Find a closed form expression for

$$\sum_{j=1}^{n} j \cdot j!$$

and prove it is true by induction for all $n \in \mathbb{N}$.

Solution:

$n = 1$: $\displaystyle\sum_{j=1}^{n} j \cdot j! = \sum_{j=1}^{1} j \cdot j! = 1 \cdot 1! = 1$

$n = 2$: $\displaystyle\sum_{j=1}^{n} j \cdot j! = \sum_{j=1}^{2} j \cdot j! = 1 \cdot 1! + 2 \cdot 2! = 1 + 4 = 5$

$n = 3$: $\displaystyle\sum_{j=1}^{n} j \cdot j! = \sum_{j=1}^{3} j \cdot j! = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! = 1 + 4 + 18 = 23$

$n = 4$:

$$\begin{aligned}
\sum_{j=1}^{n} j \cdot j! = \sum_{j=1}^{4} j \cdot j! &= 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! \\
&= 1 + 4 + 18 + 4 \cdot 24 = 23 + 96 = 119
\end{aligned}$$

$n = 5$:

$$\begin{aligned}
\sum_{j=1}^{n} j \cdot j! = \sum_{j=1}^{5} j \cdot j! &= 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! \\
&= 1 + 4 + 18 + 96 + 5 \cdot 5! = 119 + 5 \cdot 120 = 719
\end{aligned}$$

Claim: $\displaystyle\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1$

Aside: To get from $n = 5$ to $n = 6$,

$$\begin{aligned}
\sum_{j=1}^{n} j \cdot j! = \sum_{j=1}^{6} j \cdot j! \\
&= \sum_{j=1}^{5} j \cdot j! + 6 \cdot 6! \\
&= 6! - 1 + 6 \cdot 6! \\
&= 7 \cdot 6! - 1 \\
&= 7! - 1
\end{aligned}$$

Claim: $\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1$

Solution: Let $P(n)$ be the statement that

$$\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1$$

We prove $P(n)$ is true for all $n \in \mathbb{N}$ by mathematical induction.

Base Case: When $n = 1$,

$$\sum_{j=1}^{n} j \cdot j! = \sum_{j=1}^{1} j \cdot j! = 1 \cdot 1! = 1 = (1+1)! - 1.$$

Induction Hypothesis: Assume that $P(k)$ is true for some $k \in \mathbb{N}$, that is, we assume that $\sum_{j=1}^{k} j \cdot j! = (k+1)! - 1$.

Induction Step: Now, suppose that $n = k + 1$. We need to show that

$$\sum_{j=1}^{k+1} j \cdot j! = (k+1+1)! - 1 = (k+2)! - 1.$$

To do this, we start with the summation and see that

$$\begin{aligned}
\sum_{j=1}^{k+1} j \cdot j! &= \sum_{j=1}^{k} j \cdot j! + (k+1)(k+1)! \\
&= (k+1)! - 1 + (k+1)(k+1)! \qquad \text{Induction Hypothesis} \\
&= (k+1)! + (k+1)(k+1)! - 1 \\
&= (k+1)!(1+k+1) - 1 \\
&= (k+2)(k+1)! - 1 \\
&= (k+2)! - 1
\end{aligned}$$

Hence $P(k+1)$ is true. Hence $P(n)$ is true for all $n \in \mathbb{N}$ by Mathematical Induction.