- Start as early as possible, and contact the instructor if you get stuck.
- See the course outline for details about the course's marking policy and rules on collaboration.
- Submit your completed solutions to **Crowdmark**.

1. Context-free and non-context-free languages

[4] (a) Prove that the language

$$L_a = \left\{ w \in \{0,1\}^* \mid n_1(w) = n_0(w)^2 \right\}$$

is **not** context-free. Recall that $n_1(w)$ denotes the number of occurrences of the symbol 1 in the string w, and $n_0(w)$ denotes the number of occurrences of the symbol 0 in the string w.

[4] (b) Let $\Sigma = \{a, b, c\}$ be the alphabet for this part and for part 1c. Prove that the language

$$L_b = \left\{a^i b^j c^k \mid j \leq i \text{ or } k \leq i\right\}$$

is context-free.

[4] (c) Prove that the complement, L_b', is **not** context-free. (**Remark:** This proves that L_b is **not** a DCFL.)

2. Closure rules for CFLs

[2] (a) Let L be a CFL and let F be a finite language. Prove that $L \backslash F = \{w \in L \mid w \notin F\}$ is a CFL.

[2] (b) Let L be a **non**-context-free language and let F be a finite language. Prove that $L \setminus F = \{w \in L \mid w \notin F\}$ is a **non**-context-free language.

[2] (c) Let L be a **non**-context-free language and let F be a finite language. Prove that $L \cup F$ is a **non**-context-free language.

3. Computations in a Turing machine

Let M be a Turing Machine over the alphabet $\Sigma = \{0, 1\}$. Let M's tape alphabet be $\{0, 1, B\}$. Let M's states be $\{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8\}$, with q_8 being the sole final state. Let the transition function, δ, for M be defined by the following table.

q	x	$\delta(q,x)$	q	x	$\delta(q,x)$	q	x	$\delta(q,x)$
q_0	B	(q_1, B, R)	q_2	B	(q_8, B, R)	q_6	0	$(q_6, 0, R)$
q_1	0	$(q_1, 0, R)$	q_3	B	$(q_4, 0, R)$	q_6	1	$(q_6, 1, R)$
q_1	1	$(q_1, 1, R)$	q_4	0	$(q_4, 0, R)$	q_6	B	$(q_7, 1, L)$
q_1	B	(q_2, B, L)	q_4	1	$(q_4, 1, R)$	q_7	0	$(q_7, 0, L)$
q_2	0	(q_3, B, R)	q_4	B	$(q_7, 0, L)$	q_7	1	$(q_7, 1, L)$
q_2	1	(q_5, B, R)	q_5	B	$(q_6, 1, R)$	q_7	B	(q_2, B, L)

Let M begin processing in the configuration $(q_0, \underline{B}w)$, where $w \in \Sigma^*$ is the input word.

[4] (a) Draw a diagram for M.

[2] (b) Give the sequence of instantaneous descriptions of M as it processes the input word $w = 01$.

[4] (c) Give the sequence of instantaneous descriptions of M as it processes the input word $w = 100$.

[2] (d) Briefly describe the algorithm which M performs, given any input word $w \in \Sigma^*$.

4. A language which is recursive but not context-free

[8] Let $\Sigma = \{a, b, c\}$. Recall from the lectures that this language is **not** context-free:

$$L = \left\{a^i b^i c^i \mid i \geq 0\right\}.$$

Give an **algorithm** for a Turing machine that **decides** membership in the language L. **You do not need to give a detailed diagram for the Turing machine**, provided you describe your algorithm clearly enough. Argue informally why your algorithm is correct.

5. Every context-free language is recursive

[6] Let Σ be a non-empty finite alphabet. Let G be an arbitrary context-free grammar over Σ, and let $L = L(G)$. Give an algorithm for a Turing machine which decides membership in the language L. Argue informally why your algorithm is correct.