Does 7 divide $x^{12} + x^2 + 3$ for some integer x.

Solution: We are trying to determine if

$$x^{12} + x^2 + 3 \equiv 0 \mod 7$$

Let's use Fermat's Little Theorem which states that

$$a^6 \equiv 1 \mod 7$$

whenever $7 \nmid a$.

If $7 \mid x$, then $x \equiv 0 \mod 7$

$$x^{12} + x^2 + 3 \equiv (0)^{12} + (0)^2 + 3 \equiv 3 \mod 7$$

Now, we suppose that $7 \nmid x$. By FLT, we have that $x^6 \equiv 1 \mod 7$ and squaring gives $x^{12} \equiv 1 \mod 7$. Substituting gives

$$x^{12} + x^2 + 3 \equiv 1 + x^2 + 3 \equiv x^2 + 4 \mod 7$$

Trying all 6 values gives

$$\begin{aligned}(\pm 1)^2 + 4 &\equiv 5 \mod 7\\ (\pm 2)^2 + 4 &\equiv 1 \mod 7\\ (\pm 3)^2 + 4 &\equiv 6 \mod 7\end{aligned}$$

Since none of the above values are 0, we have that $7 \nmid x^{12} + x^2 + 3$.