- Start as early as possible, and contact the instructor if you get stuck.
- See the course outline for details about the course's marking policy and rules on collaboration.
- Submit your completed solutions to **Crowdmark**.

1. **Definition 1** *Let X and Y be sets. Then the* ***intersection of X and Y, denoted*** *$X \cap Y$, is the set of elements of both X and Y:*

$$X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}.$$

Definition 2 *Let X and Y be sets. Then the* ***union of X and Y, denoted*** *$X \cup Y$, is the set of elements of either X or Y (or both):*

$$X \cup Y = \{z \mid z \in X \text{ or } z \in Y, \text{ or both}\}.$$

Definition 3 *Let X and Y be sets. Then Y* ***is a subset of X, denoted*** *$Y \subseteq X$, if and only if every element of Y is also an element of X.*

Definition 4 *Let X be a set. Then the* ***power set of X, denoted*** *$P(X)$, is defined to be the set of subsets of X. In other words, $Y \in P(X)$ if and only if $Y \subseteq X$.*

[6] (a) Let E and F be sets. Prove that $P(E) \cap P(F) = P(E \cap F)$.

[3] (b) Let E and F be sets. Prove that $P(E) \cup P(F) \subseteq P(E \cup F)$.

[3] (c) Give an example of finite sets E and F such that $P(E \cup F) \not\subseteq P(E) \cup P(F)$. Briefly explain why your choice of E and F is correct.

[3] 2. Let $\Sigma = \{0, 1\}$. Let L be the language, over Σ, or words having at least two different substrings, each of length 2. For example,

- $010 \in L$, because it contains the substrings 01 and 10, and
- $000 \notin L$, because its only substring of length 2 is 00.

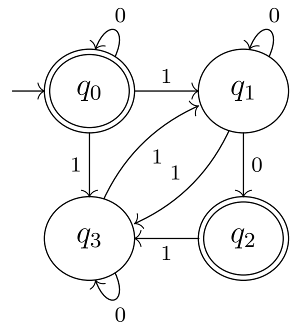
Describe L by writing a sentence of the form

$$L = \{w \in \Sigma^* \mid P(w)\},$$

where $P(w)$ is a first-order logic formula. In $P(w)$, you may use the notation

- $|x|$ to return the length of any string x,
- standard arithmetic relations $=, \neq, <, \leq$, etc.,
- standard arithmetic constants $0, 1, 2$, etc., and
- the relation $Substr(u, v)$, which is true if and only if u is a substring of v.

3. Consider the NFA, M, having alphabet $\Sigma = \{0, 1\}$ and defined by the following diagram.



For each choice of input word w given below, determine whether or not $w \in L(M)$. Briefly justify each answer.

[3] (a) $w = 101$

[3] (b) $w = 1010$

4. Draw the diagram of a DFA, NFA or ε-NFA which accepts each of the following languages over $\Sigma = \{0, 1\}$, and argue informally why your automaton accepts exactly the language given.

[4] (a) $L_a = \{w \in \Sigma^* \mid n_0(w) \geq 2\}$. Recall that $n_0(w)$ denotes the number of occurrences of the symbol 0 in the string w.

[4] (b) $L_b = \{w \in \Sigma^* \mid w \text{ begins or ends with 00 or 11}\}$.

[4] (c) $L_c = \{w \in \Sigma^* \mid n_0(w) \equiv 0 \bmod 2 \text{ and } n_1(w) \equiv 0 \bmod 2\}$.

5. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. let $\hat{\delta}$ denote the extended transition function of M, as defined in the lecture slides.

[4] (a) Prove that, for any $x, y \in \Sigma^*$, and any $q \in Q$, we have

$$\hat{\delta}(q, xy) = \hat{\delta}(\hat{\delta}(q, x), y).$$

[4] (b) Assume that for some state $q \in Q$, and for every $a \in \Sigma$, we have $\delta(q, a) = q$. Prove that $\hat{\delta}(q, x) = q$ holds for every $x \in \Sigma^*$.

[4] (c) Assume that for some state $q \in Q$, and some string $x \in \Sigma^*$, we have $\hat{\delta}(q, x) = q$. Prove that, for every $n \geq 0$, we have $\hat{\delta}(q, x^n) = q$.