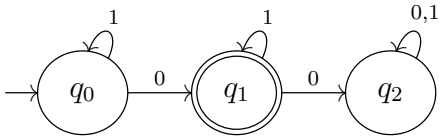
Lemma 2.1.4. *Let $D = (\Sigma, Q, q_0, F, \delta)$ be a DFA, with extended transition function $\hat{\delta}$. Let $q \in Q$ and $a \in \Sigma$ be arbitrary. Then $\hat{\delta}(q, a) = \delta(q, a)$, i.e. $\hat{\delta}$ agrees with δ for any state q and on any single alphabet symbol a.*

Proof.

$$\begin{aligned} \hat{\delta}(q, a) &= \hat{\delta}(q, \varepsilon a) \\ &= \delta(\hat{\delta}(q, \varepsilon), a) \\ &= \delta(q, a). \end{aligned}$$

□

Example: Letting M be our earlier DFA:



Show that M accepts 1011.
Solution: By the shape of M and by the refined Definition 2.1.2, we need to show that $\hat{\delta}(q_0, 1011) = q_1$. We compute

$$\begin{aligned}
\hat{\delta}(q_0, 1011) &= \delta(\hat{\delta}(q_0, 101), 1) && (1)\\
&= \delta(\delta(\hat{\delta}(q_0, 10), 1), 1) && (2)\\
&= \delta(\delta(\delta(\hat{\delta}(q_0, 1), 0), 1), 1) && (3)\\
&= \delta(\delta(\delta(\delta(\hat{\delta}(q_0, \varepsilon), 1), 0), 1), 1) && (4)\\
&= \delta(\delta(\delta(\delta(q_0, 1), 0), 1), 1) && (5)\\
&= \delta(\delta(\delta(q_0, 0), 1), 1) && (6)\\
&= \delta(\delta(q_1, 1), 1) && (7)\\
&= \delta(q_1, 1) && (8)\\
&= q_1 && (9)\\
&\in F. && (10)
\end{aligned}$$

Questions and Answers:

1. Are we not processing backwards here?
   A: No. Applying the definition of $\hat{\delta}$ is not processing anything. From line 5 onwards, we use δ repeatedly to process forward through the input word.

Definition 2.1.5. *Let $M = (Q, \delta, \Sigma, q_0, F)$ be a DFA, with extended transition function $\hat{\delta}$. The **language of** M, **denoted** $L(M)$ is the set of all words accepted by M:*

$$L(M) = \left\{ w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F \right\}.$$

Terminology: $L(M)$ can be called:

1. The language of the DFA M
2. The language **accepted** by the DFA M
3. The language **recognized** by the DFA M

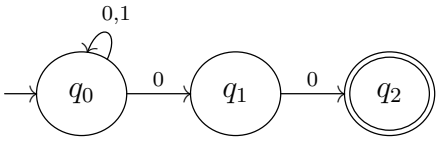
Example: See lecture slides for an example of a DFA that accepts all binary strings which are the binary representation of a multiple of 3.
We are going to prove a theorem soon that FAs accept a specific class of languages, called **regular languages**.

- That should be robust: changes to an FA should not invalidate the property.
- So we will change FAs in a variety of small and large ways.
- The first major change is **nondeterminism**.

2.2 Nondeterministic Finite Automata

- We enhnance a DFA to create and NFA: given a state and an alphabet symbol, transition to some set of states (not necessarily a single state, as in a DFA).
- We will now need to keep track of multiple threads.

An Example: $L = \{\text{all words with 00 as the last two symbols}\}$



Definition 2.2.1. *NFA defined by 5 parameters*

- Σ = *input alphabet*
- Q = *finite set of computation states*
- q_0 = *start state*
- F = *accept states*
- δ = *transition function*
  - **Important:** *In an NFA, there need* ***not*** *be a transition defined for every state and for every possible alphabet character. If a thread reaches a state which has no outgoing transition for the given input symbol, then that thread* ***crashes****; it proceeds no further.*

Differences from DFAs:

- δ: function from $Q \times \Sigma \to \{\text{subsets of } Q\}$.
  - Recall notation: $2^Q = \{\text{subsets of } Q\}$.
  - Based on a state in Q and an input letter from Σ, which states are now active in Q?
  - (Different from DFA, where it is from $Q \times \Sigma \to Q$.)
- $\hat{\delta}(q, w)$ = all states that we can reach from start state q processing the input word w.
- $\hat{\delta}$: function $Q \times \Sigma^* \to 2^Q$.
- Accepts whenever **any** state path from q_0 is to an accept state.

Definition 2.2.2.

- *Base case:* $\hat{\delta}(q, \varepsilon) = \{q\}$.
- *Recursive case: If* $|w| > 0$*, then we may write* $w = xa$*, where* $|a| = 1$.
- *Then define* $\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a)$*. (defining* $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$ *doesn't work because* $\hat{\delta}(q, x)$ *is a set, not necessarily a single state.)*
  - ***Note:*** *This definition handles threads that crash correctly.*

* *A thread that crashes has no outgoing transition from state p for input symbol a.*
* *In other words, $\delta(p, a) = \emptyset$.*
* *But then, $\delta(p, a)$ contributes nothing to the union of sets of states, reflecting the fact that the thread has crashed and proceeds no further.*

Definition 2.2.3. *An NFA M **accepts** a word w if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$.*

"When processing w, at least one thread terminates in an accept state."

Definition 2.2.4.

- *The language of the NFA $M = (\Sigma, Q, q_0, F, \delta)$ is:*

$$L(M) = \left\{ w \in \Sigma^* | \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}.$$

- *That is, all words accepted by the NFA.*

Theorem 2.2.5. *Let L be a language that is accepted by an NFA. Then L is accepted by a DFA.*

"NFAs are no more powerful than DFAs".

Outline of proof

- Given an NFA N, we will construct a DFA, D.
- Then, we will have to show that $L(D) = L(N)$.

Remember: The machine D does **not** have to have the same states as N, just the same alphabet and language! See the motivating example in the slides of the subset construction we are about to generalize for our proof.

Proof. Write $N = (\Sigma, Q_N, q_0, F_N, \delta_N)$. We will construct a new DFA $D = (\Sigma, Q_D, \{q_0\}, F_D, \delta_D)$, with these parameters:

- Q_D is the set of subsets of Q_N, which is denoted 2^{Q_N} in the lecture slides.
- $F_D = \{S \in Q_D \mid S \cap F_N \neq \emptyset\}$. (Recall by the definition of Q_D, S is some subset of the set Q_N of states of N.)
- And a more complicated transition function. For any $S \in Q_D$, and any $a \in \Sigma$, define:

$$\begin{aligned} & \delta_D(S, a) \\ = \; & \{\text{all states reachable in } N \text{ from } S \text{ when we read } a\} \\ = \; & \bigcup_{p \in S} \delta_N(p, a). \end{aligned}$$

- Let D's initial state be $\{q_0\}$, where q_0 is the initial state of N. This is a legal state in Q_D (it is a particular subset of the states Q_N).

Now we must show that the languages of the DFA D and the NFA N equal.

1. $w \in L(D)$ if and only if $\hat{\delta}_D(\{q_0\}, w) \in F_D$.
2. $w \in L(N)$ if and only if $\hat{\delta}_N(q_0, w) \cap F_N \neq \emptyset$. By the definition of F_D, 1 is equivalent to:
3. $w \in L(D)$ if and only if $\hat{\delta}_D(\{q_0\}, w) \cap F_N \neq \emptyset$.

Using 2 and 3, to show that $w \in L(D)$ if and only if $w \in L(N)$, it suffices to show that $\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$, for any word $w \in \Sigma^*$.

Proof: By induction on $|w|$.

- Base case ($|w| = 0$): Thus $w = \varepsilon$.
  - Then

$$\begin{aligned} & \hat{\delta}_D(\{q_0\}, \varepsilon) \\ \underbrace{=}_{DFA\text{-rule}} & \{q_0\} \\ \underbrace{=}_{NFA\text{-rule}} & \hat{\delta}_N(q_0, \varepsilon). \end{aligned}$$

- Inductive case ($|w| > 0$):
  - The inductive hypothesis is that, for every x with $|x| < |w|$, we have $\hat{\delta}_D(\{q_0\}, x) = \hat{\delta}_N(q_0, x)$.
  - Since $|w| > 0$, we may write $w = xa$, where $|a| = 1$.
  - Then the induction hypothesis applies to x.
  - We compute

$$\begin{aligned} \hat{\delta}_D(\{q_0\}, w) \quad & \underbrace{=}_{w=xa} && \hat{\delta}_D(\{q_0\}, xa) \\ & \underbrace{=}_{\text{Definition of } \hat{\delta}_D \text{ for the DFA } D} && \delta_D(\hat{\delta}_D(\{q_0\}, x), a) \\ & \underbrace{=}_{\text{induction hypothesis}} && \delta_D(\hat{\delta_N}(q_0, x), a) \\ & \underbrace{=}_{\text{Definition of } \delta_D} && \bigcup_{p \in \hat{\delta}_N(q_0, x)} \delta_N(p, a) \\ & \underbrace{=}_{\text{Definition of } \hat{\delta}_N} && \hat{\delta_N}(q_0, xa) \\ & \underbrace{=}_{w=xa} && \hat{\delta_N}(q_0, w). \end{aligned}$$

□

Remarks:

1. The opposite direction is easy: DFAs are NFAs! (Well, we must change δ trivially, so that the NFA transitions to a single state correctly reflect the original DFA definition.)
2. Although Theorems 2.2.5 and 3.1.7 will not appear on assessments, ideas and techniques from these proofs are fair game.

3 Lecture 03

Outline

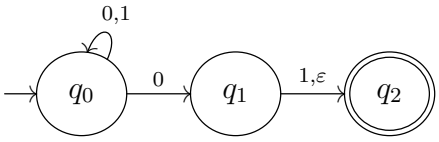
1. ε-Nondeterministic Finite Automata - M2 31-44

3.1 ε-Nondeterministic Finite Automata

- Sometimes in designing an NFA, it is handy to have transitions that happen automatically, without reading any letters of the input.
- Machine models that allow this are **ε-NFAs**.

Definition 3.1.1. *An ε-NFA is a 5-tuple, like an NFA. The only difference is transition function: δ is now a function: $Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$. (Transitions may exist that do not consume input letters.)*

Example: The ε-NFA



accepts binary words ending with 0 or with 01. (We could do this without ε-transitions, by making the second state an accept state.)
Defining the ε-closure of a state:
We want ECLOSE(p) to be all states reachable starting from p only using ε-transitions. This motivates the following definition.

Definition 3.1.2. *Let's start with ε-transitions:*

- *Define* ECLOSE(p) *recursively:*
  - *(Base)* $p \in$ ECLOSE(p)

– *(Recursive) If* $q \in \textsc{Eclose}(p)$*, then so are all of the states in* $\delta(q, \varepsilon)$*.*

The set $\textsc{Eclose}(p)$ is called the **ε-closure** of the state p.
Defining the ε-closure of a set of states

Definition 3.1.3. *Define* $\textsc{Eclose}(S)$*, for a set* S *of states via:*

$$\textsc{Eclose}(S) = \bigcup_{s \in S} \textsc{Eclose}(s).$$

Lemma 3.1.4. *For an ε-NFA E, having states Q, a subset $S \subseteq Q$ and a decomposition $S = \bigcup_i S_i$,*

$$\bigcup_i \textsc{Eclose}\left(S_i\right) = \textsc{Eclose}\left(\bigcup_i S_i\right),$$

(i.e. taking ε-closure commutes with taking set unions).

Proof.

$$\begin{aligned}
\bigcup_i \textsc{Eclose}\left(S_i\right) &\underbrace{=}_{\text{Definition of } \textsc{Eclose}(S_i)} \bigcup_i \left(\bigcup_{s \in S_i} \textsc{Eclose}(s) \right) \\
&\underbrace{=}_{S = \bigcup_i S_i} \bigcup_{s \in S} \textsc{Eclose}(s) \\
&\underbrace{=}_{\text{Definition of } \textsc{Eclose}(S)} \textsc{Eclose}(S) \\
&\underbrace{=}_{S = \bigcup_i S_i} \textsc{Eclose}\left(\bigcup_i S_i \right). \square
\end{aligned}$$

□

Definition 3.1.5. *We define $\hat{\delta}$ for ε-NFA's, also recursively.*

- *Base case:* $\hat{\delta}(q, \varepsilon) = \textsc{Eclose}(q)$*:*
  i.e. states reachable from q via ε-transitions
- *Inductive case: Suppose that $w = xa$, where $a \in \Sigma$.*
  *(**Note**: a **cannot be** ε, which is not a member of Σ.)*

- *We know that $P = \hat{\delta}(q, x)$ is the set of all states in Q that we can get to by following either edges for the letters of x or ε-transitions (including ε-transitions at the end of x).*
- *Then, we must follow the transitions for the alphabet symbol a: Let $R = \bigcup_{p \in P} \delta(p, a)$: then R has all of the states we can get to from P after following a transition for a.*
- *Last, we might have some more ε-transitions.*
- *So, $\hat{\delta}(q, w) = \textsc{Eclose}(R) = \textsc{Eclose}\left(\bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a)\right)$*

Definition 3.1.6. • *Language of an ε-NFA:*
$L = \left\{ w \in \Sigma^* \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}$

• *That is, $\hat{\delta}(q_0, x)$ includes an accept state.*

Q: Are ε-NFAs more powerful than DFAs?
A: No.

Theorem 3.1.7. *Given an ε-NFA E, there exists an ordinary DFA D such that $L(D) = L(E)$.*

Remarks:
1. This is not very surprising: we must show that we can include the ε-transitions of E in the transition function δ_D for D.
2. (The other direction is just by definition: a DFA is an ε-NFA, once we make some trivial changes to the structure of δ so that it produces a 1-element set when it reads a symbol and the empty set when it reads in ε.)
3. To prove the theorem, we must construct a DFA, D, accepting language $L(E)$.
   (a) Write $E = (\Sigma, Q_E, q_0, F_E, \delta_E)$.
   (b) We will construct $D = (\Sigma, Q_D, q_0, F_D, \delta_D)$.

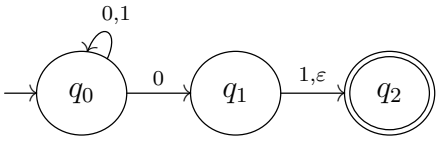
Proof. • Both machines use the same alphabet, of course.
- We use the subset construction, as when we built the DFA for an NFA.
  1. $Q_D = \{\text{all possible subsets of states from } Q_E\}$
  2. $F_D = \{S \in Q_D \mid S \cap F_E \neq \emptyset\}$
- The starting state is $q_D = \textsc{Eclose}(q_0)$. We start having implicitly taken ε-transitions from the starting state q_0 of E.
- <u>Transition function:</u>

  - From one DFA state, S (which is a set of states in E), if we process one letter, a, in the new DFA, we should mimic this behaviour from E:
    * follow any edges labelled a, and
    * take any ε-transitions
  - From any one state from E, say p, this then takes us to:
    * $\delta_E(p, a)$
    * $\textsc{Eclose}(\delta_E(p, a))$
- And we therefore want the union over all states $p \in S$:

$$\delta_D(S, a) = \bigcup_{p \in S} \textsc{Eclose}(\delta_E(p, a)).$$

Subset Construction Example:

- Here we demonstrate one step in the subset construction for the earlier small example of an ε-NFA:



- The subset construction gives us

$$\begin{aligned} Q_D &= \{\emptyset, \{q_0\}, \{q_1\}, \{q_2\}, \{q_0, q_1\}, \{q_0, q_2\}, \{q_1, q_2\}, \{q_0, q_1, q_2\}\} \\ q_D &= \textsc{Eclose}(\{q_0\}) = \{q_0\} \end{aligned}$$

- Now we determine $\delta_D(S, a)$, where $S = \{q_0, q_1, q_2\}$ (the state we reach from $\{q_0\}$ upon reading 0) and $a = 1$.
- Computing $\textsc{Eclose}(\delta_E(p, a))$ for each $p \in S$ gives

$$\begin{aligned} \textsc{Eclose}(\delta_E(q_0, 1)) &= \textsc{Eclose}(q_0) = \{q_0\} \\ \textsc{Eclose}(\delta_E(q_1, 1)) &= \textsc{Eclose}(\{q_2\}) = \{q_2\} \\ \textsc{Eclose}(\delta_E(q_2, 1)) &= \textsc{Eclose}(\emptyset) = \emptyset \end{aligned}$$

- Hence the target state coming from the subset construction is $\{q_0, q_2\}$.
- The construction says that we need to add to the transition function for D: $\delta_D(\{q_0, q_1, q_2\}, 1) = \{q_0, q_2\}$.
- Now to complete the transition function for D, we do this same construction for each of the 8 choices for S (above) and each alphabet symbol from $\Sigma = \{0, 1\}$.

Now, to show equality of the languages of the two automata, we must show that if x is accepted by E, then x is accepted by D, and vice versa.

- (One particular concern: do we do the right thing for the word ε?)

Goal: Prove that $w \in L(E)$ if and only if $w \in L(D)$.

1. $w \in L(E)$ if and only if $\hat{\delta}_E(q_0, w) \cap F_E \neq \emptyset$.
2. $w \in L(D)$ if and only if $\hat{\delta}_D(\textsc{Eclose}(q_0), w) \in F_D$. Using the definition of F_D, rewrite 2 as
3. $w \in L(D)$ if and only if $\hat{\delta}_D(\textsc{Eclose}(q_0), w) \cap F_E \neq \emptyset$.

Now using 1 and 3, to show that $w \in L(E)$ if and only if $w \in L(D)$, it suffices to prove that $\hat{\delta}_E(q_0, w) = \hat{\delta}_D(\textsc{Eclose}(q_0), w)$, for every word $w \in \Sigma^*$.

The proof is by induction on $|w|$.

Base case ($|w| = 0$): In this case, $w = \varepsilon$. We have $\hat{\delta}_E(q_0, \varepsilon) = \textsc{Eclose}(q_0)$, by definition of $\hat{\delta}_E$ in the ε-NFA.

- We therefore have

$$\begin{aligned}
\hat{\delta}_E(q_0, \varepsilon) &\underbrace{=}_{\varepsilon-NFA \text{ rule}} \textsc{Eclose}(q_0) \\
&\underbrace{=}_{\text{Definition of } q_D} q_D \\
&\underbrace{=}_{D \text{ is a } DFA} \hat{\delta}_D(q_D, \varepsilon) \\
&\underbrace{=}_{\text{Definition of } q_D} \hat{\delta}_D(\textsc{Eclose}(q_0), \varepsilon).
\end{aligned}$$

- So the base case holds.

Inductive case ($|w| > 0$):

- The induction hypothesis is that $\hat{\delta}_E(q_0, x) = \hat{\delta}_D(\textsc{Eclose}(q_0), x)$, for every $x \in \Sigma^*$ with $|x| < |w|$.
- We need to argue that $\hat{\delta}_E(q_0, w) = \hat{\delta}_D(\textsc{Eclose}(q_0), w)$.
- Write $w = xa$, where a is a single character.
- The induction hypothesis applies to x.
- Thus we may let $S = \hat{\delta}_E(q_0, x) = \hat{\delta}_D(\textsc{Eclose}(q_0), x)$.

- Now we compute

$$\begin{aligned}
\hat{\delta}_E(q_0, w) &\underbrace{=}_{w=xa} \hat{\delta}_E(q_0, xa) \\
&\underbrace{=}_{\text{Definition of } \hat{\delta}_E} \textsc{Eclose}\left(\bigcup_{p \in \hat{\delta}_E(q_0,x)} \delta_E(p,a)\right) \\
&\underbrace{=}_{\text{Definition of } S} \textsc{Eclose}\left(\bigcup_{p \in S} \delta_E(p,a)\right) \\
&\underbrace{=}_{\text{Lemma 3.1.4}} \bigcup_{p \in S} \textsc{Eclose}(\delta_E(p,a)) \\
&\underbrace{=}_{\text{Definition of } \delta_D} \delta_D(S,a) \\
&\underbrace{=}_{\text{Definition of } S} \delta_D(\hat{\delta}_D(\textsc{Eclose}(q_0), x), a) \\
&\underbrace{=}_{\text{Definition of } \hat{\delta}_D} \hat{\delta}_D(\textsc{Eclose}(q_0), xa) \\
&\underbrace{=}_{\text{w=xa}} \hat{\delta}_D(\textsc{Eclose}(q_0), w), \text{ as required.}
\end{aligned}$$

- **Remark:** You should convince yourself that the base case handles the input word ε correctly.
- We have proved by induction that the two extended transition functions agree on all input words.
- Therefore, as we argued earlier, this shows that the two automata accept precisely the same languages.
- So we are done.

□

4 Lecture 04

Outline

1. Regular Expressions - M3 1-5
2. Regular Languages - M3 6-12

4.1 Regular Expressions

Definition 4.1.1. *Let Σ be a finite alphabet. We construct the* ***regular expressions*** *over Σ (and describe the* ***language*** *which each regular expression represents, i.e. the set of words that fit into the mold defined by the regular expression) recursively, as follows.*

Base

1. *$\emptyset$ is a regular expression, and $L(\emptyset) = \emptyset$.*
2. *ε is a regular expression, and $L(\varepsilon) = \{\varepsilon\}$.*
3. *If $a \in \Sigma$ is any symbol, then a is a regular expression and $L(a) = \{a\}$.*

Induction

1. *If E and F are regular expressions, then $E + F$ is a regular expression, and $L(E + F) = L(E) \cup L(F)$.*
2. *If E and F are regular expressions, then EF is a regular expression, and $L(EF) = L(E)L(F)$.*
3. *If E is a regular expression, then E^* is a regular expression, and $L(E^*) = (L(E))^*$.*
4. *If E is a regular expression, then (E) is a regular expression, and $L((E)) = L(E)$.*

Remarks:

1. The regular expression for $L = \{w\}$ is just w itself.
2. As in algebra, there is an order of operations here:
   (a) **Parentheses** are used to override (or emphasize) the default order as needed.
   (b) *
   (c) **Concatenation**
   (d) $+$

Examples:

1. Stuff.

4.2 Regular Languages

Definition 4.2.1. *A language L is* ***regular*** *if it obeys the following recursive definition:*

Base

1. *$L = \emptyset$ is regular.*
2. *$L = \{\varepsilon\}$ is regular.*
3. *$L = \{a\}$ for some alphabet letter $a \in \Sigma$ is regular.*

Induction

1. $L = L_1 \cup L_2$, *for regular languages* L_1, L_2
2. $L = L_1 L_2$ *for regular languages* L_1, L_2
3. $L = L_1^*$, *for some regular language* L_1

No other languages are regular.

Remarks:

1. It might appear at first glance that the union part of the regular language definition says that every language is regular (every language is a possibly infinite union of 1-word languages, each of which is regular). But infinite unions are not the same as finite unions.
2. There do exist non-regular languages, e.g. $\{0^i 1^i \mid i \geq 0\}$. In L06 we will develop a technique to establish the non-regularity of a language (Pumping Lemma for Regular Languages).
3. Remember that $\emptyset \neq \{\varepsilon\}$!

Theorem 4.2.2. *Every one-word language is regular.*

Proof. See Lecture Slides. □

Theorem 4.2.3. *Every finite language is regular.*

Proof. See Lecture Slides. □

Remarks:

1. The converse of Theorem 4.2.3 is false! For example Σ^* is regular, and this is always infinite.

Examples: (trim down for class; refer to slides for the rest)

1. If $\Sigma = \{0, 1\}$, then $\Sigma^* = \{0, 1\}^*$.
   Regular expression: $(0 + 1)^*$
2. Even-length sequences:
   Can be divided into 2-letter sub-words. $(00 + 11 + 01 + 10)^*$ or $((0 + 1)(0 + 1))^*$
3. Sequences with length at most 3: $(0 + 1)(0 + 1)(0 + 1) + (0 + 1)(0 + 1) + (0 + 1) + \varepsilon$ or $(0 + 1 + \varepsilon)(0 + 1 + \varepsilon)(0 + 1 + \varepsilon)$ or $\varepsilon + 1 + 0 + 11 + 10 + 01 + 00 + 111 + \cdots$
4. Sequences with at most two zeros:
   $1^*(0 + \varepsilon)1^*(0 + \varepsilon)1^*$
5. And lots more.

Rules For Regular Languages (trim down for class; refer to slides for the rest): Basic rules that you can often use (not exciting, but true...):

- $\emptyset e = e\emptyset = \emptyset$ (If $w \in L(\emptyset e), w = w_1 w_2$ where $w_1 \in L(\emptyset)$. But nothing works for w_1.)
- $\emptyset^* = \{\varepsilon\}$ (might be surprising)
- $\{\varepsilon\}^* = \{\varepsilon\}$
- $x + x = x$ (remember, + means union)
- $(x^*)^* = x^*$ (Taking closure twice equals taking closure once)
  - Side Note: an operation for which applying the operation twice equals applying the operation once is called **idempotent**.
  - Other natural examples of idempotents are projections in linear algebra.
- $x(y + z) = xy + xz$

Tons more of these, but we will not focus on them.
The first two might be surprising, and are thus important.

5 Lecture 05

Outline

1. Kleene's Theorem - M3 13-39

5.1 Kleene's Theorem

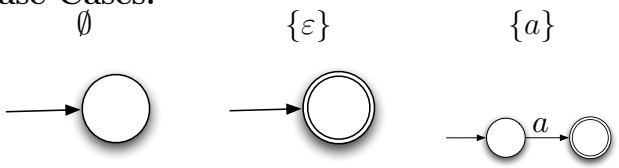
Kleene's Theorem 5.1.1. *Every regular language is the language of a DFA, and every DFA accepts a regular language.*

Remarks about Theorem 5.1.1, Before Its Proof:

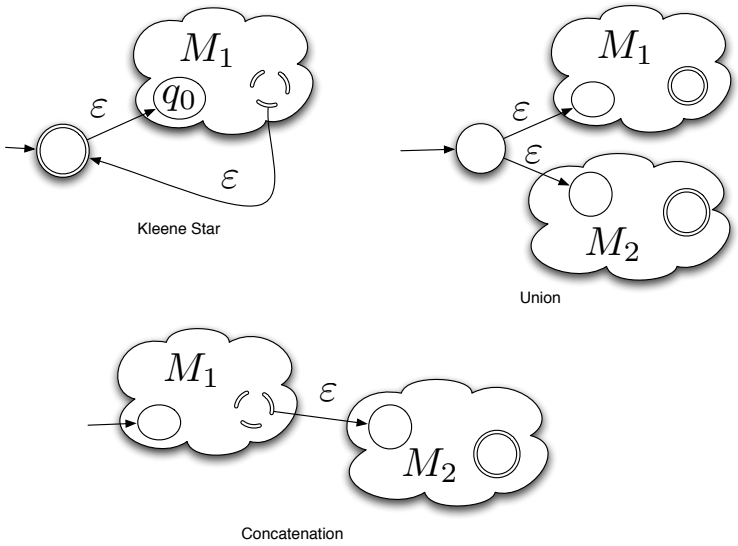
1. Recall: We saw that DFAs are equally powerful to ε-NFAs.
2. Given any regular language, we will construct and ε-NFA which accepts it.
3. We will prove that the language of any DFA is regular.
4. Together these constructions will prove the Theorem.

Forward **To Prove:** For every regular language L, there is an ε-NFA M, where $L(M) = L$. We will show an ε-NFA for the base cases (easy), then prove the other three cases by structural induction (more interesting).

Base Cases:



Inductive Cases:



An argument that this works, for the Kleene * operator:

- Consider the Kleene * construction L_1^* for some regular language L_1.
- Suppose x is accepted by the ε-NFA, M, which we have constructed as above for L_1^* (i.e. suppose $x \in L(M)$).
- Then there is a path in M for x ending in the start state.
- This path can then be broken down into sub-paths, each of which begins and ends in the start state.
- Each sub-path from the start state back to the start state corresponds exactly to a word from L_1 (M_1 accepts exactly L_1, and the only non-trivial path to the accept state of M is via an ε-transition from an accept state of M_1).
- Therefore x is a concatenation of zero or more words in L_1, in other words $x \in L_1^*$. (Note that x can be ε.)
- This shows that $L(M) \subseteq L_1^*$.

Now, suppose that $x \in L_1^*$.

- Then $x = w_1 w_2 w_3 \cdots w_k$, with all $w_i \in L_1$.

- So there is a path from start state in the new machine to an accept state for each of the w_i.
- Join the paths together, following each with the ε-transition back to the start state to get a path for x from the start state back to itself.
- The start state is an accept state, so our new machine accepts x.
- So $x \in L(M)$.
- This shows that $L_1^* \subseteq L(M)$.

Now we are done.

- We have shown that $L(M) \subseteq L_1^*$ and $L_1^* \subseteq L(M)$.
- Therefore we have $L(M) = L_1^*$, as claimed.

The other inductive cases are similar and are left as exercises.

Backward **To Prove:** For every DFA, its language is regular.

One idea: Given a DFA D, we must find a regular expression for its language. Paths through a state can be replaced by the regular expression that represents going from the previous state to the next one. This leads us to the idea of **state removal**. See the Lecture Slides for details. However we will not write our proof this way here.

Recall:

- $L(D) = \{x \in \Sigma^* \mid \hat{\delta}(q_0, x) \in F\} = \bigcup_{r \in F} \{x \in \Sigma^* \mid \hat{\delta}(q_0, x) = r\}$
- Characterize all input strings that take us from some state q to r: Define
  $L(q, r) = \{x \in \Sigma^* \mid \hat{\delta}(q, x) = r\}$.
- With this definition, $L(D) = \bigcup_{r \in F} L(q_0, r)$ (i.e. the set of words that take us from q_0 to any accept state $r \in F$)
- This is a finite union, so if all $L(q_0, r)$ are regular, then so is $L(D)$. (Prove this, by induction on $|F|$!)

What we now want:

Theorem 5.1.2. *Let $D = (\Sigma, Q, q_0, F, \delta)$ be a DFA. Let $q, r \in Q$ be any states in D. Define $L(q, r) = \{x \in \Sigma^* \mid \hat{\delta}(q, x) = r\}$. Then $L(q, r)$ is regular.*

Remarks on the Proof:

- We will prove this using structural induction.
- How do we get from q to r?
- Do we use an intermediate state p?
- If so, we go from q to p, maybe from p to itself, then from p to r.
- One term of the regular expression for $L(q, r)$ might be $L(q, p)L(p, p)^*L(p, r)$.

But structural induction needs a **base case**!

[**Remember**: structural induction goes from "simple" structures to "complex"] How can we make the base case simple enough?

- Restrict the number of states that can come between q and r, and grow this set of states.
- Number the states of the DFA M : $1, \ldots, n$.
- Define $L(q, r, k) = \{$ all words in $L(q, r)$ where all **intermediate** states between q and r are from $1, \ldots, k\}$.
  (Remember, M is a DFA, so each word has only one state path.)

More about the languages $L(q, r, k)$

- If the path from q to r for word x uses state k, then x is not in $L(q, r, k-1)$, or $L(q, r, k-2)$, or any other $L(q, r, i)$ for $i < k$.
- Formally: $L(q, r, k) = \{x \in \Sigma^* \mid \hat{\delta}(q, x) = r$ and $\hat{\delta}(q, w) \leq k$ for all proper prefixes of w of x where $w \neq \varepsilon$ and $w \neq x\}$.
- (Terminology: w is a **proper** prefix of x if w is a prefix of x and not equal to x.)
- Now it is enough to show $L(q, r, n)$ is regular, since $L(q, r) = L(q, r, n)$.
- Then it is enough show that $L(q, r, k)$ is regular for all k.
- The proof is by induction on k.

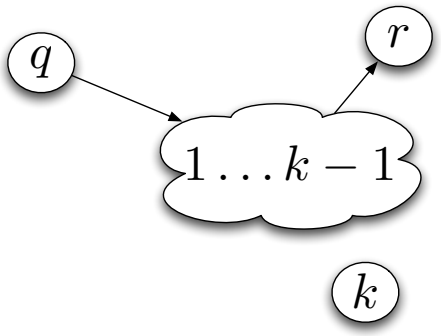
Goal: $L(q, r, k)$ is regular for all k: proof by induction on k.

Proof. Base case: $k = 0$:

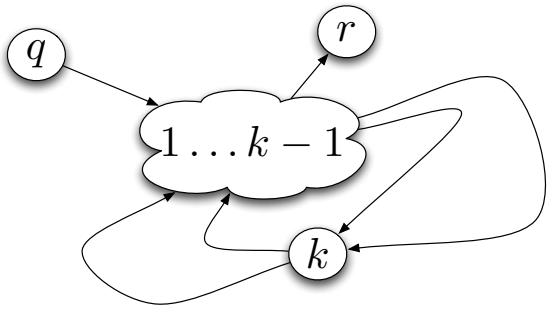
- If $x \in L(q, r, 0)$, then $\hat{\delta}(q, w) \leq 0$ for all proper prefixes w of x.
- But that means there are no proper prefixes of the word x.
- Therefore, $|x| = 0$ or 1.
- So all words in $L(q, r, 0)$ are of length 0 or 1.
- As our alphabet is finite, this implies $L(q, r, 0)$ is finite (and thus regular by Theorem 4.2.3).

Induction step: $k = n \geq 1$:

- Inductive hypothesis: $L(q, r, k-1)$ is regular for all q and r.
- We must show that $L(q, r, k)$ is regular.
  - Let x be a word in $L(q, r, k)$.
  - If the path from q to r for x never touches the state k, then $x \in L(q, r, k-1)$.



- Otherwise: k *is* on the path from q to r for x.
- Then there is a first and a last time we are in state k. Between the first and last time, we are only in states $1, \ldots, k$, regardless of whether $q > k$ or $r > k$.



So we can divide x into:

- The part from q to k the first time,
- The first loop (if any) from k to k,
- The next loop from k to k,
- ...
- The last loop from k to k,
- and the part from k to r.

Words divided in this way are exactly the words in $L(q, r, k)$ that include state k on their state path. This set of words is $L(q, k, k-1)(L(k, k, k-1))^* L(k, r, k-1)$.

So the language $L(q, r, k)$ is the union of two languages:

- $L(q, r, k-1)$, which we know is regular by the induction hypothesis, and
- $L(q, k, k-1)(L(k, k, k-1))^* L(k, r, k-1)$, which is the concatenation of three languages (each of which is regular by the induction hypothesis),

and thus is also regular.

Hence $L(q, r, k)$ is regular. We are done, but that may not be obvious yet.

- We proved that $L(q, r, k)$ is regular for all k.
- We noted that $L(q, r) = L(q, r, n)$, so $L(q, r)$ is always regular for any $q, r \in Q$.
- Therefore $L(q_0, r)$ is regular for any $r \in F$.
- Thus $\bigcup_{r \in F} L(q_0, r)$ is regular (as it is the union of a finite number of regular languages).
- But $L(M) = \bigcup_{r \in F} L(q_0, r)$ is therefore regular.

□

So $L(M)$ is regular! (Again, how long is the expression for $L(q_0, r)$?)

This is the end of the proof of Kleene's Theorem:

- Given a DFA, we have shown that its language is regular.
- Given a regular language, we can produce an ε-NFA which recognizes it.
- NFAs and ε-NFAs have the same computing power as DFAs.

Next module: the boundaries of regular languages, and closure rules for them.

6 Lecture 06

Outline

1. Pumping Lemma for Regular Languages - M4 1-20

6.1 Pumping Lemma for Regular Languages

Non-regular languages By Kleene's Theorem, a language L is **not** regular if for every DFA M, $L \neq L(M)$. How does a DFA M work?

- Suppose M has n states.
- Consider a word $x \in L(M)$ with $|x| \geq n$.
- On its path from q_0 to an accept state, it must repeat a state somewhere along the path.
- **Reason:** There are only n states in total, and the machine starts out in one of them by default, then reads $\geq n$ input characters. (Recall the **pigeonhole principle**.)
- Let's say that we repeat state r.

- Then the word x can be decomposed: $x = uvw$, where:
  - u = the part from q_0 to the first time we reach r (i.e. after processing u, we are in state r).
  - v = the loop from r to itself (i.e. after processing v, we are again in state r).
  - w = The part from the second time we reach r that leads us to an accept state
  - **Note:** it is possible that either u or w is ε, but v cannot be ε.
- This decomposition is possible for **any** word $x \in L(M)$ with $|x| \geq n$.
- **Fact**: $uvvw$ is also in $L(M)$. Why?
  - vv also takes M from r back to itself:

$$\hat{\delta}(r, vv) \underbrace{=}_{\text{A01}} \hat{\delta}(\hat{\delta}(r, v), v) = \hat{\delta}(r, v) = r.$$

- Another word in $L(M)$ is $uw = uv^0w$.
- By induction, $uv^iw \in L(M)$, for all $i \in \mathbb{N}$.
- If we choose the **first** time a state is repeated, then $|uv| \leq n$.
  - **Reason:** The machine has n states, so we must have the first repeated state by the n^{th} step.)
- And $|v| \geq 1$, since it is a DFA, and therefore has no ε-transitions.
- We can "pump out" many copies of v, for example $uvvvvvvvvvvw$ is also part of $L(M)$.

Pumping Lemma For Regular Languages 6.1.1. *For every regular language L, there **exists** some positive integer n such that all words $x \in L$ with $|x| \geq n$ can be decomposed into $x = uvw$, where:*

- $|uv| \leq n$,
- $v \neq \varepsilon$, *and*
- $uv^iw \in L$ *for all non-negative integers i.*

Remarks:

1. Think of n as being the number of states in a smallest DFA, M, accepting L.
2. This describes one feature of all long words in a regular language:
   (a) For some definition of "long", **all long words can be pumped**.
   (b) Note that, if L is finite (and therefore regular), then taking any $n > \max_{x \in L}\{|x|\}$ works (because with such an n, L contains **no** long words).

We know something about regular languages: all long words can be pumped. This provides us the following technique for proving that a language cannot be regular.

Technique for Proving that a Language Cannot Be Regular

Let L be a language. Suppose that for **any** positive integer n:

1. There exists a word $x \in L$ with $|x| \geq n$ such that
2. for **any** decomposition of x into $x = uvw$, with $|uv| \leq n$ and $v \neq \varepsilon$,
3. $uv^*w \not\subseteq L$.

Then L is **not** a regular language.

The Same Technique, Explained in English

1. "Suppose that for any value of $n > 0$, there exists a word $x \in L$ with $|x| \geq n$"... *(If there is always a long word in L)*
2. "such that for any decomposition of x into $x = uvw$, with $|uv| \leq n$ and $v \neq \varepsilon$" ... *(that cannot be decomposed into three parts where the first 2 parts are not long and the middle part is non-trivial)*
3. "$uv^*w \not\subseteq L$."... *(and the second part cannot be pumped,)*
4. Then L is not a regular language.

An example

Theorem 6.1.2. *$L = \{0^i1^i | i \geq 0\} = \{\varepsilon, 01, 0011, 000111, 00001111, \ldots\}$ is not a regular language.*

Proof.

- For any $n > 0$, choose a word $x \in L$ whose length is at least n.
- We will choose $x = 0^n1^n$. This is our **long word**.
- Now, consider all decompositions $x = uvw$, where $|uv| \leq n$, and $v \neq \varepsilon$.
- **Fact**: for any such decomposition, $uv = 0^k$ for some $0 < k \leq n$, because the first n characters of $x = uvw$ are all 0 (by the definition of x).
- Now, we must show that because of what we found, uv^*w is not a subset of L. In particular, we must find an $i \geq 0$ such that $uv^iw \notin L$. (Typically, $i = 0$ or $i = 2$.)
- Let $i = 0$. Recall that v is all 0's. Then uv^0w will have fewer 0's than 1's. So $uv^0w \notin L$.
- And hence the language L is not regular.

□

Again, how did that work?

Pumping lemma: to prove languages are not regular.

- For any definition of **long**, find a long word:
  Long: length $\geq n$. Our long word was $x = 0^n1^n$.

- Consider all breakdowns of x into $x = uvw$, where uv is short and $v \neq \varepsilon$.
  For the long word x, if $x = uvw$, and uv is short, then uv is all 0's.
- If for all of these breakdowns $x = uvw$, we cannot pump v, then L is not regular.
  No matter what v is, it must be all 0's. So if we pump v, then $uvvw$ or uw both have the wrong number of 0's. So L is not regular.
- We can also prove L is not regular by thinking of possible DFAs for L and showing that they cannot exist.
- This is hard in general. The Pumping Lemma is better.

Another example

Theorem 6.1.3. *The language $L = \{0^p | p$ is a prime $\}$ is not regular.*

Why can we not pump the primes?

Proof.
- (This language includes $00, 000, 00000, 0000000, 00000000000, \ldots$)
- Proof by Pumping Lemma. (Assume that there are infinitely many primes. There are many nice proofs of this fact.)
  - Choose a value of $n > 0$.
  - Choose $x = 0^p$, for a prime $p \geq n$.
  - Then x is a long word in L.
  - Now we must argue that no decomposition of x can be pumped.

  Consider all decompositions $x = uvw$, where $|uv| \leq n$ and $v \neq \varepsilon$.
  - Then $v = 0^k$ for some $1 \leq k \leq n$.
  - And $uv^*w = \{0^{p-k}, 0^p, 0^{p+k}, 0^{p+2k}, \ldots\}$.
  - Is it possible that all of these are in L?
  - No. One member of uv^*w is $0^{p+(pk)}$; it is the $(p+2)^{th}$ member in the above list.
  - This word is not a member of L, since $p+pk = (1+k)p$ is composite (both factors are non-trivial, as $k \geq 1$).
- For any n, we can find a long word, such that all decompositions of it cannot be pumped. Therefore L is not regular.

□

Another example: palindromes

Theorem 6.1.4. *$L = \{s \mid s = s^R\}$, the language of palindromes, is not regular.*

Remarks:

1. Examples of palindromes: $0110, 01110, \varepsilon, 1111$, etc.

Proof. Proof by Pumping Lemma.

- Given a value of $n > 0$, find a word in L of length at least n.
- How about $x = 0^n 10^n$?
- Now, consider all decompositions of this into $x = uvw$, where uv is short and v is not ε.
- Again, v must be 0^i for some $1 \leq i \leq n$.
- And the number of 0's before the only 1 in uv^2w is more than the number after it, so it cannot be a palindrome.
- So we cannot pump x, regardless of our choice of decomposition.
- So L is not regular.

□

One more example

Theorem 6.1.5. *Let* $L = \{y!z \mid |y| > |z|, y, z \in \{0,1\}^*\}$, *over* $\Sigma = \{0, 1, !\}$. *Then* L *is not regular.*

Remarks:

1. This language includes words like $111!00, 1!, 10001!111$.

Proof. Proof by Pumping Lemma.

- Consider a value $n > 0$.
- The string $x = 0^n!0^{n-1}$ is long, and in L.
- We will show that uv^0w is not in the language.
  - Decompose $x = uvw$ with uv of length at most n and nonempty v.
  - For all such decompositions, $v = 0^k$ for some $k \geq 1$.
  - And $uv^0w = 0^{n-k}!0^{n-1}$.
  - This is not a word in L: the part before the ! character is too short.
  - So v is not pumpable, no matter how we do it.
- L is not regular.

□

What can go wrong?

It is easy to misuse the Pumping Lemma.

- The existence of one bad decomposition of x does not matter.

- We must show that **all** decompositions of $x = uvw$ with $|uv| \leq n$ and $v \neq \varepsilon$ cannot be pumped.

Example:

- Obviously, $L = (01)^*$ is regular.
- For any value of $n > 0$, $(01)^n$ is a long word in L.
- Decompose into $u = 0, v = 1, w = (01)^{n-1}$.
- Then $uv^2w = 011(01)^{n-1}$ is not in L.
- So we conclude that L is not regular (?!?!?)

Clearly we have done something wrong!

- Problem: We must show that **no** decomposition can be pumped.
- The decomposition $u = \varepsilon, v = 01, w = (01)^{n-1}$ is pumpable.

More Pumping Lemma: pitfalls

- The Pumping Lemma:
  - Long words in regular languages can be pumped.
- Its contrapositive:
  - If a language has long words that cannot be pumped, it is not regular.
- **Note**: the theorem does not give a definition of regular languages. The following is **not** true:
  - If all long words in a language can be pumped, it is regular.
- In fact, some non-regular languages can be pumped.

7 Lecture 07

Outline

1. Closure Rules for Regular Languages - M4 21-28
2. Decision Problems for Regular Languages - M4 29-36

7.1 Closure Rules for Regular Languages

Definition 7.1.1. *1. A class of languages is **closed** under a binary operation if applying that operation to two languages in the class always yields a language in the class*

*2. A class of languages is **closed** under a unary operation if applying that operation to one language in the class always yields a language in the class.*

Theorem 7.1.2. *Regular languages are closed under $*$, **finite** union and **finite** concatenation.*

Proof. Obvious, by the Definition 4.2.1. □

Remarks:

1. Subsets of regular languages are **not necessarily regular**: $(0+1)^* = \Sigma^*$ is regular, so any language over $\Sigma = \{0, 1\}$ is the subset of a regular language!
2. We just saw examples of languages over Σ which are not regular.

Theorem 7.1.3. *If language L is regular, then so is its complement, L'.*

Proof. Proof using Kleene's Theorem 5.1.1.

- Since L is regular, it is the language of some DFA, M, with state set Q and accept states $F \subseteq Q$.
- Construct a new DFA, M' from M, by swapping the accept and reject states in M.
- Then M', with accept set $Q \setminus F$ accepts all words which the old DFA, M, rejected and rejects all words which M accepted.
- M is a DFA, so there is only one path for a particular word.
- The same is true for M', so M' is a new DFA.
- M' accepts L'.
- By Theorem 5.1.1, L' is regular.

□

Remarks:

1. This is much easier than constructing a regular expression for L' from a regular expression for L.

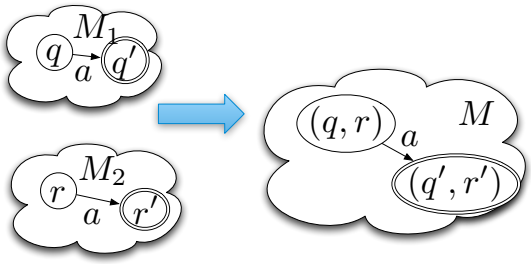
Theorem 7.1.4. *If L_1 and L_2 are regular languages, then so is $L_1 \cap L_2$.*

Remarks:

1. In other words, regular languages are closed under intersection.
2. This can be proved in lots of ways.
3. One easy way: $L_1 \cap L_2 = (L_1' \cup L_2')'$. (That can take a couple seconds to understand! Draw a suitable Venn diagram if it helps.)
4. And L_1' is regular (by Theorem 7.1.3); so is L_2'.
5. By Definition 4.2.1, so is $L_1' \cup L_2'$.
6. And again, by our Theorem 7.1.3, so is $(L_1' \cup L_2')' = L_1 \cap L_2$.

Proof. **Strategy: by building a DFA** Let M_1 and M_2 be DFAs accepting L_1 and L_2 respectively. We shall construct a new DFA, M, which simulates processing each alphabet symbol, a, through M_1 and M_2 in parallel. Define M using the ingredients

- Σ is the same as for L_1, L_2.
- $Q = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\}$
- $F = \{(f_1, f_2) \mid f_1 \in F_1, f_2 \in F_2\}$
- $q_0 = (q_{1_0}, q_{2_0})$
- $\delta : \delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$



□

Closure under reversal Recall: w^R is the **reversal** of the word w.
Given a language L, let L^R be the language that consists of all of the words of L, reversed.

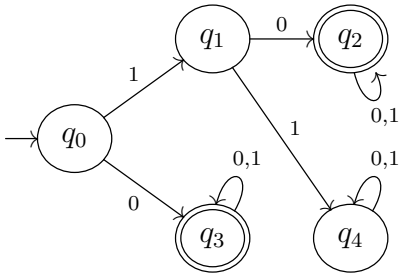
Theorem 7.1.5. *If L is regular, then so is L^R.*

Proof idea (we won't make it rigourous): Let M be a finite automaton whose language is L. Make a new finite automaton R with the same states as M, plus a new start state:

- All of the edges of R are the reversals of the edges of M.
- The sole accept state of R is the start state of M.
- The start state of R has an ε-transition to each accept state of M.

Then R reverses the automaton M: if we start at an accept state of M and work our way back to the start state of M (i.e. if M accepted x), then the new machine R accepts the word x^R, and vice versa.

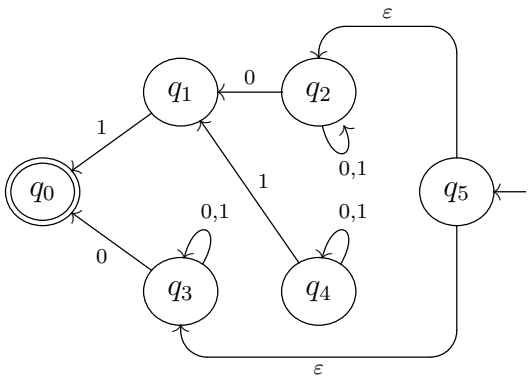
Closure under reversal Suppose we have the following DFA, M



to accept the language

$$L = \{w \mid w \text{ begins with 0 or with 10}\}.$$

Closure under reversal The construction yields this ε-NFA, R



to accept the language

$$L = \{w \mid w \text{ ends with 0 or with 01}\}.$$

Note that, although this ε-NFA R can be simplified, the construction is still correct.

Proof. We can prove this theorem by structural induction on the construction of the regular language L.

- Base cases: If $L = \emptyset, \{\varepsilon\}$ or $\{a\}$,then $L^R = L$ is regular.
- Induction cases:
  - If $L = L_1 \cup L_2$ for regular languages L_1 and L_2, then $L^R = L_1^R \cup L_2^R$, and both of L_1^R and L_2^R are regular by induction.

- If $L = L_1 L_2$ for regular languages L_1 and L_2, then $L^R = L_2^R L_1^R$, and this is the concatenation of the regular languages L_2^R and L_1^R (by induction), and hence is regular.
- If $L = L_1^*$, then
  * We want to prove that $L^R = (L_1^*)^R$ is regular.
  * The induction hypothesis is that L_1^R is regular.
  * Therefore by the definition of regular languages, it follows that $(L_1^R)^*$ is regular.
  * I claim that $(L_1^*)^R = (L_1^R)^*$. Proving the claim will complete this case.
  * For an arbitrary word w, we have

$$\begin{aligned} & w \in (L_1^*)^R \\ \Leftrightarrow \quad & w = (w_1 \cdots w_n)^R, \text{ for some } w_i s \in L_1 \\ \Leftrightarrow \quad & w = w_n^R \cdots w_1^R, \text{ for some } w_i s \in L_1 \\ \Leftrightarrow \quad & w \in (L_1^R)^*, \text{ which establishes the claim.} \end{aligned}$$

□

7.2 Decision Problems for Regular Languages

Algorithmic questions about finite automata We do not normally create algorithms in this class. Here is an exception:
Is it possible to find algorithms for the following:

1. Given a DFA M and a word x, does M accept x?
2. Given a DFA M, is $L(M)$ empty?
3. Given a DFA M, is $L(M)$ infinite?
4. Given two DFAs M_1 and M_2, is $L(M_1) \cap L(M_2)$ empty?
5. Given two DFAs M_1 and M_2, is $L(M_1) \subseteq L(M_2)$?
6. Given two DFAs M_1 and M_2, is $L(M_1) = L(M_2)$?
7. Given two regular expressions e_1 and e_2, do they generate the same language?

- Given a DFA M and a word x, does M accept x?
  - Just simulate the DFA.
  - (This may seem obvious. But we will not be able to do this for Turing machines.)
- Given a DFA M, is $L(M)$ empty?
  More fun.

Lemma 7.2.1. *If a DFA, M, having n states accepts any words, then it must accept a word with fewer than n letters.*

Proof.
- Assume $L(M) \neq \emptyset$.
- By Kleene's Theorem, $L(M)$ is regular.
- Let $x_0 \in L(M)$ be arbitrary.
- If $|x_0| < n$, then we are finished.
- Otherwise, $|x_0| \geq n$ and by the proof of the Pumping Lemma, we can decompose $x_0 = u_0 v_0 w_0$, with $u_0 w_0 \in L(M)$ and $|v_0| \geq 1$.
- If $|u_0 w_0| < n$, then we are finished.
- Otherwise, $|u_0 w_0| \geq n$ and $u_0 w_0 \in L(M)$ and so by the proof of the Pumping Lemma, we can decompose $u_0 w_0 = u_1 v_1 w_1$, with $u_1 w_1 \in L(M)$ and $|v_1| \geq 1$.
- Continuing in this way we obtain a sequence of words in $L(M)$ having strictly decreasing lengths: $x_0, u_0 w_0, u_1 w_1, \ldots, u_j w_j, \ldots$.
- As x_0 has finite length, after at most $|x_0| - n + 1$ steps, we will obtain a word in $L(M)$ with length $< n$. □

□

Now, back to the question
- Given a DFA M, is $L(M)$ empty?
  - Try **every** word of length less than n (finitely many since our alphabet is finite).
  - If no short word is accepted, then by Lemma 7.2.1, $L(M) = \emptyset$.

Is the language of an FA finite?
- Given a DFA M, is $L(M)$ infinite?

Theorem 7.2.2. *If M is a DFA with n states, then $L(M)$ is infinite if and only if $L(M)$ includes a word x satisfying $n \leq |x| < 2n$.*

Proof.
- Suppose $x \in L(M)$ and $n \leq |x| < 2n$.
- From the Pumping Lemma, x must be pumpable.
- The word $x = uvw$ can be used to generate the infinite language uv^*w, which is a subset of $L(M)$.
- So $L(M)$ is infinite.

Other direction: Assume that $L(M)$ is infinite.
- For a contradiction, suppose that there does not exist any $x \in L(M)$ satisfying $n \leq |x| < 2n$.

- In other words, every word $x \in L(M)$ with length at least n must have length at least $2n$.
- Let $x \in L(M)$ be a shortest word with length at least n (so that $|x| \geq 2n$ by the above point).
- (If there is no $x \in L(M)$ with length at least n, then $L(M)$ is finite, which cannot happen.)
- Decompose $x = uvw$, where $v \neq \varepsilon$ and $|uv| \leq n$, so that $1 \leq |v| \leq n$.
- By the Pumping Lemma, uw is also in $L(M)$.
- We have only removed at most n letters by removing v, so $|uw| \geq n$, and by construction, $|uw| < |x|$.
- Thus uw violates the choice of x as a shortest word in $L(M)$ with length at least n.
- This contradiction completes the proof.

□

- If $L(M)$ is infinite, we must have a word in $L(M)$ with length between n and $2n$.
- To see if $L(M)$ is infinite, check all words between n and $2n$ in length.
- This runs in a finite amount of time.

Disjoint languages, subset

- Given two DFAs M_1 and M_2, is $L(M_1) \cap L(M_2)$ empty?
  - First, construct a DFA for $L(M_1) \cap L(M_2)$.
  - Then use the algorithm for testing for an empty language of an FA to see if it accepts the empty language.
- Given two DFAs M_1 and M_2, is $L(M_1) \subseteq L(M_2)$?
  - If so, then $L(M_2)' \cap L(M_1)$ is empty.
    (There is nothing in $L(M_1)$ that is not in $L(M_2)$.)
  - Build the DFA for $L(M_2)'$.
  - Use it to build the DFA for $L(M_2)' \cap L(M_1)$.
  - Use the algorithm from before to test if its language is empty!

Two FAs with the same language

- Given two DFAs M_1 and M_2, is $L(M_1) = L(M_2)$?
  - Yes, if $L(M_1) \subseteq L(M_2)$ and $L(M_2) \subseteq L(M_1)$.
  - Use the algorithm for testing for subset twice.
- Given two regular expressions e_1 and e_2, do they represent the same language?
  - Construct the DFAs for each regular expression, using Kleene's Theorem.

- Then use the algorithm for testing if two FAs have the same language.

(If you like this kind of stuff, take CS 462.)

8 Lecture 08

Outline

1. Context-Free Grammars and Languages - M5 1-25

8.1 Context-Free Grammars and Languages

Definition 8.1.1. *a **context-free grammar** (CFG) is a 4-tuple: $G = (V, T, P, S)$, where*

- *V is a finite set of **variables**, usually denoted by capital letters.*
- *T is a finite alphabet, called **terminals**. T is disjoint from V. T is the alphabet for the CFG's **language**.*
- *P is a finite set of **productions**, which are definitions for the variables.*
- *Each production P is of the form $A \to \alpha$, where*
  - *A is a variable, and*
  - *α is a string of symbols from V and T, therefore $\alpha \in (V \cup T)^*$, OR α may equal ε.*
  - ***Notation**: If we have productions $A \to \alpha$ and $A \to \beta$, shorthand these two statements as $A \to \alpha \mid \beta$.*
- *S = **Start variable**: the symbol from V where the derivation of a word begins.*

An Example, which generates $L(0^*1^*)$ (Exercise - prove it!):
$G = (V, T, P, S)$, where

- $V = \{A, B, S\}$
- $T = \{0, 1\}$
- $P = \{S \to AB, A \to 0A \mid \varepsilon, B \to B1 \mid \varepsilon\}$
- $S = S$

Goal: Define what it means for a word to be in the language of a CFG.

Definition 8.1.2. *We write $\alpha \underset{G}{\Rightarrow} \beta$ if β can be produced from α via a single derivation in G.*

For example, with the above G, $00AB1 \underset{G}{\Rightarrow} 00AB11$.

Definition 8.1.3.
- *Suppose that there is a sequence of strings* $\gamma_1, \gamma_2, \ldots, \gamma_k$ *for some* $k \geq 1$ *such that:*
  - $\alpha = \gamma_1$,
  - $\beta = \gamma_k$,
  - *For all* i *from 2 to* k, $\gamma_{i-1} \underset{G}{\Rightarrow} \gamma_i$.
- *Then, we can say that* $\alpha \overset{*}{\underset{G}{\Rightarrow}} \beta$.

Example: Using the above G,

$$S \Rightarrow AB \Rightarrow 0AB \Rightarrow 00AB \Rightarrow 00\varepsilon B \Rightarrow 00B1 \Rightarrow 00\varepsilon 1.$$

This derivation says that $S \overset{*}{\underset{G}{\Rightarrow}} 001$.

Remarks on the Example:

1. We know the derivation is fiished when it contains only terminals, no variables.
2. This derivation uses the **leftmost rule** (the leftmost variable must be resolved first before moving on to other variables).

Leftmost and rightmost derivations Our convention will be to use leftmost derivations.

Language of a grammar

Definition 8.1.4. *Given a context-free grammar* $G = (V, T, P, S)$*, its* ***language*** *is:*

$$L(G) = \left\{ w \in T^* \mid S \overset{*}{\underset{G}{\Rightarrow}} w \right\}.$$

In English, $L(G)$ *is the set of words that we can derive from the start variable* S*, with no variables from* V *remaining in them.*
A language L *is* ***context free*** *if it is the language of some context-free grammar* G*.*

Clarifying the leftmost rule: Let $G : S \to AB, A \to 0A \mid \varepsilon, B \to 01$. Then we have

$$S \Rightarrow AB \Rightarrow \varepsilon B \Rightarrow \varepsilon 01$$

We cannot produce 01 using the other rule for A first. But the leftmost rule forces us to resolve A, somehow, before we resolve B. Note that the leftmost rule does not dictate **which** production for A we choose.

Why context-free languages?

- It does not matter where we see a symbol.

- Suppose P includes this production: $A \to ab$
- Given any string x_1Ax_2, we can change the A into ab.
- That is, we can do this substitution regardless of the **context** of A, which is what x_1 and x_2 are.
- Think about mad-libs from when you were a kid; they were funny because you were making sentences that were silly. They did not have context.
- Proper English (or indeed, any human language) has context.

An example $G = (V, T, P, S)$, where

- $V = \{S\}$,
- $T = \{0, 1\}$,
- $P = \{S \to 0S1 \mid S1 \mid \varepsilon\}$, and
- $S = S$.

We do not need to specify V, T or S when they are obvious.

- $G : S \to 0S1 \mid S1 \mid \varepsilon$.

The language of this CFG Let $L = \{0^i1^j$, where $i \leq j\}$

- Proof using the Pumping Lemma that L is not regular:
  - Let $n > 0$.
  - Let $x = 0^n1^n \in L$ be our long word.
  - Write $x = uvw$, where $|uv| \leq n$ and $v \neq \varepsilon$.
  - Then $v = 0^k$ for some $1 \leq k \leq n$, so $uv^2w \notin L$.

Theorem 8.1.5. *Let* $G : S \to 0S1 \mid S1 \mid \varepsilon$. *Let* $L = \{0^i1^j$, *where* $i \leq j\}$. *Then* $L(G) = L$.

Proof. To prove the Theorem, we must show $L \subseteq L(G)$ and $L(G) \subseteq L$.

Proof that $L \subseteq L(G)$**:** Let $x \in L$ be arbitrary. We must show x can be generated by G. The proof is by induction on $|x|$.

- Base case: $|x| = 0$. But then $x = \varepsilon$, and $S \to \varepsilon$ is a rule in the grammar G. Therefore we have $S \stackrel{*}{\Rightarrow} x$. So the base case holds.
- Inductive case: Let $|x| = n \geq 1$. Then $x \neq \varepsilon$. Write $x = 0^i1^j$, for some $i \leq j$.
- The induction hypothesis is that, for any $w \in L$, such that $|w| < |x|$, we have that $w \in L(G)$, i.e. that $S \stackrel{*}{\Rightarrow} w$.
- Since $x \in L$, we know that $x = 0^i1^j$ for some $i \leq j$.

We have these two cases for i:

1. $i = 0$, so $x = 1^j = 1^{j-1}1$ ($j \geq 1$ as we are not in the base case).
   - Then $1^{j-1} \in L$ by the definition of L.

- As $|w| < |x|$, we have that $w \in L(G)$ by the induction hypothesis.
- So $S \stackrel{*}{\Rightarrow} w$.
- Then derive x in G via

$$S \Rightarrow S1 \stackrel{*}{\Rightarrow} (1^{j-})1 = x.$$

- This shows that $x \in L(G)$ in this case.

2. $x = 0^i 1^j$, and $i > 0$.
   - Then $0^{i-1}1^{j-1}$ is also in L, as $i \leq j$ implies $i - 1 \leq j - 1$.
   - $w \in L(G)$ by the induction hypothesis.
   - So $S \stackrel{*}{\Rightarrow} w$.
   - Then derive x in G via

$$S \Rightarrow 0S1 \stackrel{*}{\Rightarrow} 0(0^{i-1}1^{j-1})1 = x.$$

   - This shows that $x \in L(G)$ in this case.

In either case, $x \in L(G)$. This finishes the proof that $L \subseteq L(G)$.

Proof that $L(G) \subseteq L$**:** Let $x \in L(G)$ be arbitrary. The proof is by induction on k, the number of steps in the derivation of x.

- Base case ($k = 1$): Because of the shape of G, the only 1-step derivation which produces a word is $S \to \varepsilon$. Therefore $x = \varepsilon$, which is a member of L. So the base case holds.
- Inductive case ($k > 1$): I.H.: every word $w \in L(G)$ derived in $< k$ steps is in L.
  - As we are not in the base case, the first step in the derivation of x must be $S \to 0S1$ or $S \to S1$.
  - If the first step is $S \to 0S1$,
    - Write $x = 0w1$ where $S \stackrel{*}{\Rightarrow} w$ in $< k$ steps.
    - By the induction hypothesis, $w \in L$.
    - Write $w = 0^i 1^j$, for some $i \leq j$.
    - Then $x = 0w1 = 0(0^i 1^j)1 = 0^{i+1}1^{j+1} \in L$.
  - If the first step is $S \to S1$,
    - Write $x = w1$ where $S \stackrel{*}{\Rightarrow} w$ in $< k$ steps.
    - By the induction hypothesis, $w \in L$.
    - Write $w = 0^i 1^j$, for some $i \leq j$ ($i \leq j$ implies $i \leq j + 1$).
    - Then $x = w1 = (0^i 1^j)1 = 0^i 1^{j+1} \in L$.
- In either case, $x \in L$.
- This finishes the proof that $L(G) \subseteq L$.

Put them together: $L(G) \subseteq L$ and $L \subseteq L(G)$. So $L = L(G)$ as claimed. □

This take a bit of care to set up, but theorems like this one are not hard to prove.

Another Example of a CFL

Theorem 8.1.6. *Let* $L = \{w \in \{0,1\}^* \mid n_0(w) = n_1(w)\}$.

- L *consists of all binary strings with as many* 0*s as* 1*s.*

Then L *is context free.*

Proof. We will show that L is the language of the grammar: $G = (V, T, P, S)$, where

- $V = \{S\}$,
- $T = \{0, 1\}$,
- $P = \{S \to \varepsilon \mid 0S1 \mid 1S0 \mid SS\}$, and
- $S = S$.

To prove this, we must show that the languages are subsets of each other, in both directions.

Proof that $L(G) \subseteq L$:

- Let $w \in L(G)$ be arbitrary, i.e. assume $S \overset{n}{\Rightarrow} w$, for some n.
- The proof is by induction on n.
- Base case ($n = 1$): Then $w = \varepsilon$ (the only one-step derivation in G is $S \to \varepsilon$). Note that $\varepsilon \in L$, so the base case holds.
- Inductive case ($n \geq 2$): The induction hypothesis is that for every $x \in L(G)$ which is derived in fewer than n steps, we have $x \in L$.

Inductive step in one direction Consider the first step in a derivation of w. We have these cases (which are exhaustive, as we are no longer in the base case):

- $\underline{S \to 0S1}$: Write $w = 0x1$, where $S \overset{n-1}{\Rightarrow} x$. The induction hypothesis applies to x, so $n_0(x) = n_1(x)$. Hence, by construction, $n_0(w) = n_1(w)$ too. Thus $w \in L$.
- $\underline{S \to 1S0}$: Write $w = 1x0$, where $S \overset{n-1}{\Rightarrow} x$. The induction hypothesis applies to x, so $n_0(x) = n_1(x)$. Hence, by construction, $n_0(w) = n_1(w)$ too. Thus $w \in L$.
- $\underline{S \to SS}$: Write $w = xy$, where $S \overset{*}{\Rightarrow} x$ and $S \overset{*}{\Rightarrow} y$. The induction hypothesis applies to x, so $n_0(x) = n_1(x)$. The induction hypothesis also applies to y, so $n_0(y) = n_1(y)$. Hence, by construction, $n_0(w) = n_1(w)$ too. Thus $w \in L$.

We have shown that in all cases, if $w \in L(G)$ then $w \in L$. Thus the containment $L(G) \subseteq L$ is established.

Proof that $L \subseteq L(G)$: Let $w \in L$ be arbitrary. We must show that $w \in L(G)$. The proof is by induction on $|w|$.

- Base case ($|w| = 0$): Then $w = \varepsilon$. Then $w \in L(G)$ via the production $S \to \varepsilon$ in G.
- Inductive case ($|w| \geq 1$): The induction hypothesis is that for all words $x \in L$ with $|x| < |w|$, we have $x \in L(G)$.
- Note that all words in L have even length, since they have equal numbers of 0s and 1s.
- We will have four cases, depending on the two outside letters of w.

Four cases of the induction proof

1. $w = 0x1$.
   - Since $w \in L$, therefore $n_0(w) = n_1(w)$ by the definition of L.
   - But then $n_0(x) = n_1(x)$ by construction, and so $x \in L$.
   - Also by construction, $|x| < |w|$.
   - Thus by the induction hypothesis, $x \in L(G)$, i.e. $S \stackrel{*}{\Rightarrow} x$.
   - Then we can derive w in G via

$$S \Rightarrow 0S1 \stackrel{*}{\Rightarrow} 0x1 = w, \text{ so } w \in L(G).$$

2. $w = 1x0$. Same argument, instead using the rule $S \to 1S0$ to start.
3. $w = 0x0$. We will use the rule $S \to SS$ here to produce w (see below).
4. $w = 1x1$. Here, we will use the rule $S \to SS$ analogously (see below).

For cases 3 and 4, must show that w can be decomposed into two parts $w = yz$, both of which are in $L(G)$.

Finishing cases 3 and 4 Claim: If $w = 0x0$ and $n_0(w) = n_1(w)$, then we can write $w = yz$, where $n_0(y) = n_1(y)$, $y \neq \varepsilon$, and $z \neq \varepsilon$.

Why is this useful?

- Decompose w into two parts that are both shorter and in L.
- By the inductive hypothesis, they are then both also in $L(G)$.
- Use the $S \to SS$ rule to start the derivation.

Why is it true?

- Basic idea: look at the balance between the number of 1s and 0s in prefixes of w.
- The 1-letter prefix of w is 0, which has one more 0 than 1s.
- The $(|w| - 1)$-letter prefix of w is $0x$, which has one fewer 0 than 1s. (Why? We will wind up evening out at the last letter, which is 0.)

- This balance between 0s and 1s shifts by 1 each letter. It eventually goes from $+1$ to -1.
- So at some point strictly in between, the balance is zero.

End of cases 3 and 4 Definitions:

- Let p_i be the i-letter prefix of w.
- Let $b_i = n_0(p_i) - n_1(p_i)$.

With this in mind:

- $b_1 = 1$, since $p_1 = 0$.
- $b_{|w|-1} = -1$, since $p_{|w|-1} = 0x$, where $0x0 \in L$, and
- $b_{|w|} = 0$, since $p_{|w|} = 0x0 \in L$.
- For any i, $b_i = b_{i-1} \pm 1$, depending on whether the i^{th} character is 1 or 0.
- Since we must go from $+1$ to -1 by steps of 1, there must be some i satisfying $1 < i < |w| - 1$ such that $b_i = 0$.
- Decompose $w = yz$, then, taking y to be the i-letter prefix of w and z the rest of w.
- The two substrings y and z are both shorter (and both in L by construction), so we can use $S \to SS \stackrel{*}{\Rightarrow} Sz \stackrel{*}{\Rightarrow} yz = w$ as our derivation for w.

Thus, $w \in L(G)$. (This works for Case 4 also, swapping 0 and 1.) □

We are done!

- We were given the language $L = \{w \in \{0,1\}^* \mid n_0(w) = n_1(w)\}$, and told to show it is context free.
- We gave a grammar G, and asserted that $L(G) = L$.
  - We showed that $L(G) \subseteq L$, by showing that any word derived in G had an equal number of 0's and 1's.
  - We showed that $L \subseteq L(G)$, by showing that any word with an equal number of 0's and 1's could be derived in G.
- Hence, L is context free.
- (You can easily prove that L is not regular, using the pumping lemma on 0^n1^n.)

9 Lecture 09

Outline

1. Parse Trees - M5 26-33

2. Ambiguity in Context-Free Grammars - Definitions - M5 34-43

9.1 Parse Trees

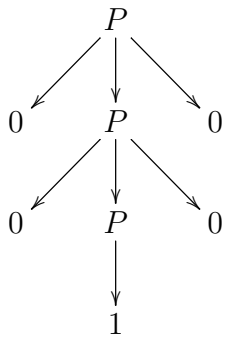
Key Idea: A **parse tree** is a visual presentation of a derivation.
Example: Consider this grammar (which generates palindromes) from the slides:

$$G : P \to \varepsilon \mid 0 \mid 1 \mid 0P0 \mid 1P1.$$

Exercise: Give a rigourous proof that $L(G)$ is actually the language of palindromes.
The derivation in this grammar

$$P \Rightarrow 0P0 \Rightarrow 00P00 \Rightarrow 00100$$

is represented by the parse tree



Parts of the Tree:

- **Root** of the tree: A variable in the grammar
- **Internal nodes** of the tree: variables generated by productions in the grammar
- **Leaves** of the tree: variables, terminals, or ε, generated by productions in the grammar

Definition 9.1.1. *Given a context-free grammar G, a rooted tree is a **parse tree** for a derivation in G if*

1. *the root of the tree is a variable in G,*
2. *the interior nodes are labeled by variables in G,*
3. *the leaves of the tree are either labeled by variables of G, terminals in the grammar G, or ε (If a leaf is labeled by ε, then it must be the only child of its parent), and*

4. *if there is an internal node labelled A, whose children are labelled $B_1, B_2, \ldots, B_k$, then there must be a production $A \to B_1 B_2 \cdots B_k$ in G.*

Any subtree of a parse tree is also a parse tree, unless it is a leaf.

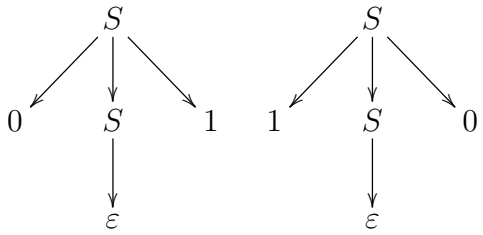
Definition 9.1.2. *Given a parse tree, the **yield** of the tree is the concatenation of the symbols at the leaves of the tree, in order from left to right. (N.B. Don't include ε in the concatenation.)*

Example: The yield of the above parse tree is 00100.

Definition 9.1.3. *A parse tree is **full** if its root is the start variable in the grammar, and if all its leaves are labelled with terminals or ε. In these trees, the yield is to a word in the language of the grammar.*

Remarks:

1. Order matters here. We read the yield of a parse tree along the leaves, from left to right. Recall this grammar from the proof of Theorem 8.1.6:
$$S \to \varepsilon \mid 0S1 \mid 1S0 \mid SS$$
These two parse trees over this grammar have different yields:



2. The above parse tree with yield 00100 is full.
3. We can go from derivations to parse trees and back.
4. As in the slides, we could formalize this correspondence, but we will not.

9.2 Ambiguity in Context-Free Grammars - Definitions

Definition 9.2.1. *A context-free grammar G is **ambiguous** if there is a word $w \in L(G)$ with more than one leftmost derivation (or equivalently more than one parse tree).*

Example: This grammar, given in the slides, is ambiguous:

$$G : S \to S * S \mid S + S \mid (S) \mid a$$

Definition 9.2.2. *A context-free grammar G is **unambiguous** if it is not ambiguous.*

Definition 9.2.3. *A context-free language L is **inherently ambiguous** if every context-free grammar G such that $L(G) = L$ is ambiguous.*

Remarks:

1. It should be clear that for parsing purposes, unambiguous grammars are desirable.
2. If a CFL is inherently ambiguous, then there is no hope.
3. Sometimes the first grammar we might write down is ambiguous, but there is a non-ambiguous alternative that generates the same language.
4. The above grammar is ambiguous, but its language is not inherently ambiguous. the example of replacing the grammar with a different, unambiguous grammar which generates the same language, demonstrates this.

Setup for the Example from the Slides:

1. We will show that the grammar $G : S \to S * S \mid S + S \mid (S) \mid a$ is ambiguous.
2. We will show that the new grammar G', with three variables:
   - E for expressions (which is the start variable): $E \to E + T \mid T$
   - T for terms, which can be added together to make expressions: $T \to T * F \mid F$
   - F for factors, which can be multiplied into products: either the single terminal a or a parenthesized expression: $F \to (E) \mid a$

   is unambiguous, and generates the same language.
3. Examples like this one tend to be tedious, but not difficult.
4. You will do an example of removing ambiguity like this one on A03.

10 Lecture 10

Outline

1. Ambiguity in Context-Free Grammars - Example - M5 44-67

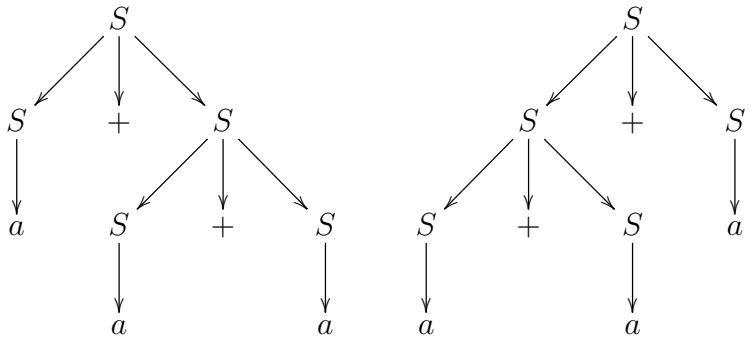
10.1 Ambiguity in Context-Free Grammars - Example

Consider the language of the grammar

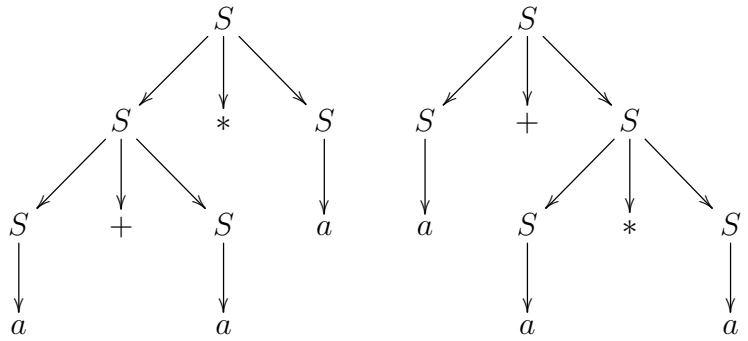
$$G : S \to S * S \mid S + S \mid (S) \mid a$$

Where does this grammar acquire ambiguity? Two ways:

1. We can derive $a + a + a$ in two different ways: no sense of order in associativity.



2. We can derive $a + a * a$ in two different ways: no sense of operator precedence.



Strategy to Remove Ambiguity:
Separate expressions being multiplied, added or parenthesized

- If we have the sum of two terms, we cannot use that as a factor in a product.
- The factors being multiplied in a product have to be either products themselves, or a or a parenthesized expression.

- If we have the sum of three terms, we have to have build the first two as a sum, and then sum this with the third (e.g. $a + a + a$ means $(a + a) + a$).

A new grammar This idea suggests a grammar G' with three variables:

- E for **expressions** (which is the start variable): $E \to E + T \mid T$
- T for **terms**, which can be part of a sum: $T \to T * F \mid F$
- F for **factors**, which can be multiplied into products: either the single terminal a or a parenthesized expression: $F \to (E) \mid a$

(**Q:** Why not $E \to T + T \mid T$? **A**: Because then there is still no **unique** way to derive $a + a + a$. We need to make sure that there is **exactly** one way of generating every expression!)

This grammar includes the order of operations. It is also unambiguous and generates the same language.

Formal statement of our result

Theorem 10.1.1. *If G is the grammar*

$$S \to S + S \mid S * S \mid (S) \mid a,$$

and G' is the grammar with rules:

- $E \to E + T \mid T$,
- $T \to T * F \mid F$,
- $F \to (E) \mid a$,

then

1. *G' is unambiguous, and*
2. *$L(G') = L(G)$.*

Remark: This is **not** going to be a short proof.

Lemma 10.1.2. *First, consider G':*

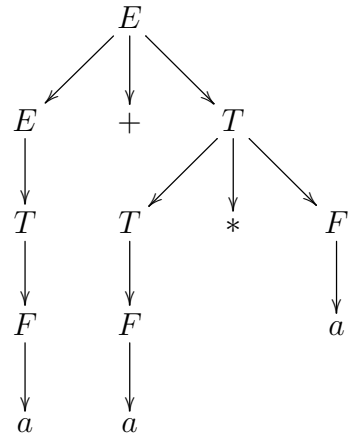
1. *If $T \stackrel{*}{\Rightarrow} x$, then x has no $+$ character outside of parentheses.*
2. *If $F \stackrel{*}{\Rightarrow} x$, then x has no $+$ or $*$ characters outside parentheses.*
3. *If x has no $+$ or $*$ characters outside parentheses, then $F \stackrel{*}{\Rightarrow} x$.*
4. *If x has no $+$ characters outside parentheses, then $T \stackrel{*}{\Rightarrow} x$ or $F \stackrel{*}{\Rightarrow} x$.*

Proof: This is clear from the rules of G'. Prove by structural induction on G' if you are not already convinced. □

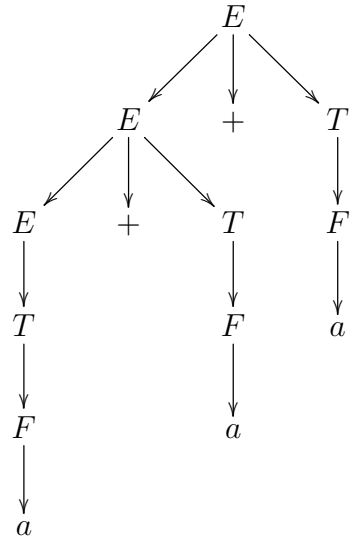
We will use these facts quite a lot.

Examples which are ambiguous in G, now unambiguous in G':

1. In this new grammar, $a + a * a$ has only one parse tree:



2. And similarly $a + a + a$ has only one parse tree.



First: G' is unambiguous We will prove something stronger:

Lemma 10.1.3. *If, for some string x containing only alphabet symbols, we have $E \stackrel{*}{\Rightarrow} x$ or $T \stackrel{*}{\Rightarrow} x$ or $F \stackrel{*}{\Rightarrow} x$, then there is a unique left-most derivation for that string from E, T or F, respectively.*

Proof. by induction on $|x|$:

- Base case ($|x| = 1$): Then $x = a$.

  - If $E \overset{*}{\Rightarrow} a$, then there is only one possible derivation: $E \Rightarrow T \Rightarrow F \Rightarrow a$.
  - If $T \overset{*}{\Rightarrow} a$, then there is only one possible derivation: $T \Rightarrow F \Rightarrow a$.
  - If $F \overset{*}{\Rightarrow} a$, then there is only one possible derivation: $F \Rightarrow a$.
- Inductive case ($|x| > 1$): Inductive Hypothesis: Any y such that $E \overset{*}{\underset{G'}{\Rightarrow}} y, T \overset{*}{\underset{G'}{\Rightarrow}} y$ or $F \overset{*}{\underset{G'}{\Rightarrow}} y$ with $|y| < |x|$ has a **unique** leftmost derivation from E, T or F, respectively.
- Now, let's look at $x \in L(G')$ and figure out how we got to where we are.

Three cases for the induction We will examine x and see what it has outside of parentheses:

1. There is a + symbol outside of parentheses in x.
2. There is no + symbol outside of parentheses in x, but there is a $*$ outside of parentheses.
3. There is neither a + symbol nor a $*$ symbol outside of parentheses in x.

Case 1 Assume there is a + symbol outside of parentheses in x.
Basic idea: Writing $x = y + z$, argue that we must start with the rule $E \to E + T$ where $E \overset{*}{\Rightarrow} y$ and $T \overset{*}{\Rightarrow} z$, and then argue that y and z must have unique derivations.

- If x has a + outside of parentheses, then $T \overset{*}{\not\Rightarrow} x$ and $F \overset{*}{\not\Rightarrow} x$ (Lemma 10.1.2 for both).
- The only remaining possibility is that $E \overset{*}{\Rightarrow} x$.
- So the first rule used in any derivation of x in G' must be $E \to E + T$.
- We know that $x \in L(G')$, so there must be some way, now, to generate $x = y + z$, where $E \overset{*}{\Rightarrow} y$ and $T \overset{*}{\Rightarrow} z$.
- But by our induction hypothesis, the ways of generating y and z are unique, so for this decomposition $x = y + z$, there exists exactly one way of generating x.
- **Q:** Could we choose y and z in different ways?
- **A:** No: we know that y is before the last + symbol outside parentheses in x.
  - If not, then z has a + symbol outside parentheses, and thus by Lemma 10.1.2 z cannot be generated from T.
  - But we know that $T \overset{*}{\Rightarrow} z$.
- So, for this case, the only leftmost derivation for x is: $E \Rightarrow E + T \overset{*}{\Rightarrow} y + T \overset{*}{\Rightarrow} y + z$, and x is unambiguously derived.

Case 2 Assume x has no $+$ outside parentheses, but does have a $*$ outside parentheses.

- We cannot start with the $E \to E + T$ rule, so we must start with $E \to T$ instead.
- But since there is a $*$ outside parentheses, by Lemma 10.1.2, $F \stackrel{*}{\not\Rightarrow} x$.
- So we cannot use $T \to F$ and we must use the $T \to T * F$ rule instead. We are going to thus be decomposing x into $x = y * z$.
- As in the proof for Case 1 (since $F \stackrel{*}{\Rightarrow} z$), z must have no $*$ outside parentheses, so it is uniquely chosen.
- Again, for $x = y * z$, y and z have unique derivations by induction, and hence there is a unique derivation for x.

Case 3 Assume x has neither * nor $+$ outside parentheses.

- But then $x = (y)$ for some y, since we are **not** in the base case, namely the case where $x = a$.
- We must start the derivation with $E \Rightarrow T \Rightarrow F \Rightarrow (E)$.
- Then we follow the derivation for y, which is unique by the induction hypothesis.

This handles all three possibilities for a word in $L(G')$. In all cases, they have a unique derivation.
Hence, the grammar G' is unambiguous. □

That is the proof that the new grammar is unambiguous. This proves part 1 of Theorem 10.1.1.
What about the proof that $L(G) = L(G')$ (part 2 of Theorem 10.1.1)?

Proof. Proof that $L(G') \subseteq L(G)$: Let $x \in L(G')$ be arbitrary. The proof is by induction on $|x|$.
Base case: If $|x| = 1$, then $x = a$, since that is the only 1-letter word in $L(G')$, and it is the only 1-letter word in $L(G)$, too.
Inductive case: Inductive Hypothesis: every $y \in L(G')$ such that $|y| < |x|$ satisfies $y \in L(G)$. Consider the **unique** derivation of x in G'.
How is x derived? Three cases for the **start** of the unique derivation of x in G':

1. $E \to E + T \stackrel{*}{\Rightarrow} x$.
2. $E \Rightarrow T \Rightarrow T * F \Rightarrow \stackrel{*}{\Rightarrow} x$
3. $E \Rightarrow T \Rightarrow F \Rightarrow (E) \stackrel{*}{\Rightarrow} x$

Case 1: x derives from $E \to E + T$

- Write $x = y + z$, where $E \stackrel{*}{\Rightarrow} y$ and $T \stackrel{*}{\Rightarrow} z$.

- Since $E \stackrel{*}{\Rightarrow} y$, therefore $y \in L(G')$.
- Since $E \Rightarrow T \stackrel{*}{\Rightarrow} y$, therefore $z \in L(G')$.
- We can derive z in G' using $E \Rightarrow T \stackrel{*}{\Rightarrow} z$.
- Both y and z are in $L(G')$ and are strictly shorter than x, so they are both in $L(G)$ by our inductive hypothesis.
- Derive x in $L(G)$ via

$$S \Rightarrow S + S \stackrel{*}{\Rightarrow} y + S \stackrel{*}{\Rightarrow} y + z = x,$$

  so that $x \in L(G)$.

Case 2: x derives from $E \Rightarrow T \Rightarrow T * F \Rightarrow \cdots$

- Then $x = y * z$, where $T \stackrel{*}{\Rightarrow} y$ and $F \stackrel{*}{\Rightarrow} z$.
- Since $T \stackrel{*}{\Rightarrow} y$, we can derive y via $E \Rightarrow T \stackrel{*}{\Rightarrow} y$, and therefore $E \stackrel{*}{\Rightarrow} y$.
- Since $F \stackrel{*}{\Rightarrow} z$, we can derive z via $E \Rightarrow T \Rightarrow F \stackrel{*}{\Rightarrow} z$, and therefore $E \stackrel{*}{\Rightarrow} z$.
- So y and z are in $L(G')$ and are shorter than x.
- By the inductive hypothesis, y and z are in $L(G)$, and we can derive x in G by

$$S \Rightarrow S * S \stackrel{*}{\Rightarrow} y * S \stackrel{*}{\Rightarrow} y * z = x.$$

Case 3: x derives from $E \Rightarrow T \Rightarrow F \Rightarrow (E) \Rightarrow \cdots$

- Then $x = (y)$, where $E \stackrel{*}{\Rightarrow} y$.
- The induction hypothesis applied to y says $S \stackrel{*}{\Rightarrow} y$.
- Therefore we can derive x in G via

$$S \Rightarrow (S) \stackrel{*}{\Rightarrow} (y) = x.$$

So we see that $L(G') \subseteq L(G)$.

Proof that $L(G') \supseteq L(G)$: Let $x \in L(G)$ be arbitrary. The proof is by induction on $|x|$.

Base case: If $|x| = 1$, then $x = a \in L(G')$; seen before.

Inductive case:

- Inductive Hypothesis: Assume all words shorter than $y \in L(G)$ such that $|y| < |x|$ satisfy $y \in L(G')$.
- Then $x = y + z$ or $x = y * z$ or $x = (y)$ for y and z also in $L(G)$.
- We will have three cases, depending on how x can be derived in G.
- As derivations in G need not be unique, we need to carefully state what the cases are.

Cases for the Inductive part of the proof:

1. x has at least one derivation of the form $S \Rightarrow (S) \stackrel{*}{\Rightarrow} (y)$, where $x = (y)$.
2. x has no such derivation, but has at least one derivation of the form $S \Rightarrow S + S \stackrel{*}{\Rightarrow} y + z$, where $x = y + z$.
3. The only derivations for x have the form $S \rightarrow S * S \stackrel{*}{\Rightarrow} x$.

Case 1: x has at least one derivation of the form $S \Rightarrow (S) \stackrel{*}{\Rightarrow} (y)$, where $x = (y)$: Suppose that one derivation for x in G is:

$$S \Rightarrow (S) \stackrel{*}{\Rightarrow} (y).$$

- Then in G' we can use the derivation: $E \Rightarrow T \Rightarrow F \Rightarrow (E) \cdots$
- Since $S \underset{G}{\stackrel{*}{\Rightarrow}} y$, and $|y| < |x|$, our inductive hypothesis tells us that $y \in L(G')$, i.e. $E \underset{G'}{\stackrel{*}{\Rightarrow}} y$.
- Therefore in G' we have $E \Rightarrow T \Rightarrow F \Rightarrow (E) \stackrel{*}{\Rightarrow} (y) = x$, which witnesses the fact that $x \in L(G')$.

Case 2: $S \Rightarrow S + S \stackrel{*}{\Rightarrow} y + z$, where $x = y + z$: Suppose that x has no derivation as in Case 1, but has at least one derivation in G of the form $S \Rightarrow S + S \stackrel{*}{\Rightarrow} y + z$, where $x = y + z$.

- **Problem**: we want to start the derivation of x in G' using the production: $E \rightarrow E + T$.
- By the induction hypothesis, $E \underset{G'}{\stackrel{*}{\Rightarrow}} y$ and $E \underset{G'}{\stackrel{*}{\Rightarrow}} z$, but to apply the desired production in G', we need to know that $T \underset{G'}{\stackrel{*}{\Rightarrow}} z$.
- **Idea**: Choose y as long as possible, so that z has no $+$ symbols outside of parentheses.
- Then, as above, we still have $E \stackrel{*}{\Rightarrow} y$, but now we can only use $E \Rightarrow T \stackrel{*}{\Rightarrow} z$ to derive z in G'.
- But then we do have $T \underset{G'}{\stackrel{*}{\Rightarrow}} z$, as required.
- Putting it all together, in G' we have $E \Rightarrow E + T \stackrel{*}{\Rightarrow} y + z = x$.
- This witnesses the fact that x is in $L(G')$.

Case 3: $S \Rightarrow S * S$, where $x = y * z$: This last case is quite similar.

- Suppose that the only derivations for x start by using the production $S \rightarrow S * S$.
- Because x has no derivation beginning with $S \rightarrow S + S$, therefore every $+$ in x must be inside parentheses.
- If x contains an exposed $+$, then we can write $x = y + z$, with $S \stackrel{*}{\Rightarrow} y$ and $S \stackrel{*}{\Rightarrow} z$, so that we can always find a derivation for x which starts with $S \rightarrow S + S$, contradicting the case that we are in.

- Write $x = y * z$, choosing y as long as possible such that we can still derive both y and z from S in G.
- Then y is a word in $L(G)$, shorter than x, and hence by the induction hypothesis is in $L(G')$.
- But then y must be derivable in G' via $E \Rightarrow T \stackrel{*}{\Rightarrow} y$, since y has no exposed + characters.
- Also, z must have no exposed + or $*$ characters, yet still be in $L(G)$.
- Therefore z is derivable in G' by $E \Rightarrow T \Rightarrow F \stackrel{*}{\Rightarrow} z$.
- So finally in G' we can derive x by $E \Rightarrow T \Rightarrow T * F \stackrel{*}{\Rightarrow} y * F \stackrel{*}{\Rightarrow} y * z$.
- Thus $x \in L(G')$.

□

Recap of the proof: Overall, we have shown:

1. G' is unambiguous
2. Words in $L(G')$ are in $L(G)$.
3. Words in $L(G)$ are in $L(G')$.

(All 3 proofs by induction.)
The third proof was probably hardest: the content was to reconstruct the way in G' to derive each word of $L(G)$.
Hence, $L(G)$ is **not** inherently ambiguous: G' is an unambiguous grammar for it.
What does this show us? Ambiguity can, sometimes, be removed:

- Identify the source of ambiguity.
- Re-organize to identify precedence or other desired grammar rules.
- Such proofs are long, but useful: we really do need unambiguous grammars in practice.

11 Lecture 11

Outline

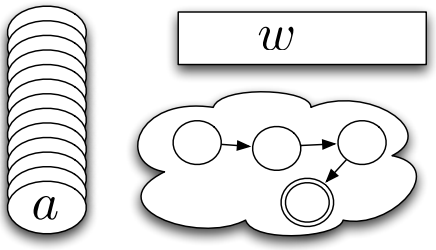
1. Introduction to Pushdown Automata - M6 1-10
2. Computations in a PDA - M6 11-16
3. Bad things in PDAs - M6 17-18
4. Language of a PDA - M6 19-25

11.1 Introduction to Pushdown Automata

From an ε-NFA to a PDA

Add memory to an ε-NFA, in the form of a **stack**. Each transition is of the form:

- If
  - top letter on the memory stack is b, and
  - the next letter in the input word is a (or ε, if we take an ε-transition), and
  - the controlling FA is in state q,
- then:
  - Go to state r,
  - take the b off the top of the memory stack,
  - eat the letter a from the input (again, a might be ε), and
  - write a word w onto the top of the memory stack.
- We will define the **stack alphabet** as part of the definition of the PDA.



Remarks:

1. This is modelled on an ε-NFA, which is **inherently nondeterministic**. Therefore every thread running in a PDA has its own stack, independent of the stacks in the other threads.

More tweaks in how they work

The stack must never be empty:

- There is a special "stack empty" symbol Z_0.
  - If Z_0 is on the top of the stack, we make sure it gets returned to bottom of stack after the next move.
- The string put on the stack
  - may be empty (then the stack gets 1 symbol shorter), or long,
  - does not have to just be 0 or 1 letters long!
  - *must* be of finite length.

(We will see later that deterministic PDAs are **less powerful** than nonde-

terministic ones!)

Definition 11.1.1. *A **pushdown automaton (PDA)** is a 7-tuple, $M = (\Sigma, Q, F, q_0, \Gamma, Z_0, \delta)$, where:*

- Σ *is an alphabet*
- Q *is a finite set of states for control*
- F *is a set of accept states for control*
- q_0 *is the start state for control*
- Γ *is a finite **stack alphabet** (often it is just* $\Gamma = \{Z_0\} \cup \Sigma$*, but* Γ *can be any finite set)*
- Z_0 *is a stack start letter (the symbol with which the stack is initialized, the "stack empty" symbol)*
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to$ *finite subsets of* $Q \times \Gamma^*$ *is the transition function*

Remarks on Definition 11.1.1:

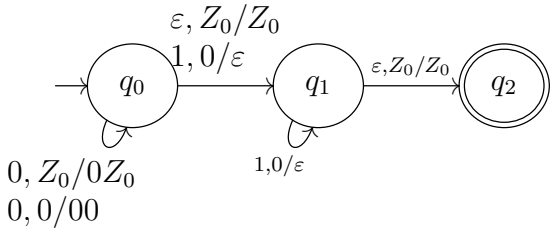
1. $Z_0 \notin \Sigma$.
2. **Q:** Why does δ map into finite subsets of $\{Q \times \Gamma^*\}$? **A:** There are an infinite number of words in Γ^* that we could put on the stack! The machine description must be finite.

PDA Diagrams

We draw PDAs like DFAs, except for the labels of edges:

1. Edges are labelled by both the input letter (e.g. a or ε) **and** the top symbol of the stack (e.g. b); these are separated by a comma.
2. We also must show what word w is pushed onto the stack: this follows a "/" character. The beginning of the string is the *top* of the stack after the transition occurs. E.g. if the stack is empty, and we push $w = 10$ onto the stack, then 1 is on top of the stack and 0 is below 1.
3. It is possible that ε is pushed onto the stack: this makes the stack become shorter.

Example:



Remarks:

1. Naming this PDA P, we will see later that $L(P) = \{0^i 1^i \mid i \geq 0\}$.

2. By the end of M6, we will prove the analog of Kleene's Theorem for CFLs and PDAs. By Kleene's Theorem, $L(P)$ is a CFL.
3. Exercise: Give a CFG, G, such that $L(G) = L(P)$.

11.2 Computations in a PDA

Definition 11.2.1. *Given a PDA, its* ***instantaneous description*** *is a triple (q, w, γ), where*

1. *q is the state the machine is in,*
2. *w is the word left to read in the input, and*
3. *γ is the contents on the stack.*

Remarks:

1. PDAs are non-deterministic: they can have multiple threads.
2. Definition 11.2.1 describes one thread only.
3. We don't attempt to define $\hat{\delta}$ for PDAs. Because of the stack, it is too cumbersome. We have to write down each thread separately.

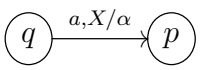
We need a way to characterize one step in the PDA's computation:

- This is more complicated than the $\hat{\delta}$ notation for finite automata: we need to show what happens to the stack.
- Notation describes what could follow the current configuration of the computation.
- Our new notation describes "one path" of the PDA's computation: to simulate the whole PDA, you would need to present all nondeterministic possibilities.

Transitions in the PDA

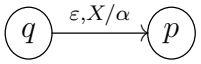
What happens if we go forward one step?

- Let (q, w, γ) and (p, z, ζ) be two instantaneous descriptions of a PDA's configuration.
- From the first to the second in one transition: $(q, w, \gamma) \vdash (p, z, \zeta)$.
- What does that really mean?
    1. If $\gamma = X\beta$ and $w = az$, where
        (a) $|X| = 1$,
        (b) $|a| = 1$,
        (c) then $\delta(q, a, X)$ contains (p, α), where $\zeta = \alpha\beta$.
        (d) I.e. there is a transition:



Remember also that α is of finite length.

2. Or if $\gamma = X\beta$ and $w = \varepsilon z = z$, where
   (a) $|X| = 1$,
   (b) then $\delta(q, \varepsilon, X)$ contains (p, α), where $\zeta = \alpha\beta$.
   (c) I.e. there is a transition



3. Then we write $(q, w, \gamma) \vdash (p, z, \zeta)$.
4. Read $\vdash$ as "produces in one step".

If the machine P needs to be indicated explicitly, then write $I \underset{P}{\vdash} J$.

Longer derivations

1 step in P: $(q, x, \beta) \underset{P}{\vdash} (q', x', \alpha)$.

An arbitrary (finite) number of steps: $(q, x, \beta) \overset{*}{\underset{P}{\vdash}} (q', x', \alpha)$.

Definition 11.2.2. *Proper inductive definition of* $I \overset{*}{\underset{P}{\vdash}} J$*:*

1. *Base case: For any instantaneous description* I, $I \overset{*}{\underset{P}{\vdash}} I$.
2. *Inductive case: If* $I \underset{P}{\vdash} K$ *and* $K \overset{*}{\underset{P}{\vdash}} J$, *then* $I \overset{*}{\underset{P}{\vdash}} J$.

(I.e. there exists a finite sequence of instantaneous descriptions $K_1, K_2, \ldots, K_n$ *such that* $I = K_1$, $J = K_n$ *and for all* $i = 1, \ldots, n-1$, $K_i \underset{P}{\vdash} K_{i+1}$.*)*

Reminder: This describes one thread only.

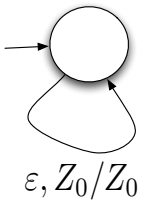
We will **not** attempt to define an analogue to $\hat{\delta}$ for PDAs.

11.3 Bad things in PDAs

There are two bad kinds of events too worry about here. Both are also possible with ε-NFAs.

1. **Running forever, using ε-transitions:**
   - There are only a finite number of possible states we can reach following ε-transitions, and they will cycle.

- But now we also can grow the stack, so the configurations are different at each step.



- $(q_0, x, Z_0) \vdash (q_0, x, Z_0Z_0) \vdash (q_0, x, Z_0Z_0Z_0) \vdash (q_0, x, Z_0Z_0Z_0Z_0) \overset{*}{\vdash} \cdots$
- One thread can run forever, or the machine can spawn infinitely many threads. This example produces infinitely many instantaneous descriptions, because the stack keeps growing.

Remark: This is a DPDA, so even DPDAs can run forever.

2. **Crashing** The PDA **crashes** if there are no available transitions from the current instantaneous description, or if the stack is empty with input characters remaining.
   - No available transitions: if the current instantaneous description is (q, x, z), where $x = aw$ and $z = X\beta$ for some alphabet symbol a and stack symbol X, and if the sets $\delta(q, a, X)$ and $\delta(q, \varepsilon, X)$ are both empty, then the PDA can make no transitions and therefore crashes.
   - Empty stack: If the current instantaneous description is (q, x, ε) for some $x \neq \varepsilon$, then the PDA can make no transitions and therefore crashes.
     - It is supposed to remove the top stack symbol as it makes its next transition, but there is no top stack symbol!
     - Note that, if the machine arrives at an instantaneous description $(q, \varepsilon, \varepsilon)$, then the machine will
       * accept if q is an accept state, and
       * reject if q is not an accept state.

11.4 Language of a PDA

A valid PDA computation remains valid if we:

1. Add string $w \in \Sigma^*$ to the end of the input for all K_i.
2. Add string $\gamma \in \Gamma^*$ to the end of the stack for all K_i.
3. Remove an unused suffix from the input for all K_i.

Why does this matter?

- Lets us isolate part of a computation.
- Adding to the end of the stack will help us with some theorems about the equivalence of PDAs and CFGs.

We can prove the first two principles in the same theorem:

Theorem 11.4.1. *Suppose that for a given PDA P, $(q, x, \alpha) \overset{*}{\underset{P}{\vdash}} (p, y, \beta)$. Then for any strings $w \in \Sigma^*$ and $\gamma \in \Gamma^*$, it is also the case that*

1. $(q, xw, \alpha) \overset{*}{\underset{P}{\vdash}} (p, yw, \beta)$, *and*
2. $(q, x, \alpha\gamma) \overset{*}{\underset{P}{\vdash}} (p, y, \beta\gamma)$.

Proof. Consider all transitions that show: $(q, x, \alpha) \overset{*}{\underset{P}{\vdash}} (p, y, \beta)$.

1. None of these transitions uses characters of w or γ.
2. As such, each move is valid if those symbols are added to the end of the input or stack.

□

Removing common suffixes Similarly, if we remove a common suffix from the input, the computation remains valid:

Theorem 11.4.2. *If $(q, xw, \alpha) \overset{*}{\underset{P}{\vdash}} (p, yw, \beta)$, then $(q, x, \alpha) \overset{*}{\underset{P}{\vdash}} (p, y, \beta)$.*

Proof. This is easily shown by following all of the transitions in the first derivation.

1. None of them consumes any of the letters of w, since we can only remove symbols from the input of a PDA, never add to the input.
2. As such, these transitions are also valid to show that $(q, x, \alpha) \overset{*}{\underset{P}{\vdash}} (p, y, \beta)$.

□

How does a PDA accept?

Definition 11.4.3. *A PDA, P **accepts** word w if*

$$(q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (q, \varepsilon, y),$$

for some accept state $q \in F$, and some string $y \in \Gamma^*$.

- This is called **acceptance by final state**.
- Later: we will accept if the stack empties at the same moment that the last input symbol has been processed (called **acceptance by empty stack**).

Definition 11.4.4. *The **language of a PDA** P is*

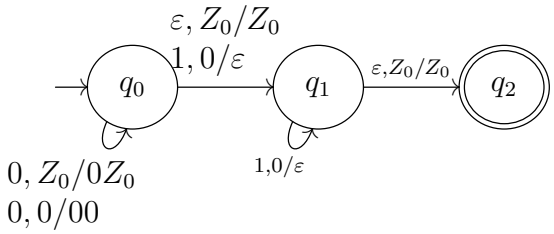
$$L(P) = \{w \in \Sigma^* \mid P \text{ accepts } w\},$$

as in Definition 11.4.3.

An example: $L = \{0^i 1^i\}$

A PDA for $L = \{0^i 1^i \mid i \geq 0\}$:

- Recall that this language is not regular.
- PDAs can be hard to draw!
- This one has three states:
  - q_0 for pushing 0s onto the stack, while reading them from the input word,
  - q_1 for popping 0s off of the stack, while reading corresponding 1s from the input word, and
  - q_2 for accepting at end of word.



How does this work?

For words in L:

- Start with stack Z_0.
- Add 0 symbols from w to the stack.
- ... until we reach the first 1 symbol. Then, we absorb 0s from the stack and corresponding 1s the input.
- ... until we are done, and then we follow the ε-transition to q_2 and accept.

For example, on the input 0011, a computation witnessing acceptance is:

$$\begin{aligned}(q_0, 0011, Z_0) &\vdash (q_0, 011, 0Z_0)\\ &\vdash (q_0, 11, 00Z_0)\\ &\vdash (q_1, 1, 0Z_0)\\ &\vdash (q_1, \varepsilon, Z_0)\\ &\vdash (q_2, \varepsilon, Z_0)\end{aligned}$$

Some formality

1. Only words $w \in 0^*1^*$ can have the property that $(q_0, w, y) \overset{*}{\vdash} (q_2, \varepsilon, z)$ for any strings y and z, since we transition from q_0 to q_1 by consuming a word from $0^*(1 + \varepsilon)$, and we transition from q_1 to q_2 by consuming a word from $1^*\varepsilon$; the concatenation of these is 0^*1^*.
2. The only valid computation after reading in 0^i for $i > 0$ is $(q_0, 0^i w, Z_0) \overset{*}{\vdash} (q_0, w, 0^i Z_0)$.
3. Now consider what occurs upon processing 0^i1^k.
4. If $k < i$, then all threads crash after reaching q_1. So from now on, we will consider $k \geq i$.
5. The only valid computation after reading in 0^i1^k is $(q_0, 0^i1^k w, Z_0) \overset{*}{\vdash} (q_1, w, 0^{i-k}Z_0)$, unless $k = i$, at which point we can transition to (q_2, ε, Z_0). If $k > i$, then the machine has no valid computations that read the first $i + k$ symbols.
6. We can only accept if after reading in 0^i1^i, there are no more symbols to read, so we need not fear accepting 0^i1^k when $k > i$.

Another example: a PDA for palindromes See the slides for the details.
Moral: By the Theorem at the end of M6, the language of palindromes is a CFL.

12 Lecture 12

Outline

1. Acceptance By Empty Stack - M6 31-36

12.1 Acceptance By Empty Stack

Current model of acceptance:

- The language of the machine is

$$L(P) = \left\{ w \mid (q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (p, \varepsilon, \alpha) \text{ for somee } p \in F, \alpha \in \Gamma \right\}$$

New model of acceptance:

- The language of the machine is

$$N(P) = \left\{ w \mid (q_0, w, Z_0) \overset{*}{\underset{P}{\vdash}} (p, \varepsilon, \varepsilon) \text{ for } \textbf{any} \text{ state } p \right\}.$$

This is called **acceptance by empty stack**, for obvious reasons.
A PDA accepting by empty stack does not have accept states, so is a 6-tuple: $(\Sigma, Q, q_0, \Gamma, Z_0, \delta)$.
Key fact about acceptance by empty stack

Theorem 12.1.1. *1. Suppose that P_F is a PDA that accepts by final state. Then there exists a PDA, P_N, which accepts $L(P_F)$ by empty stack.*

2. Suppose that P_N is a PDA that accepts by empty stack. Then there exists a PDA, P_F, which accepts $N(P_N)$ by final state.

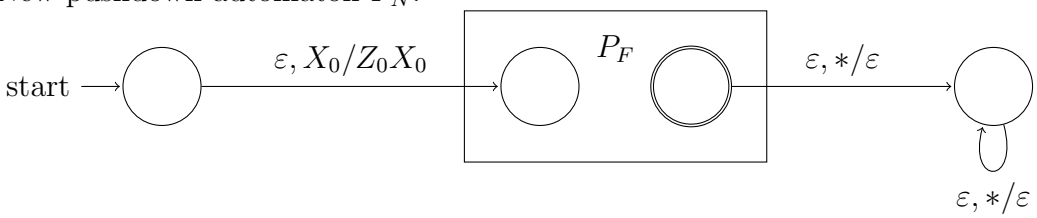
Remark on Why We Care About Theorem 12.1.1 We will use acceptance by empty stack to show that PDAs accept exactly the class of context-free languages.

Proof. **From final state to empty stack**
Suppose we are given a PDA P_F whose language, using acceptance by final state, is L. We will construct a new PDA, P_N, accepting by empty stack, such that P_N accepts exactly L.

- To make P_N accept whenever P_F accepts:
  - From every accept state in P_F, take an ε-transition to a "drain" state that empties the stack.
- To prevent P_N from accepting whenever P_F does not accept:
  - If there are input letters remaining when the above ε-transition is taken, then the thread crashes.
  - Before doing any computation, we put a special character X_0 at the bottom of the stack, below even the Z_0 character.
  - It is crucial that X_0 **not** be in the stack alphabet for P_F.

- We only remove the Z_0 and X_0 characters when we are transitioning to or already in our new "drain" state. I.e. we need X_0 so that, if P_F crashes because of empty stack, the constructed machine P_N will not accept.
- All computations in P_F happen as before, so we can only remove Z_0 and X_0 from the stack when P_F would have already accepted.

New pushdown automaton P_N:



Remark: Both $*$s here include X_0 in the set of all stack symbols in the constructed machine, P_F.

From empty stack to final state

A similar technique can be used to construct a PDA, P_F, with $L(P_F) = L$, given a PDA P_N with $N(P_N) = L$.

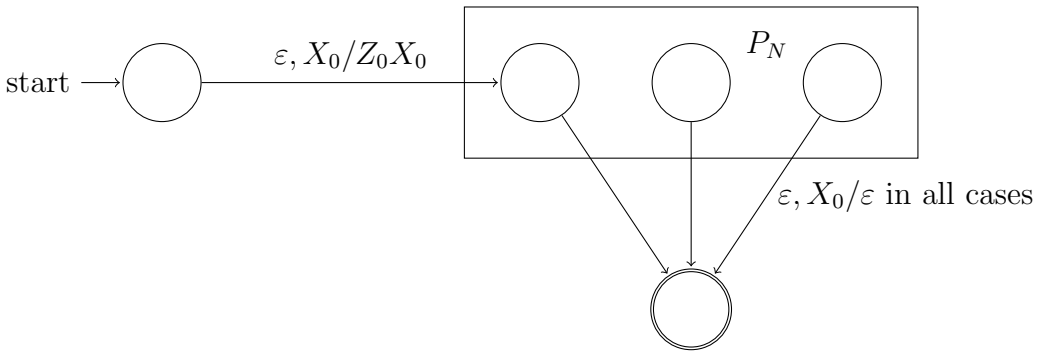
Again, we need a special character X_0 added to the end of the stack that only gets removed after P_N would have accepted.

- Again, it is crucial that X_0 **not** be in the stack alphabet for P_N.
- Here, the X_0 character alerts us that we have removed all of the stack characters inside of P_N.
- Without this special symbol, we would crash trying to detect empty stack.
- When we see this special character, we take an ε-transition to the machine's only accept state.

To ensure that the new PDA P_F only accepts (by final state) the words P_N accepts:

- P_F accepts precisely when we exhaust the input string at the same moment that we empty the stack.
- No outgoing transitions from the new accept state: if we have not processed the entire input word, the machine crashes instead of accepting.

The actual construction New pushdown automaton P_F:



Note: We have not rigourously proved either construction. Both proofs are in the text, not especially enlightening. □

13 Lecture 13

Outline

1. Equivalence of PDAs and CFGs - M6 37-56

13.1 Equivalence of PDAs and CFLs

Theorem 13.1.1. *1. Given a CFG G whose language is L, there exists a PDA P such that $N(P) = L$.*

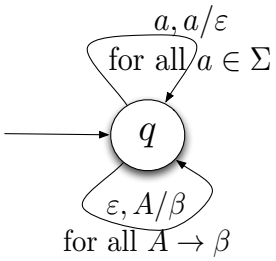
2. Given a PDA P where $N(P) = L$, there exists a CFG G such that $L(G) = L$.

Note: We will prove using acceptance by empty stack in both cases.
How to construct a PDA from a grammar
Start with the grammar $G = (V, T, P, S)$. Construct the PDA

$$P = \{T, \{q\}, q, V \cup T, S, \delta\},$$

where δ has the following members:

- For all rules $A \to \beta$ in the grammar, $\delta(q, \varepsilon, A)$ includes (q, β). There are no other members of $\delta(q, \varepsilon, A)$.
- For all terminals $a \in T$, $\delta(q, a, a) = \{(q, \varepsilon)\}$.
- All other values of $\delta(q, x, y)$ are $\emptyset$.



Theorem: This PDA, P, accepts $L(G)$ by empty stack. Two parts: $L(G) \subseteq N(P)$, and $N(P) \subseteq L(G)$. We will begin with showing that $L(G) \subseteq N(P)$.

- Let $w \in L(G)$ be arbitrary.
- Then there exists some leftmost derivation for w in G :

$$S = \gamma_1 \underset{lm}{\Rightarrow} \gamma_2 \underset{lm}{\Rightarrow} \gamma_3 \underset{lm}{\Rightarrow} \cdots \underset{lm}{\Rightarrow} \gamma_n = w.$$

- For all of the γ_i except γ_n, γ_i has a a leftmost variable.
  - For $i < n$, write $\gamma_i = x_i\alpha_i$, where α_i begins with the leftmost variable, A_i, in γ_i.
  - Since $\gamma_i \overset{*}{\underset{lm}{\Rightarrow}} w$, and since x_i contains only terminals, therefore x_i is a prefix of w.
  - Write $w = x_i y_i$, for some string $y_i \in \Sigma^*$.
- **Claim:** In the PDA, because the transitions are defined from the productions of G, we have $(q, w, S) \overset{*}{\underset{P}{\vdash}} (q, y_i, \alpha_i)$ for all $1 \leq i \leq n$. (We prove this claim rigourously, below.)
- Then in particular, taking $i = n$ gives $(q, w, S) \overset{*}{\underset{P}{\vdash}} (q, \varepsilon, \varepsilon)$, and so $w \in N(P)$, because P accepts w by empty stack.

Proof of the claim, by induction on i:

- Base case ($i = 1$): $x_1 = \varepsilon, y_1 = w, \alpha_1 = S$, so we have that $(q, w, S) \overset{*}{\underset{P}{\vdash}} (q, y_i, \alpha_i)$ holds trivially.
- Inductive case ($1 < i \leq n$): The inductive hypothesis is that $(q, w, S) \overset{*}{\underset{P}{\vdash}} (q, y_j, \alpha_j)$, for all $j < i$.
  - By the induction hypothesis $(q, w, S) \overset{*}{\underset{P}{\vdash}} (q, y_{i-1}, \alpha_{i-1})$, where
    - $w = x_{i-1}y_{i-1}$,
    - $\gamma_{i-1} = x_{i-1}\alpha_{i-1}$ and
    - $\alpha_{i-1} = A_{i-1}\eta_{i-1}$ for some variable A_{i-1} and some suffix η_{i-1}.

- Everything constructed here comes from a leftmost derivation of w in G, so there exists a production $A_{i-1} \to \beta_{i-1}$ in G, for some β_{i-1} (a string of terminals and/or variables).
- The production $A_{i-1} \to \beta_{i-1}$ gives that $\delta(q, \varepsilon, A_{i-1})$ contains (q, β_{i-1}) so $(q, y_{i-1}, \alpha_{i-1}) \underset{P}{\vdash} (q, y_{i-1}, \beta_{i-1}\eta_{i-1})$.
- By construction, $y_{i-1} \in \Sigma^*$.
- As everything constructed here comes from a derivation of w in G, y_{i-1} and β_{i-1} must have a (possibly empty) prefix of matching terminals.
- The leftmost variable in $\beta_{i-1}\eta_{i-1}$ is A_i, which begins α_i.
- Match all of the terminals at the top of the stack against the terminals in the input y_{i-1} (using $\delta(q, a, a) = \{q, \varepsilon\}$ for all $a \in T$).
- Consume all matching terminals to obtain $(q, y_{i-1}, \beta_{i-1}\eta_{i-1}) \overset{*}{\underset{P}{\vdash}} (q, y_i, \alpha_i)$
- Thus $(q, w, S) \overset{*}{\underset{P}{\vdash}} (q, y_i, \alpha_i)$.

The proof in the other direction We have proved $L(G) \subseteq N(P)$. Now we prove $N(P) \subseteq L(G)$.
Suppose that $w \in N(P)$. To show that $w \in L(G)$, we will show something more general:

- For any variable A, if $(q, w, A) \overset{*}{\underset{P}{\vdash}} (q, \varepsilon, \varepsilon)$, then $A \overset{*}{\underset{G}{\Rightarrow}} w$.
- This is sufficient:
  - By definition $w \in N(P)$ means that $(q, w, S) \overset{*}{\underset{P}{\vdash}} (q, \varepsilon, \varepsilon)$.
  - Then applying the above implication, with $A = S$ gives us that $S \overset{*}{\underset{G}{\Rightarrow}} w$, in other words, $w \in L(G)$.

The proof is by induction on n, the length of the chosen computation that takes (q, w, A) to $(q, \varepsilon, \varepsilon)$:

- Base case ($n = 1; n = 0$ is impossible since $A \neq \varepsilon$):
  - In the construction of the PDA P, the only transitions defined for a non-terminal A at the top of the stack are ε-transitions corresponding to productions for A in G.
  - So our computation must take one of these ε-transitions as its first (and hence its only) step.
  - As this ε-transition takes w to ε, therefore $w = \varepsilon$.
  - Also $(q, \varepsilon) \in \delta(q, \varepsilon, A)$, so by construction there is a rule $A \to \varepsilon$ in G.

– Therefore $A \overset{*}{\underset{G}{\Rightarrow}} w$ as desired.

The inductive case($n > 0$): The inductive hypothesis is that for all variables A, if $(q, w, A) \overset{*}{\underset{P}{\vdash}} (q, \varepsilon, \varepsilon)$ in fewer than n steps, then $A \overset{*}{\underset{G}{\Rightarrow}} w$. Now consider an n-step computation $(q, w, A) \overset{*}{\underset{P}{\vdash}} (q, \varepsilon, \varepsilon)$.

- The first step in the computation must use a rule of the form $A \to Y_1 Y_2 \cdots Y_k$ (where the Y_i are in $T \cup V$), since those are the only rules valid when A is on top of the stack (and the transition is an ε-transition). So $(q, w, A) \vdash (q, w, Y_1 \cdots Y_k)$.
- Decompose the computation from this time forward into phases: there will be some first point when the stack contains just $Y_2 \cdots Y_k$, then some first point when the stack contains just $Y_3 \cdots Y_k$, and so on, until the first time it has just Y_k.
  (Each step removes at most one symbol from the stack, and we eventually empty the stack.)
- Consider when the stack contains $Y_2 \cdots Y_k$: we know that $(q, w, A) \overset{}{\underset{P}{\vdash}} (q, w, Y_1 Y_2 \cdots Y_k) \overset{*}{\underset{P}{\vdash}} (q, w', Y_2 \cdots Y_k)$, for some suffix w' of w, where $w = w_1 w'$.
- Rewriting the last statement, we have $(q, w_1 w', Y_1 Y_2 \cdots Y_K) \overset{*}{\underset{P}{\vdash}} (q, w', Y_2 \cdots Y_k)$.
- Because we can delete the unread suffix w' of the input word without affecting the validity of the computation, we therefore have that $(q, w_1, Y_1 Y_2 \cdots Y_k) \overset{}{\underset{P}{\vdash}} (q, \varepsilon, Y_2 \cdots Y_K)$.
- Moreover, since in the computation, we never touch the symbols of $Y_2 \cdots Y_k$ on the stack, we also have that $(q, w_1, Y_1) \overset{}{\underset{P}{\vdash}} (q, \varepsilon, \varepsilon)$.
- If Y_1 is a terminal, then by the shape of the PDA, $(q, w_1, Y_1) \overset{}{\underset{P}{\vdash}} (q, \varepsilon, \varepsilon)$ implies that $w_1 = Y_1$ is also a terminal, so that $Y_1 \overset{*}{\underset{G}{\Rightarrow}} w_1$ holds trivially.
- Otherwise Y_1 is a variable, and by construction, this computation witnessing $(q, w_1, Y_1) \overset{}{\underset{P}{\vdash}} (q, \varepsilon, \varepsilon)$ takes fewer than n steps, therefore $Y_1 \overset{*}{\underset{G}{\Rightarrow}} w_1$, by our inductive hypothesis.
- To summarize, whether Y_1 is a terminal or a variable, we have $Y_1 \overset{*}{\underset{G}{\Rightarrow}} w_1$.
- We then apply this argument to each subsequent Y_i, obtaining a derivation $Y_i \overset{*}{\underset{G}{\Rightarrow}} w_i$ for all i, with $w = w_1 w_2 \cdots w_k$.
- Finally, we have a derivation for w, by combining these together: $A \Rightarrow$

$$Y_1 \cdots Y_k \overset{*}{\underset{G}{\Rightarrow}} w_1 Y_2 \cdots Y_k \overset{*}{\underset{G}{\Rightarrow}} w_1 w_2 Y_3 \cdots Y_k \overset{*}{\underset{G}{\Rightarrow}} w_1 \cdots w_k = w$$

- This shows that $A \overset{*}{\underset{G}{\Rightarrow}} w$, as desired.

Next, from a PDA to a CFG

- We have shown the easier case: for a CFG G with language $L(G) = L$, we can exhibit a PDA P such that $N(P) = L$.
- Now, given a PDA P (which accepts by empty stack), we must exhibit a grammar G such that $N(P) = L(G)$.
- This is harder.
  - Suppose that, in P,

$$(q, w, X) \overset{*}{\vdash} (p, \varepsilon, \varepsilon),$$

    for some $X \in \Gamma$.
  - How does that happen?

How do we consume one symbol from the stack?

- Suppose the first step in the computation is $(q, az, X) \vdash (r, z, Y_1 Y_2 \cdots Y_k)$.
- We may be following a transition that does or does not consume the first symbol of w, so $a \in \Sigma$ or $a = \varepsilon$.
- The process begins by removing X from the stack, replacing it with some string $Y_1 \cdots Y_k$, and moving to state r.
- Then, we go from state r to some state r_1, eventually removing all of the symbols of Y_1 from the stack, so that it consists of just $Y_2 \cdots Y_k$; this process may consume some input letters,from w, or it might not.
- Eventually, we wind up in state p, in the instantaneous description $(p, \varepsilon, \varepsilon)$, having read all of the letters of w and all of the symbols in $Y_1 \cdots Y_k$.

Making a grammar from this

To construct a grammar which generates the same language that the PDA accepts, we need to construct our productions to mimic the action of the transitions in the PDA.

- We will have a non-terminal $[qXp]$ for every stack symbol X and every pair of states q and p.
- Each of these $[qXp]$ triplets is a variable in the new grammar we are building.
- A string generated by the non-terminal $[qXp]$ in G corresponds with a (possibly partial) computation in the PDA. It will capture the input letters consumed so far and the current stack contents.

- We want: $[qXp] \overset{*}{\underset{G}{\Rightarrow}} w$ if and only if $(q, w, X) \overset{*}{\underset{P}{\vdash}} (p, \varepsilon, \varepsilon)$.
- Now we need to construct the productions of the grammar to mimic the transitions in the given PDA.
- Examine each rule of the transition function, one at a time.
- Suppose that one element of $\delta(q, a, X)$ is $(r, Y_1 Y_2 \cdots Y_k)$.
  - **Note**: a can be ε, and so can $Y_1 \cdots Y_k$.
  - If the output pair is (r, ε), so that $k = 0$, then add a production $[qXr] \to a$.
    - Read a, pop X, move to state r.
  - Otherwise, if $k \geq 1$, then add productions $[qXr_k] \to a[rY_1r_1][r_1Y_2r_2] \cdots [r_{k-1}Y_kr_k]$ to the grammar, for **all** choices of $r_1, \ldots, r_k$ from our state set Q.
    - Read a, pop X, push $Y_1 \cdots Y_k$ onto the stack, allowing for **any** sequence $r_1, \ldots, r_k$ of states to be used to pop $Y_1 \cdots Y_k$ in the end, move to state r.
- The language of the PDA is the union of the language corresponding to emptying the stack in **any** possible state, so for **every** state p we add a rule: $S \to [q_0 Z_0 p]$.
- Then by construction $(q_0, w, Z_0) \overset{*}{\vdash} (p, \varepsilon, \varepsilon)$ if and only if $S \Rightarrow [q_0 Z_0 p] \overset{*}{\Rightarrow} w$, as we had wanted.

How does this work? The basic idea, again: Each PDA transition

- either consumes a symbol from the input, or does not,
- changes state and
- takes one symbol off the stack, and then puts some new symbols on the stack, or does not.

To mimic this in one step of a leftmost derivation in the grammar which we have just constructed:

- Look at the right hand side of a particular production.
- The terminal (or ε) at the beginning of the right hand side is exactly what we expect to read from the input, according to the transition function in the given PDA.
- Then the string of nonterminals, each of the form $[rY_1r_1]$ or $[r_{i-1}Y_ir_i]$ indicate, in the Y_i characters, the symbols that get pushed onto the stack.
- The first state in the first non-terminal (namely r) is the state that the PDA goes into.
- The following states correspond to the states the automaton will be in when we have shrunk the stack.

The proof Actually, we are not going to do it.

- The basic structure is not especially interesting; it is in Section 6.3.2 of the text.
- The really interesting thing is the idea; the proof is just about filling in the details.

One thing to note: this construction is finite-size.

- This may not be obvious, but it is because the strings put on the stack are always of finite length. This means we add a finite number of rules, though potentially enormous. (How big?)

14 Lecture 14

Outline

1. Deterministic PDAs - M6 57-68

Stuff.

14.1 Deterministic PDAs

In a **DPDA**, every transition is determined.

- Must be at most 1 transition from every state for every (input letter, stack letter) pair.
- Can have ε-transitions, but each must be the only valid transition for the starting state and stack symbol.
- If there is a transition from q_0 with X on the top of the stack that does consume input letters, then there are no ε-transitions for the same stack symbol.

More formally, for a given state q_0:

- $|\delta(q_0, a, X)| \leq 1$ always,
- and if $|\delta(q_0, a, X)| = 1$ for some letter a, then $|\delta(q_0, \varepsilon, X)| = 0$.

Motivation: Why do we care about DPDAs?

- **A:** Some CFLs are **not** accepted by DPDAs.

One important difference versus DFAs:

- It is possible that no transition is available.
- Then the machine crashes if there are more input symbols left.

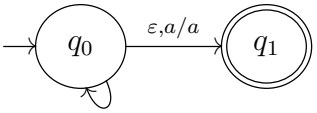
An example of a language accepted by a DPDA Ordinary PDA for the language: $L = \{x \in \{a, b\}^* \mid n_a(x) > n_b(x)\}$.

- (Words with more a's than b's.)

As we scan the word left-to-right, there will be more of one letter than the other.

- Keep track of the number of letters that we have more of.
- (Example: read in *aaaabab* so far, then we should have three a's on the stack, since we have 5 a's and 2 b's.)
- Accept when there is still an a on the top of the stack at the end.

A nondeterministic PDA for L This gives a 2-state PDA like this:



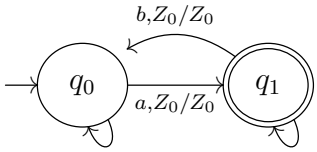
$a, Z_0/aZ_0$
$b, Z_0/bZ_0$
$a, b/\varepsilon$
$b, a/\varepsilon$
$a, a/aa$
$b, b/bb$

- First two transitions in state q_0: prefix is balanced
- Second two transitions in state q_0: reducing imbalance
- Last two transitions in state q_0: adding to imbalance
- Nondeterministic: ε-transition to state q_1.

Can we make this PDA deterministic? The current PDA is nondeterministic, because we have transitions when we pull an a off the stack that do eat input letters, and that do not.

Can we make small changes to make it deterministic?

- As before, but with states for "balanced or accumulating b's" and for "accumulating a's".
- The second of these is the accept state.



$b, Z_0/bZ_0$ $\quad$ $a, Z_0/aZ_0$
$a, b/\varepsilon$ $\quad$ $b, a/\varepsilon$
$b, b/bb$ $\quad$ $a, a/aa$

This is a deterministic PDA:

- There are no ε-transitions.
- There is always at most one choice for what to do next.
- In fact, there is exactly one choice:
  - There are two states.
  - There are two alphabet symbols.
  - By the shape of the machine, there are two possible top-of-stack symbols possible in each of q_0, q_1.
  - Thus there are $2^3 = 8$ transitions possible, each of which is defined in our picture.

But the stack still remembers the imbalance: we have just divided the transitions from before between the two states.
We jump from one state to the other when we switch from more a's than b's to not, or the other way around.

Not all CFLs are accepted by DPDAs

Fact (not proved in the text):
There exist context-free languages that are not accepted by DPDAs.

- Nondeterminism specifically makes PDAs more powerful.
- (This is not true for either Turing machines or for finite automata.)

Two different ways to prove this:

- Directly.
- Indirectly (Module 7)

We will talk a little about the first, but rely on the second.

Theorem 14.1.1. *Any regular language L is accepted by a DPDA.*

Proof. The DPDA just ignores the stack (i.e. every transition pops Z_0 and re-pushes it), and simulates the DFA that accepts L, which exists by Kleene's Theorem 5.1.1. □

An Example to show that DPDAs do not accept every CFL

We know that the language $L = \{x \in \{a, b\}^* | x \text{ is a palindrome}\}$ is a CFL, and hence accepted by a PDA.
We can argue that it is not accepted by any DPDA.

- Why? There has to be a way to identify the palindrome's middle.
- After reading the first 8 letters of the string *aabaabaa*, must both be ready to accept, if that is the end of the string, and be ready to read in more letters, in case it is just the first half of the string.
- In a DPDA, that is just not possible.

- Especially bad - we need to keep track of lots of other possibilities: the 8-letter string is the beginning of the 11-letter palindrome *aabaabaabaa*, etc.
- We cannot keep track of all of these choices for the middle: only one transition available at a time.

This can be made into a proof if we know how many states the DPDA is claimed to have; try it.

DPDAs and ambiguity

Theorem 14.1.2. *If P is a DPDA, then $N(P)$ has an unambiguous grammar.*

Proof.
- Follow the grammar produced in the theorem that transformed a PDA into a CFG.
- The only place of some concern is that transitions where $\delta(q, a, X)$ included multiple symbols, $(p, Y_1Y_2 \cdots Y_n)$ turned into a pile of distinct new rules in the grammar.
- However, since the PDA is deterministic, we can see that only one derivation in the grammar will actually lead to the word that is accepted by the DPDA.

□

You can modify the proof to work for acceptance by final state, too. (That is Theorem 6.21 in the text.)

Overall hierarchy of languages We now know that:
- All finite languages are regular.
- All regular languages can be accepted by a DPDA, so are DCFLs.
- There are non-regular DCFLs (we just saw that $L = \{x \in \{a, b\}^* \mid n_a(x) > n_b(x)\}$ is a DCFL - it is an exercise to prove that L is not regular).
- DCFLs are all not inherently ambiguous (Theorem on previous slide).
- There are languages that are not accepted by DPDAs that are not inherently ambiguous (e.g. palindromes).
- Some CFLs that are not inherently ambiguous are not DCFLs (e.g. palindromes).
- All CFLs are accepted by PDAs, and the languages accepted by PDAs are exactly the CFLs.

15 Lecture 15

Outline

1. A Normal Form For CFGs - M7 1-19

15.1 A Normal Form For CFGs

The motivation for introducing such a normal form is that it will enable us (via Theorem 15.1.8) to develop a Pumping Lemma for CFLs.

Theorem 15.1.1. *For any context-free grammar G, there is a context-free grammar G' such that $L(G) = L(G')$ (with the possible exception of ε), where all rules of the grammar G' are of one of the following two forms:*

1. *$A \to BC$ where A, B and C are variables*
2. *$A \to a$, where A is a variable and a is a terminal*

*Grammars in this form are in **Chomsky Normal Form**, or CNF.*

Outline of the proof of Theorem 15.1.1 We need to replace many kinds of now-forbidden rules:

1. $A \to \varepsilon$
2. $A \to B$
3. $A \to ABC$
4. (a) $A \to Ab$
   (b) $A \to ab$

Some of these simplifications are easier than others; none is especially hard.

1. **Removing ε rules**

   Defintion 15.1.2. *A variable A in a grammar G is **nullable** if $A \overset{*}{\underset{G}{\Rightarrow}} \varepsilon$.*

   If A is nullable, and there is a rule in the grammar $B \to AC$, we add a rule $B \to C$ to the grammar, and the language of the grammar does not change.

   - Previous derivation: $B \Rightarrow AC \overset{*}{\Rightarrow} C \cdots$
   - New derivation $B \Rightarrow C \cdots$

   We can do this for *any* nullable variable.

   - The only word lost is ε, in the case where S is nullable.

   Theorem 15.1.3. *To identify nullable variables, apply this test:*
   (a) If there exists a rule $A \to \varepsilon$, then A is nullable.

(b) *If there exists a rule* $A \to B_1 B_2 \cdots B_m$*, and all* B_i *are nullable, then* A *is nullable.*
(c) *No other variables are nullable.*

Remark: We lose no generality by assuming that all the B_is are variables: If any B_i is a terminal, then the derivation cannot produce ε from A.

Proof. First, suppose that the test discovers the variable A. Then the shape of the test provides an explicit derivation $A \stackrel{*}{\Rightarrow} \varepsilon$, which witnesses that A is nullable. It remains to prove that, if A is nullable, then the test must discover A.
We need to show that every nullable variable A is discovered by this test. Suppose that there is a k-step derivation $A \stackrel{k}{\Rightarrow} \varepsilon$. The argument is by induction on k, the length of the derivation.

- Base case ($k = 1$): Then there is a rule $A \to \varepsilon$, and so the above test discovers that A is nullable.
- Induction case ($k > 1$): The induction hypothesis is that for any strictly shorter derivation $B \stackrel{*}{\Rightarrow} \varepsilon$, the test discovers that B is nullable.
  - The first step in the derivation of ε from A is $A \Rightarrow B_1 \cdots B_m$, where all the derivations $B_i \stackrel{*}{\Rightarrow} \varepsilon$ have strictly fewer than k steps. (No terminals can occur in the first step, as the derivation ends in the empty word.)
  - So by the induction hypothesis, the test discovers the B_is are all nullable, and hence that A is nullable also.

□

We now must add new rules to the grammar corresponding to the nullable variables.

- Suppose $A \to aBcD$, with B and D nullable.
- Add new rules: $A \to acD$, $A \to aBc$, and $A \to ac$ to the grammar, corresponding to the cases where B generates ε, where D does, and where both do.

In general: from a rule with m nullable variables on the right hand side, add at most $2^m - 1$ new rules, removing each possible subset of the list of nullable variables. (There are 2^m ways of including / excluding the m nullable variables, and we already have the original rule in which all m of them are included.)

Then, remove null productions $A \to \varepsilon$ from the grammar.

Theorem 15.1.4. *Let G_1 be the grammar constructed in this way from the original grammar G. Then either $L(G_1) = L(G)$ or $L(G_1) \cup \{\varepsilon\} = L(G)$.*

- We will not show both directions of the proof (See Theorem 7.9 in the text).
- Here is the proof that if $w \in L(G)$ and $w \neq \varepsilon$, then $w \in L(G_1)$ (i.e. a proof that $L(G) \setminus \{\varepsilon\} \subseteq L(G'))$.
- We will show more generally that, for **any** variable A, if $A \underset{G}{\overset{*}{\Rightarrow}} w$, then $A \underset{G_1}{\overset{*}{\Rightarrow}} w$.
- This is sufficient because we may then take $A = S$.
- The proof is by induction on k, the number of steps in the derivation $A \underset{G}{\overset{k}{\Rightarrow}} w$.
- Base ($k = 1$):
  - Then $A \to w$ is a production in G.
  - Since $w \neq \varepsilon$, therefore $A \to w$ is a production in G_1 also.
  - Therefore we have $A \underset{G_1}{\overset{*}{\Rightarrow}} w$, as required.
- Induction ($k > 1$): Suppose that we have a k-step derivation $A \underset{G}{\overset{k}{\Rightarrow}} w$, for $k > 1$.
  - The induction hypothesis is that for all derivations $B \underset{G}{\overset{\ell}{\Rightarrow}} x$ with $\ell < k$, we have $B \underset{G_1}{\overset{*}{\Rightarrow}} x$.
  - The first step in the derivation of w in G is $A \Rightarrow B_1 B_2 \cdots B_m$, where each B_i is a variable or a terminal.
  - At least one variable remains after the first step, as we are not in the base case.
  - Write $w = w_1 w_2 \cdots w_m$, where $B_i \underset{G}{\overset{*}{\Rightarrow}} w_i$ for all i. (If B_i is a terminal, say $B_i = w_i$, then $B_i \underset{G}{\overset{*}{\Rightarrow}} w_i$ trivially.)
  - Some of the w_i may be ε, but not all, as $w \neq \varepsilon$. Let $C_1, \ldots, C_n$ be the B_i that correspond to the non-ε subwords of w.
  - Since the other B_is are nullable, by construction there exists a derivation in G_1 that starts with $A \Rightarrow C_1 \cdots C_n$.
  - Each C_i yields its corresponding w_i in G, in fewer than k steps.

– So, by induction, $C_i \overset{*}{\underset{G_1}{\Rightarrow}} w_i$, for all i. Then derive w in G_1 via

$$A \underset{G_1}{\Rightarrow} C_1 \cdots C_n \overset{*}{\underset{G_1}{\Rightarrow}} w_1 C_2 \cdots C_n \overset{*}{\underset{G_1}{\Rightarrow}} \cdots \overset{*}{\underset{G_1}{\Rightarrow}} w_1 \cdots w_n = w.$$

2. **One-variable transformations**

Defintion 15.1.5. *A **unit production** is a production of the form $S \to A$, with only one variable on the right hand side.*

We want to get rid of unit productions. Why?

- One reason: avoid cycles like $S \Rightarrow A \Rightarrow B \Rightarrow S \Rightarrow \cdots$.

Easy:

- Basic idea: find all of the variables we can get to from a given variable.
- If $S \overset{*}{\Rightarrow} A$, then add all of A's productions directly to S's productions.

Defintion 15.1.6. *Variables (A, B) are a **unit pair** if $A \overset{*}{\Rightarrow} B$.*

Theorem 15.1.7. *We can find unit pairs by a simple recursive definition:*

(a) (A, A) is a unit pair for any pair A.
(b) If (A, B) is a unit pair and there is a rule $B \to C$ in our grammar, where C is a variable, then (A, C) is a unit pair.
(c) No other pairs are unit pairs.

Easy proof (another induction, which we will not do; it is Theorem 7.11 in the text) that this method finds all unit pairs.

Removing unit productions If $S \overset{*}{\Rightarrow} A$ in our grammar G, add the productions for A to the productions for S.

Then, remove all unit productions.

Denote the new grammar by G_1.

- Any production that previously used the derivation in G starting from $S \overset{*}{\Rightarrow} A \Rightarrow B_1 B_2 \cdots B_m$ can now use the rule $S \to B_1 B_2 \cdots B_m$ in the new grammar G_1.
- This shows that $L(G) \subseteq L(G_1)$.
- Now, consider a derivation of a word w in $L(G_1)$.
  - Suppose we use a rule $S \to B_1 B_2 \cdots B_m$ in G_1 for a variable S that came from a rule $A \to B_1 B_2 \cdots B_m$ in G, where (S, A) is a unit pair in G.

- Take derivation $S \overset{*}{\underset{G}{\Rightarrow}} A \Rightarrow B_1 B_2 \cdots B_m$.
- Then the rest of derivation follows; any word we can derive in G_1, we can also derive in G.

- This shows that $L(G_1) \subseteq L(G)$.
- Therefore we have $L(G_1) = L(G)$.

3. **Remaining bad kinds of rules** For $A \to B_1 B_2 \cdots B_m$, where $m > 2$, create a cascading sequence of rules:
   - Only two symbols on right hand side for each rule.
   - If we take the first rule for A, then we will produce (eventually) all of $B_1 B_2 \cdots B_m$.

   This is not hard. Create $m - 2$ new variables $C_1, \ldots, C_{m-2}$, and these rules:

$$\begin{aligned} A &\to B_1 C_1 \\ C_1 &\to B_2 C_2 \\ C_2 &\to B_3 C_3 \\ &\vdots \\ C_{m-2} &\to B_{m-1} B_m \end{aligned}$$

   The new derivation is: $A \Rightarrow B_1 C_1 \Rightarrow B_1 B_2 C_2 \overset{*}{\Rightarrow} B_1 B_2 \cdots B_m$.
   If some of the B_i are terminals, then some of the rules we have just added are still are not allowed in a CNF grammar.
   We will correct this in the next (and last) step.
4. **The last step** In Chomsky Normal Form, a grammar has two kinds of rules:
   - $A \to BC$, for variables A, B and C
   - $A \to a$, for variables A and terminals a

   If we start with an arbitrary grammar, and we:
   - Remove ε-productions
   - Remove unit productions
   - Remove long productions

   then the only possible remaining obstacle to being in CNF is that we might still have rules of the form $A \to bc$ or $A \to Bc$, with one terminal on the right hand side of the arrow, but two symbols.
   The last step, finished This is easy:
   - For a rule of the form $A \to bc$:
     - Add two new variables:

        * X_b, and
        * X_c.
      - Add three new productions:
        * $A \to X_b X_c$,
        * $X_b \to b$, and
        * $X_c \to c$.
  - For a rule of the form $A \to Bc$:
    - Add the variable: X_c.
    - Add the productions:
      * $A \to BX_c$
      * $X_c \to c$

  Throughout, we must make certain that the new variable names added are not duplicates of any existing variable names. Then since the new variables are only used in these derivations, so they do not change the language of the grammar.

  The new grammar fits the desired framework.

Chomsky Normal Form algorithm From a general CFG:

1. Remove ε-productions.
   (a) Find nullable variables.
   (b) Change rules using them
   (c) Then remove all ε-productions.
2. Remove one-variable productions.
   (a) Find unit pairs (A, B) for each variable A.
   (b) Add B's rules to A.
   (c) Then remove one-variable productions
3. Remove long productions.
   (a) Create cascading sequence of definitions.
4. Remove terminals from two-letter rules.
   (a) Create a new variable for each terminal, and substitute it into the rules

Example: Let

$$G : S \to AB, A \to \varepsilon|0, B \to 1.$$

Note that $L(G) = \{1, 01\}$. Construct G', in Chomsky Normal Form, such that $L(G') = L(G)$.

Solution:

1. nullable variables A is nullable. Hence to construct G_1 fron G, we
   (a) add $S \to B$ and

(b) remove $A \to \varepsilon$.

$$G_1 : S \to AB|B, A \to 0, B \to 1.$$

2. unit pairs (S, B) is a unit pair. Thus we
   (a) add $S \to 1$ and
   (b) remove $S \to B$.

$$G_2 : S \to AB|1, A \to 0, B \to 1.$$

   Observe, G_2 is already in CNF.
3. long productions No changes: $G_3 = G_2$.
4. terminals in two-letter productions No changes: $G' = G_4 = G_3$.

It is now clear that $L(G') = L(G) = \{1, 01\}$.

Theorem 15.1.8. *Let G be a CNF grammar. Let $w \in L(G)$ be arbitrary. Then **any** derivation of w in G takes $2|w| - 1$ steps.*

Proof. By induction on $|w|$. We will instead prove that for any variable A in G, if $A \stackrel{*}{\Rightarrow} w$, then the derivation must be of length $2|w| - 1$ steps. This is sufficient because we may then take $A = S$.

- The grammar cannot make ε, so the base case is $|w| = 1$.
- Base ($|w| = 1$):
  - I claim that the only step in the derivation is $A \to w$.
  - There are no nullable variables, so if we instead started with a rule of form $A \to BC$, we would have to produce at least 2 letters in the end.
  - So the only derivation of a 1-letter word takes 1 step.
  - Since $1 = 2(1) - 1$, therefore the base case holds.
- Induction ($|w| > 1$):
  - The induction hypothesis is that for all words x satisfying $A \stackrel{*}{\Rightarrow} x$ and $|x| < |w|$, the derivation of x takes $2|x| - 1$ steps.
  - As we are not in the base case, the first step in the derivation of w must be of the form $A \to BC$.
  - We know that $B \stackrel{*}{\Rightarrow} w_1$ and $C \stackrel{*}{\Rightarrow} w_2$, where $w = w_1 w_2$, and neither of w_1 or w_2 is ε.
  - Since w_1 and w_2 are both shorter than w, by the induction hypothesis, the derivations for them are of lengths $2|w_1| - 1$ and $2|w_2| - 1$.

– So the overall derivation, first using the $A \to BC$ rule, and then the derivations for w_1 and for w_2, takes $2|w_1| - 1 + 2|w_2| - 1 + 1 = 2(|w_1| + |w_2|) - 1 = 2|w| - 1$ steps.

□

16 Lecture 16

Outline

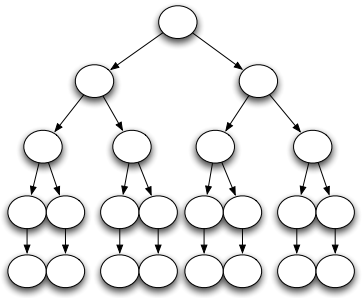
1. Pumping Lemma for CFLs - M7 20-37

16.1 Pumping Lemma for CFLs

Toward a pumping lemma for CFGs

Suppose we have a CNF grammar G with p variables.

- Given k leaves, the height of the tree (number of edges in a longest path from the root of the tree to a leaf) is **at least** $1 + \log_2 k$.



In detail, Theorem 7.17: Suppose we have a parse tree according to a CNF grammar G and suppose the yield of the tree is a word w. If the height of the tree is ℓ, then $|w| \leq 2^{\ell-1}$.

- The proof is by induction on ℓ.
- Base ($\ell = 1$): The length of a path is one less than the number of nodes on the path (count the edges).
- Thus a tree with height 1 consists of only a root and a leaf.
- Therefore $|w| = 1$, and $1 \leq 2^{1-1} = 2^0 = 1$ holds.
- Induction ($\ell > 1$):
- The induction hypothesis is that any parse tree of height $q < \ell$ has yield of length at most 2^{q-1}.

- The root of the tree must use a production of the form $A \to BC$ (as we are not in the base case).
- The induction hypothesis applies to the subtrees rooted at B and C, so these subtrees have yields of lengths at most $2^{\ell-2}$.
- The yield of the tree is the concatenation of the yields of these two subtrees, thus its length is at most $2^{\ell-2} + 2^{\ell-2} = 2^{\ell-1}$.

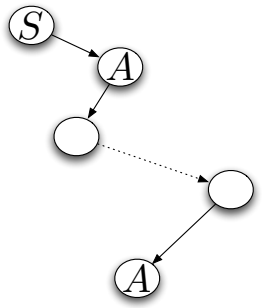
In detail (completed), The Theorem implies that, for a word $z \in L(G)$ with length at least 2^p, a parse tree for z has height at least $p + 1$.

- Suppose that $2^p \leq |z|$.
- Then by the Theorem we have $2^p \leq |z| \leq 2^{\ell-1}$, where ℓ is the height of a parse tree for z.
- Then we must have $p \leq \ell - 1$, or in other words $p + 1 \leq \ell$.

The Theorem also implies that the height of a parse tree for a word of length k is at least $1 + \log_2 k$.

- By the Theorem we have $k \leq 2^{\ell-1}$, where ℓ is the height of a parse tree.
- Then we must have $\log_2 k \leq \ell - 1$, or in other words $\log_2 k + 1 \leq \ell$.

Repeated variables on the parse tree Now in a parse tree of height at least $p+1$, there must be a repeated variable (call it A) on a path from root to any terminal on the bottom tree level.



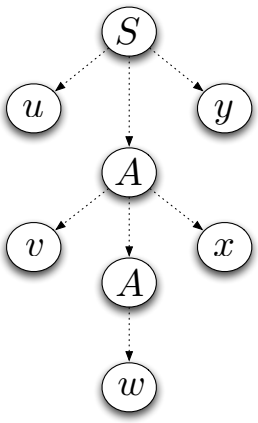
- There are p variables in grammar.
- There are $p+1$ variables and one terminal on a path starting from the root.
- By the pigeonhole principle, there must be a repeated variable.

What does that mean?

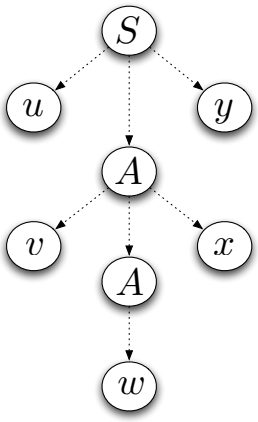
Repeated variables, in the derivation One derivation of the word z in

G is of the form:

$$\begin{aligned} S &\overset{*}{\Rightarrow} uAy \\ &\overset{*}{\Rightarrow} uvAxy \\ &\overset{*}{\Rightarrow} uvwxy = z \end{aligned}$$



Making a pumping lemma
Important: $A \overset{*}{\Rightarrow} vAx$ **and** $A \overset{*}{\Rightarrow} w$.
So $A \overset{*}{\Rightarrow} vAx \overset{*}{\Rightarrow} vvAxx \overset{*}{\Rightarrow} v^iAx^i \overset{*}{\Rightarrow} v^iwx^i$, for any choice of $i \geq 0$!
This will give our pumping lemma for CFLs.
Note: it cannot be the case that $vx = \varepsilon$, as a **non-trivial** repetition of A occurs, and unit productions $A \overset{*}{\Rightarrow} A$ are not allowed in a CNF grammar.



One more trick Choose a pair of repeated variables **near the bottom of the parse tree**.

- By assumption $|z| \geq 2^p$, so that a parse tree for z has height at least $p+1$.
- So there exists a terminal in z with a path of length at least $p+1$ above it.
- By the pigeonhole principle, there is a (non-trivially) repeated variable (Say A) in this path, no more than $p+1$ levels above the leaf.
- Then we have $A \stackrel{*}{\Rightarrow} vAx$ and $A \stackrel{*}{\Rightarrow} w$, i.e.
  - the yield of the subtree rooted at the lowest A is the word w, and
  - the yield of the subtree rooted at the second lowest A is the word vwx.
- By construction the subtree rooted at the second lowest A has height at most $p+1$.
- Applying Theorem 7.17, we have $|vwx| \leq 2^{(p+1)-1} = 2^p$.
- As the repetition of A is non-trivial, therefore v and x are not both ε (unit productions $A \stackrel{*}{\Rightarrow} A$ are not allowed in a CNF grammar).

A full statement of the CFL Pumping Lemma

Lemma: Let G be a CFG in Chomsky Normal form, with p variables.

- Any word $z \in L(G)$ of length at least 2^p can be decomposed as $z = uvwxy$, where
- $|vwx| \leq 2^p$,
- v and x are not both ε, and
- and for all nonnegative i, $uv^iwx^iy \in L(G)$.

As with the pumping lemma for regular languages, we can remove the dependency on a specific choice of CFG for the CFL, since all CFLs have a CNF grammar.

Revised version of the pumping lemma

Let L be a context-free language.

- There exists an $n > 0$ such that any word $z \in L$ where $|z| \geq n$ can be decomposed as $z = uvwxy$, where
- $|vwx| \leq n$,
- v and x are not both ε, and
- for all nonnegative i, $uv^iwx^iy \in L$.

What does this say about non-context-free languages?

Pumping Lemma for CFLs (the way we will use it)

Pumping Lemma for CFLs 16.1.1. *Let L be a language.*

- *Suppose that for any $n > 0$, there exists a word $z \in L$ with $|z| \geq n$ such that:*
  - *For **any** decomposition $z = uvwxy$, where $|vwx| \leq n$ and $|vx| > 0$,*
  - *There exists an $i \geq 0$ such that that uv^iwx^iy is **not** in L.*
- *Then L is **not** context free.*

Turning it into English
Let L be a language. Suppose that for any $n > 0$ (Suppose that for any definition of long):

- there exists a word $z \in L$ with $|z| \geq n$ (There is a long word), such that
- for any decomposition $z = uvwxy$, where $|vwx| \leq n$ and $|wx| > 0$, (for which every decomposition)
- there exists an $i \geq 0$ such that $uv^iwx^iy \notin L$. (is not pumpable.)

Then L is not context free.
This gives us a recipe for proving that a given language is not context free.
To prove a language is not context free Our recipe:

- Find a long word.
- Look at its decompositions.
- Show they cannot be pumped.

Or, formally:

- For given $n > 0$, find a word $z \in L$ at least n letters long.
- Look at all decompositions $z = uvwxy$, with $|vwx| \leq n, vx \neq \varepsilon$.
- Say something useful about the decompositions
- For each decomposition, find an i such that uv^iwx^iy is not in L.
- Then the language L is not context free.

Examples:

1. $L = \{a^ib^ic^i \mid i \geq 0\}$
   I claim that the language $L = \{a^ib^ic^i \mid i \geq 0\}$ is **not context-free**.
   - For each $n > 0$, find a word $z \in L$ that is at least n letters long. We choose the long word $z = a^nb^nc^n$. Clearly $z \in L$.
   - Consider decompositions $z = uvwxy$ with $|vwx| \leq n$ and $vx \neq \varepsilon$.
   - Say something useful about all such decompositions.
     - All such decompositions have one or two types of letters in vwx, but not all 3.
     - (Why? The smallest consecutive substring with all 3 symbols is ab^nc; it has length $n + 2$.)

- In particular, vx omits one or two letters of the set $\{a, b, c\}$.
- For each decomposition, find an i such that uv^iwx^iy is not in L.
  - Consider uwy (i.e. take $i = 0$). Observe that uwy does not have the same number of a's, b's and c's, since one of these letters is not in vx, and at least one is!
  - Hence, $uwy = uv^0wx^0y$ is not in L. (Neither is $uvvwxxy$).

We have shown that z cannot be pumped, and hence, L is not context free.

2. Consider $L = \{a^ib^jc^k \mid i < j, i < k\}$.
   - Let $n > 0$ be arbitrary.
   - Long word: $z = a^nb^{n+1}c^{n+1} \in L$.
   - Consider decompositions $z = uvwxy$ with $|vwx| \leq n$ and $vx \neq \varepsilon$. In all of them, vx has either no a's, or has a's but no c's.
     - Case 1: No a's.
       Then uwy has fewer b's or fewer c's than z, but there are not fewer a's.
       So uwy does not have fewer a's than both b's and c's, and therefore uwy is not in L.
     - Case 2: a's, but no c's.
       $uvvwxxy$ has as at least as many a's as c's, so $uvvwxxy$ is not in L.
   - So **no** decomposition of our long word $z = a^nb^{n+1}c^{n+1}$ can be pumped.

   And, thus, L is not context free.

3. Somewhat surprising, maybe:
   $L = \{ss \mid s \in \{a, b\}^*\}$.
   L includes words like aa or $abbabb$ or ε or $abaaba$.
   - For a given $n > 0$, find a long word. We will use $z = a^nb^na^nb^n$. (This choice might not be so obvious.)
   - Decompose into $z = uvwxy$, with $|vwx| \leq n$ and $vx \neq \varepsilon$. Then uwy must have at least one a or one b removed from one of the two copies of the identical string.
   - But when we remove vx from $uvwxy$ to form uwy, and lose a letter from the copied word, we cannot lose the corresponding letter on the other side; it is too far away.
   - Therefore $uwy \notin L$.
   - So L is not context-free.

   (See Example 7.21 of the text for all the gory details.)

4. **A bit surprising**
   The very similar-looking $L = \{ss^R \mid s \in \{a, b\}^*\}$, of even-length palindromes, is context free, with this grammar:
   - $S \to aSa \mid bSb \mid \varepsilon$

   However the previous example is still not context-free; we cannot keep all the information available whenever it is needed. (PDAs, which only recognize CFLs, have trouble with doing this.)

17 Lecture 17

Outline

1. Closure Rules for CFLs - M7 38-45
2. Quick Review of Decidability/Undecidability - CS 245
3. Decision Problems for CFLs - M7 46-50

17.1 Closure Rules for CFLs

CFLs are:

- Closed under union, concatenation, Kleene star and reversal.
- **Not** closed under intersection or complementation.

The easy ones

1. Union:
   - Grammar $G_1 : S_1 \to \cdots$
   - Grammar $G_2 : S_2 \to \cdots$ (with all different variable names)
   - Construct: $G : S \to S_1 | S_2 \cdots$
2. Concatenation:
   - Grammar $G_1 : S_1 \to \cdots$
   - Grammar $G_2 : S_2 \to \cdots$ (with all different variables)
   - Construct: $G : S \to S_1 S_2 \cdots$
3. Kleene star:
   - Grammar $G_1 : S_1 \to \cdots$
   - Construct: $G : S \to \varepsilon | S_1 S$
4. **Reversal** Reversal is not hard, either.
   - Given a grammar G, construct a new grammar G', by reversing the outputs of all of the productions in G.
   - For example, if G has a production $S \to XYZ$, then add the rule $S \to ZYX$ to G'.

- (If the grammar is in CNF, this works especially easily.)
- Now for a given derivation of a word $w \in L(G)$, apply the corresponding rules in G' to generate $w^R \in L(G')$.
- Then by construction, $L(G') = L(G)^R$.
- Then since $L(G)^R$ is the language of a context-free grammar, therefore $L(G)^R$ is a context-free language.

5. **Intersection** We have already seen a language that shows that the intersection of two CFLs is not always a CFL.
   $L = \{a^i b^i c^i \mid i \geq 0\}$ (we saw that this language is **not** context-free).
   - $L = L_1 \cap L_2$, where:
     - $L_1 = \{a^i b^i c^j \mid i, j \geq 0\}$.
     - $L_2 = \{a^i b^j c^j \mid i, j \geq 0\}$.
   - L_1 and L_2 are each the concatenation of two context-free languages, so context free.
   - In detail, define
     - $L_{11} = \{a^i b^i \mid i \geq 0\}$ (a CFL, with grammar $G : S \to aSb|\varepsilon$), and
     - $L_{12} = \{c^j \mid j \geq 0\} = L(c^*)$ (regular, and thus a CFL).
     - Then $L_1 = L_{11}L_{12}$.
     - And $L_{21} = \{a^i \mid i \geq 0\} = L(a^*)$ (regular, and thus a CFL).
     - $L_{22} = \{b^j c^j \mid j \geq 0\}$ (a CFL, with grammar $G : S \to bSc|\varepsilon$), and
     - Then $L_2 = L_{21}L_{22}$.

   Therefore, the class of context-free languages is **not** closed under intersection!
6. **Intersection of a CFL with a *regular* language**
   If L_1 is context free and L_2 is regular, then $L_1 \cap L_2$ is context free.
   - Suppose that a PDA M accepts L_1 by final state, and that a DFA D accepts L_2.
   - Let R be the states of M, and $F_M \subseteq R$ be the accept states.
   - Let S be the states of D, and $F_D \subseteq S$ be the accept states.
   - Define a PDA, P, which accepts by final state, with
     - States $Q = R \times S$,
     - Accept states $F = F_M \times F_D$,
     - and transition function δ, defined from the transition functions δ_M for M and δ_D for D (ignoring stack manipulations

for the moment):

$$\delta(a,(r,s)) = \begin{cases} \{(\delta_M(\varepsilon,r),s)\} & \text{if } a=\varepsilon \\ \{(\delta_M(a,r),\delta_D(a,s))\} & \text{if } a\neq\varepsilon \end{cases}$$

  - Manipulate the stack in P exactly as it was manipulated in M.
- Then P is a PDA, and from construction, we have that
  - P accepts w
  - if and only if M accepts w and D accepts w
  - if and only if $w \in L_1 \cap L_2$, so that $L_1 \cap L_2$ is a CFL.

Note, this construction will **not** work for intersection of two arbitrary CFLs: both PDAs would need editing access to the one stack.

7. **Complementation**

The following example will show that **the class of context-free languages is not closed under taking complements**. Let

$$\begin{aligned} L &= \{a^i b^i c^i \mid i \geq 0\} \\ L_1 &= \{a^i b^j c^k \mid i \neq j \text{ or } k \neq j\} \end{aligned}$$

- Then L_1 is context-free, as it is the union of the four CFLs:
  i. $L_{11} = \{a^i b^j c^k \mid i < j\} = \{a^i b^j \mid i < j\}\{c^k \mid k \geq 0\}$,
  ii. $L_{12} = \{a^i b^j c^k \mid i > j\} = \{a^i b^j \mid i > j\}\{c^k \mid k \geq 0\}$,
  iii. $L_{13} = \{a^i b^j c^k \mid j < k\} = \{a^i \mid i \geq 0\}\{b^j c^k \mid j < k\}$, and
  iv. $L_{14} = \{a^i b^j c^k \mid j > k\} = \{a^i \mid i \geq 0\}\{b^j c^k \mid j > k\}$.
- For example, a grammar for $\{a^i b^j \mid i < j\}$ is $G : S \to b|Sb|aSb$.

Now, consider $L_2 = L(a^*b^*c^*)'$. That is, L_2 is the set of words that are **not** of the form $a^i b^j c^k$, for any choice of i, j, k.

- Then L_2 is regular, as it is the complement of a regular language. (Exercise: What is a regular expression for L_2?)
- Then L_2 is a CFL.

Then $L_3 = L_1 \cup L_2$ is context-free, as it is the union of two context-free languages.

Note that words in L_3 are:

- of the form $a^i b^j c^k$ for some i, j, k, but not having $i = j = k$, or
- not of the form $a^i b^j c^k$, for any choice of i, j, k.

Now, consider L_3'. I claim that

$$L_3' = \{a^i b^i c^i \mid i \geq 0\}.$$

We have

$$L_3' \quad = \quad (L_1 \cup L_2)' \\ \underbrace{=}_{\text{DeMorgan}} \quad L_1' \cap L_2'.$$

- $L_2' = (L(a^*b^*c^*)')' = L(a^*b^*c^*)$ is the set of words that **can** be written in the form $a^ib^jc^k$, for some choice of i, j, k,
- and L_1' is the set of such words for which $i = j = k$,
- and therefore our description of L_3' is correct.
- We have already seen that L_3' is **not** context free.

L_3 is context free, and its complement is **not** context-free.

Therefore the class of context-free languages is **not** closed under complementation.

Contrasts with DCFLs

DCFLs **are** closed under complementation.

- Proving this is non-trivial.
- See the additional notes for Module 7.

Simple proof that there are context-free languages that are not DCFLs: we just saw one. L is context-free, while L' is not context-free (and hence not a DCFL)

17.2 Quick Review of Decidability/Undecidability

Definition 17.2.1. *A **decision problem** is a problem which calls for an answer of either yes or no, on some input.*

Examples:

1. Given a program P and input I, will P halt when run with input I? (the **Halting Problem**)

Definition 17.2.2. *A decision problem is **decidable** if there exists an algorithm that, given an input to the problem,*

- *outputs yes (or true) if the input has answer yes, and*
- *outputs no (or false) if the input has answer no.*

*A decision problem is **undecidable** if it is not decidable.*

Remarks:

1. It is important to point out that the algorithm must always complete after **finitely many steps**.

2. To prove that a decision problem is decidable, **write down an algorithm to decide it**.
3. A decision problem may be undecidable, and yet have particular choices of input for which the correct yes/no answer can be determined. To say that a decision problem is undecidable is to say that no algorithm exists to give the correct yes/no answer **for every input**.
4. It is proved in CS 245 that **the Halting Problem is undecidable**. We will re-prove that fact soon, using Turing Machines.

17.3 Decision Problems for CFLs

Decision Problems for CFLs
Lots of the analogs to the problems we saw in Module 4 for regular languages are **not** solvable by computers.
What we can do: membership

1. Given a CFG, does its language include the word w?
   (a) If $w = \varepsilon$, then test the starting variable, S, for nullability. If S is nullable, then yes; otherwise no.
   (b) Otherwise, $w \neq \varepsilon$. The rest of the algorithm is to handle the case $w \neq \varepsilon$.
   (c) Turn the provided CFG into CNF.
   (d) Try all derivations of length $2|w| - 1$.
   (e) Does any of them derive w?
2. Given a PDA, does its language include the word w?
   (a) Turn it into a CFG (algorithm in slides).
   (b) Use algorithm #1 on the CFG produced. (Note: This is an example of a **reduction**. Reductions will be crucial when working with Turing machines at the end of the course.)
   (c) We cannot just run the PDA: it might run forever!
3. Given a CFG, is its language empty? First turn the CFG, G, it into CNF.

   Theorem 17.3.1. *If a CFG in CNF, G, having p variables generates any words, then it must generate a word with fewer than 2^p letters.*

   Proof. (a) Assume $L(G) \neq \emptyset$.
   (b) Let $z_0 \in L(G)$ be arbitrary.
   (c) If $|z_0| < 2^p$, then we are finished.

(d) Otherwise, $|z_0| \geq 2^p$ and by the proof of the Pumping Lemma, we can decompose $z_0 = u_0v_0w_0x_0y_0$, with $|v_0w_0x_0| \leq 2^p$, $v_0x_0 \neq \varepsilon$ and $u_0w_0y_0 \in L(G)$.
(e) Let $z_1 = u_0w_0y_0$.
(f) If $|z_1| < 2^p$, then we are finished.
(g) Otherwise, $|z_1| \geq 2^p$ and $z_1 \in L(G)$ and so by the proof of the Pumping Lemma, we can decompose $z_1 = u_1v_1w_1x_1y_1$, with $|v_1w_1x_1| \leq 2^p$, $v_1x_1 \neq \varepsilon$ and $u_1w_1y_1 \in L(G)$. Let $z_2 = u_1w_1y_1$.
(h) Continuing in this way we obtain a sequence of words in $L(G)$ having strictly decreasing lengths: $z_0, z_1, z_2, \ldots$.
(i) As z_0 has finite length, after at most $|z_0| - 2^p + 1$ steps, we will obtain a word in $L(G)$ with length $< 2^p$.

□

Enumerate **all** of them, and test membership for each. This is unbelievably slow, but it will work.

Undecidable problems Other sensible problems are **undecidable**:

1. Given two CFGs, do their languages have any words in common?
2. Given two CFGs, are their languages equal?
3. Is the language of a CFG equal to Σ^*?
4. Given two CFGs, is the language of one a subset of the other's?
5. Is a given CFG ambiguous? (**Note**: this is about the **grammar**, not the **language**.)
6. Is a given CFL **inherently** ambiguous?

18 Lecture 18

Outline

1. Limits of Computation - M8 1-8
2. Introduction to Turing Machines - M8 9-12

18.1 Limits of Computation

1. In CS 245, you saw that **the Halting Problem is undecidable**.
2. In the slides, there are examples of
   (a) Two programs about which we can ask: does the program halt and return the output 1?, and for which
      i. one program clearly does while

ii. the second program runs forever.

(b) a variation on the proof from CS 245 about the Halting Problem.

Moral: Not every decision problem can be answered by a computer.
We will come back to all of this in Module 9, when we can talk about Turing machines, not Python programs.

18.2 Turing machines

1. We have been studying simple models of computation.
2. We need a model of computation which is more similar to our understanding of how people and computers work.
3. In particular, we need to be able to access memory more robustly than we can in a pushdown automaton.

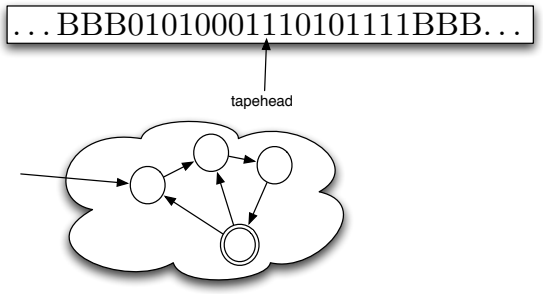
How do computers compute?
A simpler question: how do people compute?

1. We pull information from long-term memory into short term.
2. We work with the contents of short-term memory.
3. When finished, we write results to the long-term memory,
4. Until we have solved our problem.

As an automaton model
Turing's model of a computing machine has two parts:

1. A finite automaton
   (a) has short-term memory
   (b) tells us what to do with the short-term information, and what to write to long-term memory
2. A one-dimensional tape which represents the long-term storage
   (a) This machine does not access memory like a regular computer, but they are equivalent.

Church/Turing thesis: Anything we can do with a Turing machine we can do with any other reasonable computing model.
"If an algorithm exists, then a TM can be built to carry out that algorithm."
Turing machine = Finite automaton plus memory tape



- Long-term memory is only accessed at the tape head: we can only see one letter of memory at a time.

1 step in the TM:

- Given the current state in the FA, and the letter at the tape head
  - Move to a different state
  - Maybe change the letter at the tape head (or leave it alone)
  - Move tape head left or right

19 Lecture 19

Outline

1. Formal Definition of a Turing Machine - M8 13-27
2. Programming Turing machines - M8 28-30

19.1 Formal Definition of a Turing Machine

Formalization

To describe this, we need a 7-tuple:

1. Finite automaton control:
   (a) Σ: the alphabet for candidate words
   (b) Q: finite set of states
   (c) F: the subset of final states
   (d) q_0: initial state
2. Tape control:
   (a) Γ, the tape alphabet ($\Sigma \subseteq \Gamma$, since we start with $w \in \Sigma^*$ on the tape)
   (b) B, the blank symbol for the tape
   (c) Note: $B \in \Gamma$, but $B \notin \Sigma$.

3. And a transition function:
   (a) δ: transition function. $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$
   (b) If $\delta(q, a) = (p, b, L)$ (for $a, b \in \Gamma$), then the machine
      i. switches to state p,
      ii. overwrite a with b on the tape, and
      iii. moves the tape head to the **left**.
   (c) This δ need not be a full function: if there is no value of $\delta(q, a)$, the machine crashes in state q upon seeing the tape symbol a.

The machine **halts** when we reach an accept state or crash.
It can also run forever.

More specifics about TMs

- The machine has an infinite tape.
- When we launch the TM with input $w \in \Sigma^*$:
  - $|w|$ consecutive positions of the tape are filled with w,
  - the tape head points to w_1, the first character of w, and
  - all of the rest of the tape is filled with the blank symbol B.

Remember: the input could be of arbitrary size, but is always a finite string.

Instantaneous descriptions for Turing machines

Instantaneous description of the TM's state after some number of steps of computation:

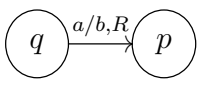
- Current state, q.
- The contents of the tape, $X_1X_2 \cdots X_k$. Surrounding it is an infinite number of Bs in both directions.
- The current position of the tape head. We will underline the current position; your text does something much grosser.
- If we want the tape head pointing to the first symbol, w_1 of a sequence of symbols, $w = w_1w_2 \cdots w_n$, we will underline w, referring to the first symbol in it.

So, at the beginning of the execution of the TM, the instantaneous description is written as $(q_0, \underline{w_1}w_2 \cdots w_n)$, or as $(q_0, \underline{w})$.

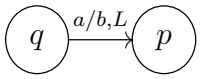
How to transition in the TM

If the current instantaneous description of the machine is $(q, x\underline{a}y)$:

- If $\delta(q, a) = (p, b, R)$, then the new instantaneous description is $(p, xb\underline{y})$. Shorthand this by writing $(q, x\underline{a}y) \vdash (p, xb\underline{y})$.



- If $\delta(q, a) = (p, b, L)$, then the new instantaneous description of the machine is $(p, x_1 \cdots x_{k-1}\underline{x_k}by)$, and we write $(q, x\underline{a}y) \vdash (p, x_1 \cdots x_{k-1}\underline{x_k}by)$.



If there is no value for $\delta(q, a)$, then the machine crashes.

Special transitions in the Turing machine

Special cases:

- Tape head at leftmost nonblank character and we move left: the tape head will move to a new blank character, B: $(q, \underline{a}y) \vdash (p, \underline{B}by)$, if $\delta(q, a) = (p, b, L)$.
- Tape head at rightmost nonblank character: similarly, the tape head will move to a new blank character, B: $(q, x\underline{a}) \vdash (p, xb\underline{B})$, if $\delta(q, a) = (p, b, R)$.
- Erasing the last letter on the tape. For example, if $\delta(q, b) = (p, B, L)$, then $(q, xa\underline{b}) \vdash (p, x\underline{a})$.

Acceptance in the TM

We also have multi-step computations: in the Turing machine M, we write $(q, x\underline{a}y) \overset{*}{\underset{M}{\vdash}} (p, v\underline{b}x)$ if we can move from the first configuration to the second in some finite number of steps.

- To keep the notation uncluttered, we only indicate the machine M when necessary.

The Turing machine M **accepts** $w = w_1w_2 \cdots w_n$ exactly if $(q_0, \underline{w_1}w_2 \cdots w_n) \overset{*}{\underset{M}{\vdash}} (p, x\underline{a}y)$ for any accept state $p \in F$ and any string $xay \in \Gamma^*$.

- The tape need not be empty, and
- the input need not have been fully examined.

Now we can define the language of a Turing machine:

- The **language** of the machine $M, L(M)$, is the set of words which M accepts.

Definitions:

1. A language L is **recursively enunerable** if it is the language of some TM.
2. A language L is **recursive (decidable)** if it is the language of some TM **which halts for every input**.

Rejection in the TM, running forever

How does a TM **reject** an input?

- If the machine winds up in a state q, pointing at a, and $\delta(q, a)$ is not defined, then it crashes and rejects the word.
- This is called "Halting and rejecting".
- (Later, once we move beyond implementation details and into describing algorithms for Turing machines, we will simply make our algorithms explicitly crash when necessary.)

A Turing machine can also fail to accept an input word w by running forever while processing it.

- This is possible with a PDA, too, or even with an ε-NFA.

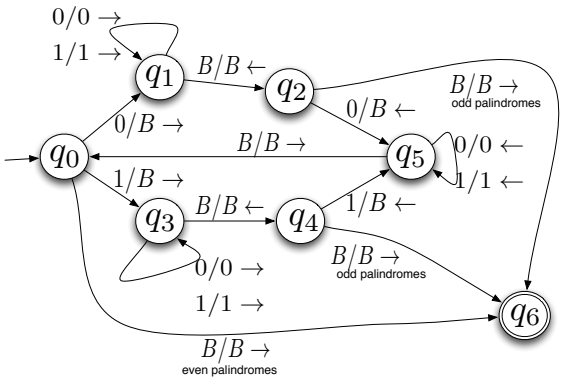
If M either accepts w or crashes on it, we say that M **halts** on input w.

- Halting on every input is a good property for a Turing machine.
- If a language L is the language accepted by a Turing machine which halts on every input, then we say that L is **recursive**, or **decidable**.

An example: palindromes

Let's construct our first TM, to accept the language of palindromes, $L = \{x \mid x = x^R\}$.

Draw like a DFA, but with a new character for the tape and the arrow for the tape direction after a slash. (Or use L and R instead)



Explanation of the Diagram

1. Top Branch (q_1, q_2, q_5): Match a 0 at the start with a 0 at the end; delete both.
2. Bottom Branch (q_3, q_4, q_5): Match a 1 at the start with a 1 at the end; delete both.
3. Odd Length Palindromes:
   (a) If the last (middle) character is 0, then delete it as we move $q_0 \to q_1$. Then the tape is all B, so go $q_2 \to q_6$ and accept.
   (b) If the last (middle) character is 1, then delete it as we move $q_0 \to$

q_3. Then the tape is all B, so go $q_4 \to q_6$ and accept.

4. Even Length Palindromes: We arrive back at q_0 after the last characters are deleted. Then the tape is all B, so go to q_6 and accept.
5. Rejecting All Non-Palindromes:
   (a) Start with 0 on the left; crash when we read a 1 in q_2.
   (b) Start with 1 on the left; crash when we read a 0 in q_4.

An accepting computation

Consider the palindrome $w = 010$.

$$\begin{aligned}
(q_0, \underline{0}10) &\vdash (q_1, \underline{1}0) \\
&\vdash (q_1, 1\underline{0}) \\
&\vdash (q_1, 10\underline{B}) \\
&\vdash (q_2, 1\underline{0}) \\
&\vdash (q_5, \underline{1}) \\
&\vdash (q_5, \underline{B}1) \\
&\vdash (q_0, \underline{1}) \\
&\vdash (q_3, \underline{B}) \\
&\vdash (q_4, \underline{B}) \\
&\vdash (q_6, \underline{B})
\end{aligned}$$

The machine accepts, having deleted the whole word.

A word not in L

Consider the non-palindrome $w = 0100$.

$$
\begin{aligned}
(q_0, \underline{0}100) &\vdash (q_1, \underline{1}00) \\
&\vdash (q_1, 1\underline{0}0) \\
&\vdash (q_1, 10\underline{0}) \\
&\vdash (q_1, 100\underline{B}) \\
&\vdash (q_2, 10\underline{0}) \\
&\vdash (q_5, 1\underline{0}) \\
&\vdash (q_5, \underline{1}0) \\
&\vdash (q_5, \underline{B}10) \\
&\vdash (q_0, \underline{1}0) \\
&\vdash (q_3, \underline{0}) \\
&\vdash (q_3, 0\underline{B}) \\
&\vdash (q_4, \underline{0})
\end{aligned}
$$

...and the machine crashes, rejecting $w = 0100$.
TMs can also accept non-context-free languages, like

$$L = \{s!s \mid s \in \{a, b\}^*\}\,.$$

We are not going to draw the machine that accepts L. (See §8.3.2 in the text.)
Idea: match letters from one copy of s with those in the other copy.

1. First, we ensure that there is exactly one ! character in the word.
2. Then, we match the first "remaining" character on each side of the ! character, and remove them.
3. Until there is nothing left.

19.2 Programming Turing machines

1. Store a small (finite) amount of information in the state (e.g. in the palindrome TM, which symbol to match)
2. Tag letters of the word with more bits of information, expanding the alphabet as needed.
3. Use subroutines where needed.

Remarks:

1. Our machine for palindromes **decides** the language of palindromes, since it accepts palindromes and crashes on all non-palindromes.
2. If our machine instead ran forever on some non-palindromes, it would only **accept** the language of palindromes.
3. If a Turing machine M accepts the language L, but does not halt on all inputs, then this does not imply that L is not recursive.
   (a) You may just have the wrong TM.
   (b) It might be possible to choose an improved TM which halts on all inputs and accepts L.

20 Lecture 20

Outline

1. Computable functions - M8 31-37
2. Using Subroutines - M8 38-46

20.1 Computable functions

Idea: To construct a TM which computes a function, say $y = f(x)$, for some positive integers x and y:

1. Start with x on the tape, encoded as 1^x.
2. Finish by entering a final state, with the convention that y will then be encoded on the tape as x was above.

Definition 20.1.1. *A function $f(x)$, such that there exists a Turing machine to compute it as above, is called a* ***computable function****.*

Remarks:

- Our assumption that this happens for every input $x \in \Sigma^*$ was very strong. We can weaken this assumption and still get something useful.
- Note that f need not be **total**: there could be words in Σ^* for which f is not defined. On those inputs, M may crash or run forever.
- The function can have multiple arguments or be multi-valued.
- If $f : \Sigma^* \times \Sigma^* \to \Gamma^*$, then we will have the two arguments for f next to each other on the tape when M starts running, with a blank character B between them.

Our first computable functions will be **unary** (i.e. they will take a single input), and return a single output:

- Suppose $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a function, and for all $x \geq 1$, we have $(q_0, \underline{1^x}) \overset{*}{\underset{M}{\vdash}} (q_F, \underline{1^{f(x)}})$, where q_F is any accept state.
- Then M **computes** f, and f is **computable**.
- Again, for multi-argument functions, the arguments on the tape are separated by blanks.

A simple example

Consider the addition function: $+ : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$.

- We encode the positive integer i on the tape as 1^i.
- Then $x+y$ is the concatenation of the two arguments 1^x and 1^y, namely 1^{x+y}.
- This $+$ is computable: we can certainly concatenate two strings with a Turing machine.

Exercise: Construct a Turing machine that performs this computation.

Characteristic functions

- In set theory, the **characteristic function of the subset** $X \subseteq S$ (for some "universe" S) is the function:

$$\begin{array}{rccl} \chi_X & : & S & \to & \{0,1\} \\ & & s & \mapsto & \begin{cases} 1 & \text{if } s \in X \\ 0 & \text{if } s \notin X \end{cases} \end{array}$$

  which indicates whether an arbitrary $s \in S$ lies in X or not (like an indicator function in Stat 230).
- Hence the **characteristic function of the language** $L \subseteq \Sigma^*$ is the function:

$$\begin{array}{rccl} \chi_L & : & \Sigma^* & \to & \{0,1\} \\ & & w & \mapsto & \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{if } w \notin L \end{cases} \end{array}$$

- This function answers the question, "is the word x in the language L?"
- If the Turing machine M computes χ_L:
  - On any input $x \in \Sigma^*$, M **always** accepts.
  - When M halts, either 1 or 0 is left on the input, with the tape head pointing at that symbol.

If we can compute χ_L, then membership in L is decidable

Suppose M computes χ_L.

Then we can build a machine M' to decide membership in L:

On input x:

- Since M always accepts, M' always gets through the M module.
- Then M' either
  - accepts (if the tape head is pointing at a 1 when M accepts), so $x \in L$, or
  - crashes (if the tape head is pointing at a 0 when M accepts).

Then M' decides membership in L.
If f is computable (or respectively if χ_L is computable), then we say that f (or respectively L) **has an algorithm**.

20.2 Using Subroutines

How to use subroutines (assuming a subroutine TM is created and ready to use):

1. Set up the input for the subroutine.
2. Transition into the first state of the subroutine.
3. Wait until the subroutine halts (accepts).
4. Note that we must correctly handle the possibility that the subroutine runs forever (every subroutine is a Turing machine after all).

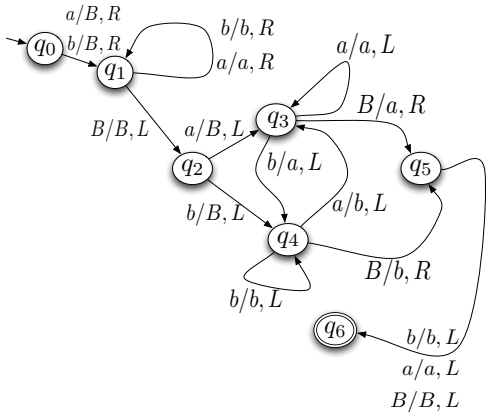
We do **not** use subroutines in Turing machines to shorten the program code. We care about ease of understanding, not program length.
Examples:

1. **Inserting a Character**
   The specification of our subroutine for inserting the character a at the current position of the tape head:
   - $(q_0, y\underline{z}) \overset{*}{\vdash} (q_F, ya\underline{z})$, where q_F is an accept state in the subroutine machine and $a \in \Sigma$.
   - Constraint: z does not have the blank character in it, so that we know when we have read the entire string z and moved it one character to the right.

   We might use this subroutine if we want to insert a different character into a string, by first inserting the character a, and then replacing the a with that character.

2. **Deleting a Character**
   Delete the character the tape head is pointing at:
   - $(q_0, y\underline{a}z) \overset{*}{\vdash} (q_F, y\underline{z})$, where q_F is again an accept state in the subroutine machine and $a \in \Sigma$.
   - Again, constrain z to not contain any blank character.

   Deletion machine



   How it works

   (a) In state q_0, we delete the current tape head character and move right.
   (b) In state q_1, we move to the right until we have read the entire word on the input.
   (c) Then, we remember the last letter, and put it into the position where the second-to-last letter was, remembering that.
   (d) In state q_3, the previous symbol was a.
   (e) In state q_4, the previous symbol was b.
   (f) Push the whole way to the beginning of the string, and copy in the last character as we move to state q_5.
   (g) From state q_5, move the tape head back to the correct position.
   (h) Accept in q_6.

   Remark: If z were B, we would crash once we reach q_2. So clarify the specification to enforce $|z| \geq 1$.
   Exercise: Enhance the provided TM so that it won't crash if z is B.

Example of Computation: processing $a\underline{b}ba$

$$\begin{aligned}(q_0, a\underline{b}ba) &\vdash (q_1, aB\underline{b}a)\\ &\vdash (q_1, aBb\underline{a})\\ &\vdash (q_1, aBba\underline{B})\\ &\vdash (q_2, aBb\underline{a})\\ &\vdash (q_3, aB\underline{b})\\ &\vdash (q_4, a\underline{B}a)\\ &\vdash (q_5, ab\underline{a})\\ &\vdash (q_6, a\underline{b}a)\end{aligned}$$

Storage in the state
Another trick, implicitly just used: augment a TM with a finite amount of memory.
In our previous example, we could combine q_2, q_3 and q_4 into a single state by storing the previous symbol seen in the memory of M.

21 Lecture 21

Outline

1. Variations on a Turing machine - M8 47-69
   (a) Multiple Tapes - M8 48-52
   (b) Multiple Tape Heads - M8 53-55
   (c) Non-Determinism - M8 56-68

21.1 Variations on a Turing machine

21.1.1 Multiple Tapes

In a **multi-tape** Turing machine, we have k tapes (for some positive integer k), each with its own tape head.

1. instantaneous description:
   (a) what is stored on each of the k tapes,
   (b) the position of all k tape heads, and
   (c) the state in the finite automaton.
2. Transition function is of form: $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$.
3. Conventions:

(a) the (finite) sequence of input symbols is placed onto the first tape (with the first tape head pointing to the leftmost character), and
(b) all other tapes start off filled with blanks (with their tape heads pointing to arbitrary positions).

Q: Can we do more with a multi-tape machine than with a single-tape machine?

Theorem 21.1.1. *1. If a language is decidable by a multi-tape Turing machine, then it is decidable by a one-tape Turing machine.*
2. If a language is accepted by a multi-tape Turing machine, then it is accepted by a one-tape Turing machine.

Proof. We will focus on a language **accepted** by a 2-tape Turing machine, M, and show it is also accepted by a 1-tape Turing machine, M'. (This argument generalizes to languages decided by M', or to machines with more than two tapes, with few changes.) First, some housekeeping:

1. Suppose that M uses the tape alphabet Γ. (Recall that $B \in \Gamma$.)
2. Then let the tape alphabet for M' be $\Gamma \times \{B, *\} \times \Gamma \times \{B, *\}$.
3. I.e. each position of the single tape for M' is two symbols from Γ, where each position is either tagged with a $*$ (to indicate tape head positions) or not.
4. We may think of the single tape for M' as having four "tracks", to remember these four pieces of information.
5. For example, part of the tape might look like

B	0	1	0	B
B	*	B	B	B
B	1	1	1	B
B	B	B	*	B

Initial Setup: See slides for the details - they agree with your intuition.
Using the one-tape TM M' to simulate one transition in the multi-tape TM M

1. Store M's current state in M''s finite memory.
2. Locate the two tape heads in M, by searching from the left of M''s tape, for the $*$ characters in the second and fourth positions of the four-tuple that is a letter in the tape alphabet of M'.
3. **Store the two corresponding characters** in M''s finite memory.
4. Now, M' knows the state in M's finite automaton, and the two symbols pointed to by the two tape heads of M.

5. Therefore M' can determine which transition M will make.
6. Then, M' updates the two values at the sites of the tape heads, (locating them by searching from left to right, again).
7. And M' marks the proper positions on the tape to indicate the new positions of the simulated tape heads of M.
8. Store M's new state in M''s finite memory.
9. Rewind the tape and repeat.
10. Declare the accepting states of M' to coincide with those for M, so that M' will accept exactly when M does.

□

Why multi-tape TMs are useful:
Example: Consider $L = \{w!w \mid w \in \{0,1\}^*\}$.

1. We can create a one-tape TM to test membership in this language.
2. But it is easier to create a multi-tape TM.
3. See the details in the slides.

21.1.2 Multiple Tape Heads

1. We could have multiple tape heads on one tape.
2. Similarly to multiple tapes, we can simulate a multi-tape-head TM using an ordinary TM.
3. See the details in the slides.
4. **Other memory alterations**
   (a) 2-dimensional memory (tape head moves up, down, left or right),
   (b) or even higher dimensions,
   (c) or a multi-tape machine where we store an address in the first tape that we can use to access the second tape: we can then model something like random-access memory, instead of sequential.

Example: Create a two-tape Turing Machine to recognize palindromes over $\{0,1\}$.
Solution: We will use

1. underlining, e.g. $\underline{a}$ to indicate the position of the first tape head, and
2. overlining, e.g. $\overline{a}$ to indicate the position of the second tape head.

By convention we start with tape contents

	B	$\overline{\underline{w_1}}$	w_2	$\cdots$	w_n	B	

where $w = w_1 \cdots w_n$. We have the following two cases, depending on the parity of n.

1. $\underline{n \text{ is odd:}}$ Write $n = 2k+1$, for some k. Then is clear that we can arrive at the tape contents

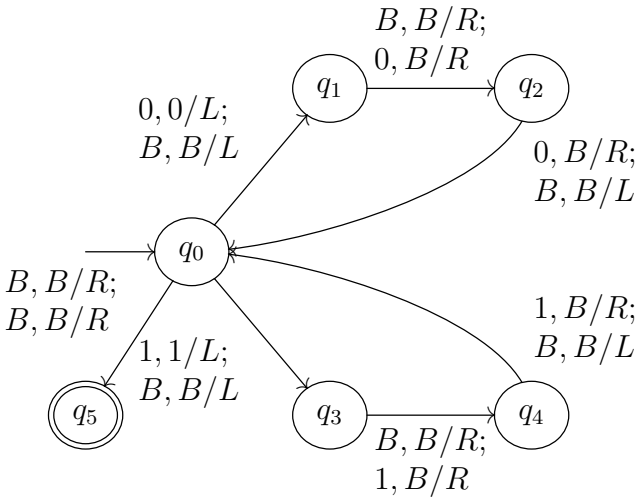
	B	$\underline{w_1}$	w_2	$\cdots$	$\overline{w_{2k+1}}$	B	

and from here we can easily decide whether w is a palindrome or not by matching symbols on the left and right until we reach the middle of the word, and accept when the left tape head sees B. We won't show any additional details for this case.

2. $\underline{n \text{ is even:}}$ Write $n = 2k$, for some k. Then is clear that we can arrive at the tape contents

	B	$\underline{w_1}$	w_2	$\cdots$	w_{2k}	$\overline{B}$	B	

and from here we need to demonstrate how we can decide whether w is a palindrome or not. We will still match symbols on the left and right until we reach the middle of the word. But we will need to move both tape heads to detect matched symbols, unlike in the previous case. The following Turing machine will carry out the matching algorithm in this case. We label each transition showing the actions at the two tape heads explicitly.



Exercise: Show the sequence of instantaneous configurations of this Turing machine while it processes:

(a) $w = 0110$

(b) $w = 0100$

21.1.3 Non-Determinism

A **nondeterministic** Turing machine can have multiple values for $\delta(q, a)$.

1. There can only be a **finite** number of entries in $\delta(q, a)$.
2. We write $(q, y\underline{x}) \vdash (p, w\underline{z})$ if **one** transition in $\delta(q, x)$ gets us to the second configuration.
3. The non-deterministic machine accepts input x if a valid computation exists that brings us from the starting configuration to an accepting configuration.
4. It is **not** true that **all** paths must lead to an accepting configurations, any more than in NFAs or PDAs.
5. These nondeterministic TMs can only accept languages, rather than deciding them.
   (a) What should it mean if one thread accepts while another crashes or runs forever?
6. They also can not compute functions.
   (a) What should it mean if two valid computations disagree about the value of $f(x)$?

Example: See the slides for a sketch of a two-tape head nondeterministic TM which accepts $L = \{ww \mid w \in \{a, b\}^*\}$.

How powerful is nondeterminism? If we have n steps in the computation, and 2 choices at each level, there could be 2^n branches! Are nondeterministic TMs more powerful?

- No.
- There is the possibility that they are faster (see the P *vs.* NP problem in 341).
- But we only care about what they **can** do, **not how quickly** they do it.

Theorem 21.1.2. *If L is accepted by a nondeterministic Turing machine M, then there exists a deterministic Turing machine M' that also accepts L.*

Proof. See the slides.

Key Idea: For each $n = 0, 1, 2, 3, 4, \ldots$: model all computations of n steps.

- if one computation accepts, then accept
- otherwise, increment n and repeat.

If one computation accepts, then we will eventually discover it. If not, then we will run forever. □

Remark: Your text (Theorem 8.11) gives a very interesting proof that uses a queue, instead.
Also Useful: Slides 59-60 show that a TM with tape head moves $\{L, S, R\}$ (where S stands for "stay put") is no more powerful than an ordinary Turing machine, with only $\{L, R\}$.

22 Lecture 22

Outline

1. An Undecidable Language - M9 1-13
2. Other Undecidable Languages - M9 14-24

22.1 An Undecidable Language

A diagonalization argument The argument is similar to showing there are more real numbers than integers. This is counterintuitive: is one type of infinity **bigger** than another? **Yes.**

1. We say that sets S and T have **equal cardinality** if there exists a bijection f between S and T.
2. (Recall that a function $f : S \to T$ is a **bijection** if it is both injective and surjective.)
3. If S and T are both finite, then the definition of equal cardinality agrees with our intuition: the sets are of equal cardinality if they have the same number of elements.
4. For infinite sets, this can feel counter-intuitive: for example, if
   (a) $S = \{\text{even integers}\}$, and
   (b) $T = \{\text{all integers}\}$, and
   (c) $f(x) = \frac{x}{2}$,
   then $f : S \to T$ is a bijection.
5. S and T are of equal cardinality, even though our intuition might tell us that S is "half" the size of T.

Are there bigger and smaller infinities?

Countable sets

1. A set S is **countably infinite**: of equal cardinality to $\mathbb{Z}$ (or any infinite subset of $\mathbb{Z}$).
2. A set S is **countable**: of equal cardinality to a subset of $\mathbb{Z}$.

3. A countable set S can be described by listing: $S = \{s_1, s_2, s_3, \ldots, s_n\}$ for finite S, or $S = \{s_1, s_2, s_3, \ldots\}$ for infinite S.
4. A set S is **uncountable** if it is not countable.
5. If S is uncountable, then any listing $S = \{s_1, s_2, s_3, \ldots\}$ misses some members of S.

We know that countable sets exist, e.g. $\mathbb{Z}$, or { even integers }. The positive integers are also countable.

But do uncountable sets exist? Consider $S = [0, 1] \subset \mathbb{R}$. Is S countable?

The real numbers between 0 and 1 are uncountable. See the proof in the slides. It uses a **diagonalization** argument.

Building to a proof about Turing machines

1. We need to identify each Turing machine M (over the binary alphabet $\{0, 1\}$) with a binary string.
2. See the technique in the slides.
3. We encode a binary string to identify the Turing machine M; treat it as an integer index for each machine M. $f(M)$ is the integer that we produce in this way.
4. There can be multiple codes for M (but only finitely many). We could renumber the states, for example.
5. **The set of Turing machines is countable.** We just built a bijection to a subset of $\mathbb{Z}$.
6. There are only countably many Turing machines, hence there are only countably many recursively enumerable languages.
7. See the proof in the slides.

A language about Turing machines

Every Turing machine M has an identifier $w = f(M)$.

1. Consider the language

$$L_{SA} = \{w \mid \text{ the Turing machine } M \text{ represented by } w \text{ accepts } w\}.$$

2. L_{SA} is the language of identifiers for machines that accept when given their own identifiers as inputs.
3. L_{SA}: The subscript SA indicates **self-accepting**.
4. It might seem that we should not be allowed to make this definition, but why is this? Lots of programs process other programs! Remember: $w = f(M)$ is basically the code of the program for M.
   (a) Parsers, compilers, interpreters, profilers, *etc.*
   (b) For example, a syntax parser whose input is the parser should accept: "Yes, that program uses proper syntax."

It will turn out that L_{SA} is not decidable, but L_{SA} will not be our first example of an undecidable language.

An undecidable language

Instead, consider

$L_{NSA} = \{w \mid$ the Turing machine M represented by w does **not** accept $w\}$.

1. L_{NSA} is the language of identifiers for Turing machines that **do not** accept when given their own identifiers as inputs.
2. For example, a program that finds syntax errors: it will not accept itself, since it does not have any syntax errors! If M is a properly coded syntax error detector represented by w, then $w \in L_{NSA}$.

Is L_{NSA} the language of any Turing machine?

1. Suppose that $L_{NSA} = L(M)$, for some Turing machine M.
2. Does M accept its own identifier w? Is $w \in L_{NSA}$?
   (a) Suppose M accepts its own identifier w.
      i. Then w is not in L_{NSA}, which is the language of M.
      ii. So M does not accept w.
   (b) Suppose M does not accept its own identifier w.
      i. Then w is in L_{NSA}, which is the language of M.
      ii. So M accepts w.

This tells us that L_{NSA} is not the language of any Turing machine.

1. We formalize this via a retread of Cantor's diagonalization argument.
2. For any Turing machine M, L_{NSA} differs from $L(M)$ in at least one position.
   (a) If the Turing machine M represented by w accepts w (so that $w \in L(M)$), then $w \notin L_{NSA}$.
   (b) If the Turing machine M represented by w does not accept w (so that $w \notin L(M)$), then $w \in L_{NSA}$.
3. L_{NSA} differs from the languages of all Turing machines.
4. By definition, L_{NSA} is **not recursively enumerable**.
5. Then L_{NSA} is also **not decidable**.

(Your text denotes L_{NSA} as L_d, where the subscript d reminds us of **diagonalization**.)

1. Now we see why the uniqueness of the identifier for a given Turing machine is not crucial for this result.
2. **No** identifier can represent a Turing machine M such that $L(M) = L_{NSA}$.

22.2 Other Undecidable Languages

1. Some undecidable languages **are** recursively enumerable.
2. Consider L_{SA}, the language of encodings for machines that **do** accept their encoding.
3. L_{SA} is recursively enumerable, but not recursive.
4. We need a new Turing machine, the Universal Turing Machine: an **interpreter**.

What will the Universal Turing Machine do?

The Universal Turing Machine U simulates a Turing machine M.

1. Input: a pair of strings, (e, w). (If you like, the machine's tape alphabet can have parentheses and commas.)
2. If e is not the encoding of any Turing machine, then U rejects (e, w).
   (a) Many identifiers correspond to no Turing machine at all, e.g. 00110 does not start with 1, and
   (b) 11001011101001010011 is not valid because it has three consecutive 1s.
3. If $e = f(M)$ for some Turing machine M, and M accepts w, then U accepts (e, w).
4. If $e = f(M)$ and M rejects w, then U rejects (e, w).
5. If $e = f(M)$ and M runs forever on w, then U runs forever on (e, w).

Define $L_u = L(U)$: the **universal language**; it includes all pairs (e, w), where

1. $e = f(M)$ for some Turing machine M, and
2. $w \in L(M)$.

Does U exist?

U will have four tapes:

1. One tape keeps (e, w), which really is (M, w)
2. One tape maintains the tape for M
3. One tape maintains the current state q of M
   (a) (state q_i is indicated by 0^i)
4. One tape is used for scratch work

To make a transition in M, we:

1. Rewind U's first tape, and find the right transition in M for (q, a), where the second tape head points to a.
2. Important: we can find the correct values for $\delta(q, a)$ on the first tape (that binary string encodes all the logic from M after all).
3. Copy the new state for the machine onto the third tape.

4. Update the second tape and move the tape head.

Some last details

1. If we reach an accept state in M, then U accepts immediately.
2. If M crashes, then U must crash as well.
3. We will need a Turing machine to parse a possible identifier for a Turing machine to determine whether it is proper.
4. As we know how the identifiers are constructed, we can write an algorithm for this.
5. By the Church-Turing Thesis, we can create a Turing machine to execute the algorithm.

What good is this?

We may simulate one step of M's execution by many steps of U.

1. We only care that this simulation **can** be done, not how slowly it will run.

An interesting idea: we can run U on U's encoding. U is a Turing machine, after all.

Is L_u decidable?

The universal language, L_u is **recursively enumerable**.

1. Trivial: U is a Turing machine; its language is r.e.

L_u is not **decidable**.

1. For a contradiction, suppose that L_u is decidable. Suppose there is a Turing machine U' which decides L_u.
2. Recall that L_{NSA}: machine descriptions w of machines M that do not accept w as input, is not r.e., and thus not decidable.
3. We will use U' to construct a new Turing machine M_{NSA} that decides membership of w in L_{NSA}:
   (a) Run U' on input (w, w).
   (b) If U' accepts (w, w), then reject.
   (c) If U' rejects (w, w), then
      i. If w is the encoding of a Turing machine (which we can test), then accept.
      ii. Else reject.
   (d) By our assumption, there is no possibility that U' runs forever.

What does that get us?

We still need to argue that M_{NSA} decides membership in L_{NSA}.

1. Suppose that w represents some Turing machine M.
2. If M accepts w, then U' accepts (w, w). So M_{NSA} rejects w, as it should.

3. If M does not accept w, then U' rejects (w, w). So M_{NSA} accepts w, as it should.

Therefore M_{NSA} decides membership in L_{NSA}. But L_{NSA} is not decidable! This is not possible! Therefore U' cannot exist. This contradiction proves that U is not decidable.

We have proved that **the universal language, L_u, is recursively enumerable, but not decidable**.

Another recursively enumerable but not recursive language

1. L_{SA} is r.e.
   (a) Construct a Turing machine M, which, on input w,
   (b) runs U on (w, w),
   (c) accepts if U accepts (w, w), and
   (d) rejects if U rejects (w, w).
   (e) Note, U can run forever on input (w, w), causing M to run forever on input w.
   (f) Then by the definition of U, it is clear that $L(M) = L_{SA}$.
2. L_{SA} is not decidable.
   (a) If it were, then we could decide membership in L_{NSA} easily:
   (b) Check membership in L_{SA} and do the opposite.
   (c) but since L_{NSA} is not decidable, this is impossible.

23 Lecture 23

Outline

1. Closure Rules for TM languages - M9 25-28
2. Reductions - M9 29-31
3. Other Undecidable Problems about Turing machines - M9 32-44
4. Rice's Theorem - M9 45-49

23.1 Closure Rules for TM languages

1. Complements
   Theorem: If L is decidable, then so is its complement L'.
   (a) Proof: See the slides.
   (b) Idea: Decide whether $w \in L$, then negate the answer!
   Remark: This result is **false** if we replace 'decidable' with 'recursively enumerable':

2. **Theorem:** If both L and L' are recursively enumerable, then L is recursive.
   Proof:
   (a) Suppose that M accepts L and M' accepts L'.
   (b) Create a 3-tape Turing machine, M_L, to simulate running M and M' in parallel, in which:
      i. tape #1 controls whether the machine is currently simulating a step of M or a step of M' (after executing a step in one machine, it alternates to the other),
      ii. tape #2 simulates the tape of M, and
      iii. tape #3 simulates the tape of M'.
   (c) For any word w, either M or M' (and not both) must accept w in a finite number of steps.
      i. If M accepts w, then M_L accepts w.
      ii. If M' accepts w, then M_L rejects w.
   (d) Then L is decided by M_L.
   Remark: If the complement of every recursively enumerable language was recursively enumerable, then this Theorem would imply that every recursively enumerable language is decidable, and we already have examples that show that this is not true.
3. Intersections
   Theorem: The intersection of two r.e. languages is r.e..
   (a) Proof: See the slides.
   (b) Idea: Check whether $w \in L_1$ and whether $w \in L_2$ (if so, both will be confirmed in finite time).
4. Unions
   Theorem: The union of two r.e. languages is r.e.:
   (a) Proof: See the slides.
   (b) Idea: Non-deterministically check in parallel whether $w \in L_1$ or whether $w \in L_2$ (if so, at least one will be confirmed in finite time).

23.2 Reductions

Proper definition of reduction

1. Suppose we have two decision problems P_1 and P_2.
2. Suppose also that we have an algorithm A that transforms instances of P_1 into instances of P_2 such that:

(a) "Yes" instances of P_1 get mapped to "yes" instances of P_2.
(b) "No" instances of P_1 get mapped to "no" instances of P_2.
(c) The algorithm A always takes finite time.

3. Then we say that A **reduces** P_1 to P_2.

Using a reduction to prove a language is undecidable

Theorem 9.7: If there is a reduction from P_1 to P_2, then

1. If P_1 is undecidable, then P_2 is also undecidable.
2. If P_1 is non-recursively enumerable, then P_2 is also non-recursively enumerable.

Proof: See the slides.

23.3 Other Undecidable Problems about Turing machines

Instructor Note: You will only have time to present the details of at most one of these in class, I think. Present the details for L_{ne} and L_e only.

1. Empty language: Given a Turing machine M, does it accept $\emptyset$?
   (a) **Nonempty language:** Given w identifying the Turing machine M, is the language of M non-empty?
   More formally: Let

   $$L_{ne} = \{w \mid w \text{ identifies the Turing machine } M \text{ and } L(M) \neq \emptyset\}.$$

   L_{ne} is recursively enumerable. Here is an algorithm for a Turing machine M_{ne} that accepts L_{ne}:
   - Given the identifier w for the Turing machine M, guess a word x (nondeterministically) that M might accept.
   - (As our alphabet is finite, we can systematically test all possible words starting with the shortest ones first.)
   - Nondeterministically execute M on all possible choices of x.
   - If M accepts any choice x, then M_{ne} accepts w.

   L_{ne} is **not decidable**. We give a proof using Theorem 9.7. For a "bare hands" proof, see the slides.
   Let Σ be a non-empty finite alphabet (e.g. $\Sigma = \{0, 1\}$).
   Proof Using Theorem 9.7:
   - Define P_1: Is (e, w) in L_u?, and P_2: Is w' in L_{ne}?
   - The following algorithm reduces P_1 to P_2:
     – Let (e, w) be an arbitrary instance for P_1.

- Construct a new Turing machine, M', such that, for any input $x \in \Sigma^*$, M' will run U on (e, w).
- If U accepts (e, w), then M' accepts x.
- If U rejects (e, w), then M' rejects x.
- U may also run forever on (e, w).
- Then we have

$$L(M') = \begin{cases} \Sigma^* & \text{if } U \text{ accepts } (e, w) \\ \emptyset & \text{otherwise} \end{cases}$$

- Let w' represent M'.
- Now take w' as the corresponding instance for P_2.

- Then "yes" instances of P_1 are sent to "yes" instances of P_2, and "no" instances of P_1 are sent to "no" instances of P_2.
- We have **reduced** membership testing in L_u, the universal language, to membership testing in L_{ne}.
- As L_u is undecidable, therefore by Theorem 9.7, so is L_{ne}.

(b) **Simpler notation from now on:** $L_{ne} = \{M \mid L(M) \neq \emptyset\}$.
- (From now on, we stop talking about f, the encoding, as much.)

(c) **Empty Language:** Let

$$L_e = \{M \mid L(M) = \emptyset\}.$$

Is L_e r.e.? Is L_e decidable?
- Neither.
- The complement of L_{ne} is

$$L'_{ne} = L_e \cup \{w \mid w \text{ is not the encoding of any Turing machine }\}.$$

- As $\{w \mid w$ is not the encoding of any Turing machine $\}$ is decidable, therefore it is also recursively enumerable.
- Then if L_e is r.e., then so is L'_{ne} (as it is the union of two r.e. languages).
- But if a language and its complement are both r.e., then the language is decidable.
- And we know that L_{ne} is not decidable.
- So L_e cannot be r.e..
- Therefore L_e also cannot be decidable.

2. Finite language: Given a Turing machine M, is $L(M)$ finite?
   (a) **Infinite language:** Let

$$L_\infty = \{M \mid L(M) \text{ is infinite } \}.$$

   - Then L_∞ is undecidable:
   - Our earlier reduction of L_u to L_{ne} also reduces L_u to L_∞.
   - From a candidate instance (M, w) for the universal language, we produce a new machine M' whose language is infinite exactly when M accepts w.
   - Then "yes" instances of L_u are mapped by our algorithm to "yes" instances of L_∞ and "no" instances of L_u are mapped by our algorithm to "no" instances of L_∞.
   - As we cannot decide L_u, therefore By Theorem 9.7 we cannot decide L_∞ either.
   - The same reduction also shows that $L_A = \{M \mid L(M) = \Sigma^*\}$ is undecidable.

   (b) **Finite language:** I claim that $L_{fin} = \{M \mid L(M) \text{ is finite } \}$ is also undecidable.
   Proof:
   - For a contradiction, suppose that L_{fin} is decidable
   - Then its complement, $(L_{fin})'$, is also decidable.
   - Writing

$$(L_{fin})' = L_\infty \cup \underbrace{\{w \mid w \text{ is not the encoding of any Turing machine } \}}_{\text{decidable}},$$

     we will have the desired contradiction by applying the following Lemma (as we already know that L_∞ is undecidable).

   (c) **Turing Machines with a Finite Language**
   Lemma: If sets A and B satisfy $A \cap B = \emptyset$, and if $A \cup B$ and B are both decidable, then A is decidable.
   Proof:
   - Let x be a candidate for membership in A.
   - Test x for membership in $A \cup B$.
     - If $x \notin A \cup B$, then reject x,
     - otherwise, test x for membership in B.
       * If $x \in B$, then reject x,
       * otherwise, accept x.

- As $A \cap B = \emptyset$, this algorithm will accept x if and only if $x \in A$, as required. $\square$

We then get the desired result in the previous point by taking

$$\begin{aligned} A &= L_\infty, \text{ and} \\ B &= \{w \mid w \text{ is not the encoding of any Turing machine }\}. \end{aligned}$$

3. Regular language: Given a Turing machine M, is its language regular?
   (a) **Turing Machines with a Non-Regular Language**
   - Let $L_{nreg} = \{M \mid L(M) \text{ is not regular}\}$.
   - We will reduce membership in L_u to membership in L_{nreg}, then apply Theorem 9.7.
   - Given a candidate instance (M, w) for L_u, we construct a new machine M' that accepts a non-regular language if M accepts w, and a regular language if M does not accept w.
   - For any input x, our new machine M' simulates M on w.
     - If M accepts w, then M' accepts x if and only if $x = 0^i 1^i$, for some $i \geq 0$ (and rejects x otherwise).
     - (We know that the set of all such words is **not** a regular language.)
     - If M does not accept w, then M' rejects x.
     - (The empty language is regular by definition.)
   - Then

$$L(M') = \begin{cases} \{0^i 1^i \mid i \geq 0\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{otherwise} \end{cases}$$

   - We have reduced membership in L_u to membership in L_{nreg}.
   - But L_u is undecidable.
   - So by Theorem 9.7, L_{nreg} is undecidable, too.

   (b) **Turing Machines with a Regular Language**

   I claim that $L_{reg} = \{M \mid L(M) \text{ is regular }\}$ is also undecidable.

   Proof:
   - For a contradiction, suppose that L_{reg} is decidable
   - Then its complement, $(L_{reg})'$, is also decidable.
   - Writing

$$(L_{reg})' = L_{nreg} \cup \underbrace{\{w \mid w \text{ is not the encoding of any Turing machine }\}}_{\text{decidable}},$$

we will have the desired contradiction by applying the previous Lemma (as we already know that L_{nreg} is undecidable) with:

$$\begin{aligned} A &= L_{nreg}, \text{ and} \\ B &= \{w \mid w \text{ is not the encoding of any Turing machine }\}. \end{aligned}$$

23.4 Rice's Theorem:

These results suggest that any "interesting" property of Turing machine languages is not decidable. We make this notion precise in the following Theorem.

Rice's Theorem: Let P be a property of some, but not all, recursively enumerable languages. (I.e. P is some non-empty proper subclass of the class of r.e. languages.) Then the language $L_P = \{M | L(M) \in P\}$ is undecidable.

Remark: Here we only work with the identifiers for legal Turing machines.

Proof: Cases

1. Suppose that the empty language, $\emptyset$, is not in P.
   - Let L be a non-empty recursively enumerable language that is in P.
   - Let M_L be a Turing machine that accepts L.
   - We will reduce membership in L_u to membership in L_P, then apply Theorem 9.7.
   - Let (M, w) be a candidate instance for L_u.
   - Create a new machine, M' that, on any input x, simulates U on (M, w).
     - If U accepts (M, w), then we simulate M_L on x.
       * If M_L accepts x, then M' accepts x.
       * If U rejects (M, w) or if M_L rejects x, then M' rejects x.
       * If U runs forever on (M, w), or if U accepts (M, w) and M_L then runs forever on x, then M' runs forever on x.
   - The above construction gives us that the language of M' is

$$L(M') = \begin{cases} L & \text{if } U \text{ accepts } (M, w) \\ \emptyset & \text{otherwise} \end{cases}$$

   - Let w' identify M'.
   - Take w' as our candidate for membership in L_P.

- Then "yes" instances for L_u are sent to "yes" instances for L_P (as we chose $L \in P$ and $L \neq \emptyset$), and
- "no" instances of L_u are sent to "no" instances for L_P (as $\emptyset \notin P$).
- We have reduced membership in L_u to membership in L_P.
- But since L_u is undecidable, then by Theorem 9.7, L_P is also undecidable.

2. Suppose that the empty language, $\emptyset$, is in P.
   - Consider the property P', which is the negation of property P.
   - Then since $\emptyset \notin P'$, therefore testing membership in $L_{P'}$ is undecidable, by the proof of Case 1.
   - Since every Turing machine accepts a recursively enumerable language, therefore $(L_P)' = L_{P'}$.
   - For a contradiction, suppose that L_P is decidable.
   - Then $(L_P)' = L_{P'}$ is also decidable.
   - But this contradicts the fact that testing membership in $L_{P'}$ is undecidable.
   - Therefore L_P must be undecidable.

Decidable Problems About TMs

1. "Does this Turing machine have fewer than k states?"
2. "Does this Turing machine ever move the tape head left on any input?"
3. Most decidable problems are not interesting.

Problems About Two TMs

There are obvious problems about two languages, too: given M_1 and M_2,

1. is $L(M_1) = L(M_2)$?
2. is $L(M_1) \subseteq L(M_2)$?
3. is $L(M_1) \cap L(M_2) = \emptyset$, i.e. are the languages **disjoint**?

These are all undecidable, too.

1. Suppose we could decide any of these problems.
2. Then suppose we wanted to decide, for an arbitrary machine M, whether $M \in L_e$. (Recall: that problem is undecidable.)
3. Make new machines M_2 that rejects all inputs, and M_3 that accepts all inputs.
4. If $L(M) = L(M_2)$, then $M \in L_e$ (else $M \notin L_e$).
5. If $L(M) \subseteq L(M_2)$, then $M \in L_e$ (else $M \notin L_e$).
6. If $L(M) \cap L(M_3) = \emptyset$, then $M \in L_e$ (else $M \notin L_e$).

In all three cases, given a machine that decides the desired equality or containment, we could then decide membership in L_e, which is undecidable.

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24 Lecture 24

Outline

1. Decision problems about CFGs and CFLs - M9 50-59
   (a) Post's Correspondence Problem - M9 51-53
2. Course Wrap-up - M9 60-69
3. Course Evaluations

24.1 Decision problems about CFGs and CFLs

We did not answer these questions, before:

1. CFG-intersection:
   (a) Given: Two grammars G_1 and G_2.
   (b) Question: Is $L(G_1) \cap L(G_2) \neq \emptyset$?
2. CFG-ambiguous:
   (a) Given: Grammar G.
   (b) Question: Is G ambiguous?

To show that they are both undecidable, we need to define a new problem.

24.1.1 Post's Correspondence Problem

1. It is a funny undecidable game (sort of).
2. It dates to roughly WWII.
3. We are given a finite set of "tiles", where each tile contains two strings over a finite alphabet Σ.

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n).$$

4. We want a non-empty string x, where it is possible to join together a sequence of tiles from the set (allowing repetition), and where the concatenation of the a_i strings and the concatenation of the b_i strings are both equal to x.

Example of PCP

Tiles:

$T_1 : (00, 001), T_2 : (11, 10)$ and $T_3 : (011, 1)$.

Want: string x to obtain from both parts of the tiles.

1. (**Note**: we do not know what x is!)
2. Guess x. Suppose $x = 001100011$.
3. Tile sequence: (T_1, T_2, T_1, T_3).

4. Look at the first strings: $00 + 11 + 00 + 011 = 001100011$
5. And the second strings: $001 + 10 + 001 + 1 = 001100011$
6. These are the same.
7. We have exhibited a solution to this instance of PCP.
8. There are instances of PCP for which no solution exists. See Example 9.14 on p402 of the text.
9. So the question naturally arises: "Is there an algorithm to decide whether any given instance of PCP has a solution or not?"
10. In other words, "Is PCP decidable?"

PCP is undecidable

Theorem 24.1.1. *PCP is not decidable. (In other words, given any instance of PCP, no algorithm exists to determine whether that instance can be solved.)*

Proof. This Theorem is proved by reducing membership in the universal language to deciding PCP:

1. Given an instance (M, x) of the universal language L_u.
2. Compute a set of tiles, such that if M accepts x, it also is a "yes" instance of the PCP, and vice versa.
3. See the text for details; it is pretty.

□

So what?

We can apply this fact to prove that the above two CFG problems are undecidable:

1. Deciding CFG-intersection decides PCP.
   (a) Given **any** PCP instance $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$.
   (b) Let $L(A)$ be all strings we can obtain by concatenating some words from A together, ending with a string of tags c_i to indicate the reversed order of the tiles used.
      i. Example: suppose we append a_1, a_3, a_7. Then $a_1a_3a_7c_7c_3c_1$ needs to be in $L(A)$.
      ii. The reversed string of c_i indicates which tiles were included.
   (c) $L(A)$ is context free:
      i. $G_A : A \to \varepsilon \mid a_1Ac_1 \mid a_2Ac_2 \mid \cdots \mid a_nAc_n$.
   (d) And similarly we define B, $L(B)$ and G_B.

(e) If $L(A)$ and $L(B)$ have non-empty words in common, then we can solve the instance of PCP:
   i. Suppose that some word in $L(A)$ equals a word in $L(B)$.
   ii. Then we must have used the same set of tiles (because they both end with the same string of c_is).
   iii. We also created the same word before the c_is.

(f) If $L(A)$ and $L(B)$ have no non-empty words in common, then we cannot solve the instance of PCP.

(g) So if we could decide CFG-intersection, then we could decide PCP, which is undecidable.

(h) Therefore CFG-intersection is undecidable.

2. Deciding CFG-ambiguity decides PCP.

(a) Consider the grammars G_A for $L(A)$ and G_B for $L(B)$.

(b) Construct a new grammar G_{AB} with
   i. variables A, B and S (with S as the start variable),
   ii. productions $S \to A|B$,
   iii. all the productions from G_A and
   iv. all the productions from G_B.

(c) With this construction completed, we now have the desired result by the following Theorem.

(d) **Theorem:** G_{AB} is ambiguous if and only if the instance (A, B) of PCP has a solution.

(e) **Proof:** ("If")
   i. Suppose that the indices $i_1, i_2, \ldots, i_m$ are a solution to this instance of PCP.
   ii. Then we have these derivations in G_{AB}:

$$S \Rightarrow A \Rightarrow a_{i_1}Ac_{i_1} \Rightarrow a_{i_1}a_{i_2}Ac_{i_2}c_{i_1} \Rightarrow \cdots \Rightarrow a_{i_1}\cdots a_{i_m}c_{i_m}\cdots c_{i_1}$$

$$S \Rightarrow B \Rightarrow b_{i_1}Bc_{i_1} \Rightarrow b_{i_1}b_{i_2}Bc_{i_2}c_{i_1} \Rightarrow \cdots \Rightarrow b_{i_1}\cdots b_{i_m}c_{i_m}\cdots c_{i_1}$$

   iii. By assumption, we have that $a_{i_1}\cdots a_{i_m} = b_{i_1}\cdots b_{i_m}$, i.e both derivations yield the same terminal string.
   iv. Since the derivations are distinct by construction, therefore we conclude that G_{AB} is ambiguous.

(f) ("Only if")
   i. Assume that G_{AB} is ambiguous.
   ii. The grammars G_A and G_B are unambiguous, because of the trailing tile markers.

iii. So the only way that a terminal string can have two different derivations in G_{AB} is if one derivation starts with $S \to A$ and the other starts with $S \to B$.
iv. The string with two different derivations has a tail $c_{i_m} \cdots c_{i_1}$, for some $m \geq 1$.
v. This tail gives a solution to the instance of PCP, because what precedes the tail is $a_{i_1} \cdots a_{i_m}$ in the first derivation and $b_{i_1} \cdots b_{i_m}$ in the second, and by assumption these must be equal.

24.2 Course Wrap-up

Stuff.

24.3 Course Evaluations

Stuff.