Four friends: Alex, Ben, Gina and Dana are having a discussion about going to the movies. Ben says that he will go to the movies if Alex goes as well. Gina says that if Ben goes to the movies, then she will join. Dana says that she will go to the movies if Gina does. That afternoon, exactly two of the four friends watch a movie at the theatre. Deduce which two people went to the movies.

Solution: Logically,

$$A \Rightarrow B$$

$$B \Rightarrow G$$

$$G \Rightarrow D$$

If Alex goes, Then Ben goes, then Gina goes and Dana goes. Thus Alex doesn't go.

If Ben goes then Gina goes and Dana goes. Thus Ben doesn't go.

If Gina goes then Dana goes and that's it. So Gina and Dana go to the movies.

Negation Examples:
Example: Negate $(A \Leftrightarrow B)$.

$$\begin{aligned}
\neg(A \Leftrightarrow B) &\equiv \neg((A \Rightarrow B) \wedge (B \Rightarrow A)) \\
&\equiv \neg(A \Rightarrow B) \vee \neg(B \Rightarrow A) && \text{DeMorgan's Law} \\
&\equiv (A \wedge \neg B) \vee (B \wedge \neg A)
\end{aligned}$$

Prove 4 consecutive integers is one less than a perfect square.

Proof: Let n be an integer. We want to show that

$$(n-1)n(n+1)(n+2)+1=y^2$$

for some integer y.

$$\begin{aligned}(n-1)n(n+1)(n+2)+1 &= (n^2-n)(n^2+3n+2)+1\\ &= n^4+2n^3-n^2-2n+1\\ &= (n^2+n-1)^2\end{aligned}$$

Napkin:

$$(n^2+an+b)^2 = n^4+2an^3+(2b+a^2)n^2+2abn+b^2$$

If this is going to work, the above must be equal to $n^4+2n^3-n^2-2n+1$. So $2an^3=2n^3$ (for all n) Thus $a=1$ is the only possibility. Comparing the n^2 terms shows that $b=-1$.

Division Algorithm

Compute the quotient and remainder when

1. $a = 172$ and $b = 5$.
2. $a = -172$ and $b = 5$.
3. $a = 172$ and $b = -5$.
4. $a = -172$ and $b = -5$.

Solutions:

1. $172 = 5(34) + 2$ Thus $q = 34$ and $r = 2$
2. $-172 = -(5(34)+2) = 5(-34) - 2 + 5 - 5 = 5(-34) - 5 + 3 = 5(-35) + 3$. Thus $q = -35$ and $r = 3$.
3. $172 = -5(-34) + 2$. Thus $q = -34$ and $r = 2$.
4. $-172 = -5(34) - 2 = -5(34) - 2 + 5 - 5 = -5(35) + 3$. Thus $q = 35$ and $r = 3$.

Alice and Bob sticks game. Goal: Show that the first player wins if and only if 4 doesn't divide n.

Solution: Let $P(n)$ be the statement "the first player wins if and only if 4 doesn't divide n". We prove $P(n)$ by strong induction.

Base Cases: $n = 1, 2, 3$ are easy (take all sticks) and $n = 4$, if the first player takes f sticks then the second player can take $4 - f$ sticks and since $1 \leq f \leq 3$ then $1 \leq 4 - f \leq 3$.

Induction Hypothesis Assume $P(i)$ is true for all $1 \leq i \leq k$ and $k \geq 4$.

Inductive Step: With $k + 1$ sticks, use the division algorithm to write $k + 1 = 4q + r$. Now $0 \leq r \leq 3$. If $r = 0$, then $4 \mid (k + 1)$. First player takes f sticks and player two takes $4 - f$ sticks. As before this is 4 sticks total so we reduce to the case $k - 3$ and we know $P(k - 3)$ is true by induction. Thus the first player loses since $4 \mid k - 3$.

If $r \neq 0$ then the first player should take r sticks. Player two is left with $k + 1 - r$ sticks which is divisible by 4 and hence the induction hypothesis $P(k + 1 - r)$ says that player 2 loses.

Prove that the sum of a rational number and irrational number is irrational.

Assume that the sum of a rational and irrational number is rational. Say $q + k = r$ where $q, r \in \mathbb{Q}$ and $k \in \mathbb{R} - \mathbb{Q}$. Now $k = r - q \in \mathbb{Q}$. This is a contradiction. Thus the sum of a rational and irrational is always irrational.

Negate the following:

$$\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \exists c \in \mathbb{Z}, ((a+b) > c \Rightarrow b > c).$$

Solution

$$\begin{aligned} &\neg(\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \exists c \in \mathbb{Z}, ((a+b) > c \Rightarrow b > c)) \\ \equiv\ &\exists a \in \mathbb{Z}\ \exists b \in \mathbb{Z}\ \forall c \in \mathbb{Z}\ ((a+b) > c \wedge b \leq c) \end{aligned}$$

Let $b \in \mathbb{Z}$. Let S be the statement $(\forall k \in \mathbb{N}\ b \mid k) \Rightarrow b = 0$. Negate S.

Negation of S: $(\forall k \in \mathbb{N}\ b \mid k) \wedge b \neq 0$.

Contrapositive: If $H \Rightarrow C$ then the contrapositive is $\neg C \Rightarrow \neg H$.

Contrapositive of S: $b \neq 0 \Rightarrow \exists k \in \mathbb{N}$ b does not divide k.

Show that there is a unique minimum value to $x^2 - 4x + 11$.

Solution Uniqueness: Suppose that $m^2 - 4m + 11 = n^2 - 4n + 11$ are both the minimum values. Isolating and factoring shows that $m^2 - n^2 + 4(n - m) = 0$. Factor again

$$(m - n)(m + n) - 4(m - n) = 0.$$

Factor one more time:

$$(m - n)(m + n - 4) = 0.$$

Thus either $m = n$ or $m + n = 4$. In the first case, we've shown that m is unique. Assume that $m + n = 4$... (Let's use existence). Since a minimum occurs at 2, starting with $m = 2$ yields $n = 2$ in both cases.

Existence: Since $x^2 - 4x + 11 = x^2 - 4x + 4 - 4 + 11 = (x - 2)^2 + 7$, we know that a minimum must occur at $x = 2$ and the minimum value is 7.

Written cleaner

Since $x^2 - 4x + 11 = x^2 - 4x + 4 - 4 + 11 = (x - 2)^2 + 7$, we know that a minimum must occur at $x = 2$ and the minimum value is 7. If another value n is such that $n^2 - 4n + 11 = 7$, then isolating for 0 gives us that

$$0 = n^2 - 4n + 4 = (n - 2)^2$$

and so $n = 2$ is the only possible value.

Cartesian Product

If $S = \{1, 2\}$ and $T = \{1, 3, 4\}$ then

$$S \times T = \{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4)\}$$

Note: $(1, 2) \notin S \times T$ but $(2, 1) \in S \times T$.