Let $\{f_n\}$ be the Fibonacci sequence (that is, $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$). Show that $f_n < \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{N}$.

Solution: We proceed by strong induction.

Base Case: For $n = 1$, we have that $f_1 = 1 < \frac{7}{4}$.

For $n = 2$, we have that $f_2 = 1 < \frac{49}{16} = \left(\frac{7}{4}\right)^2$.

Induction Hypothesis: Assume that $f_i < \left(\frac{7}{4}\right)^i$ for all $1 \leq i \leq k$ for some $k \in \mathbb{N}$ and $k \geq 2$.

Induction Step: Now, we want $f_{k+1} < \left(\frac{7}{4}\right)^{k+1}$. For $k+1$, we have

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} && \text{Valid since } k+1 \geq 3 \\
&< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} && \text{Induction Hypothesis} \\
&= \left(\frac{7}{4}\right)^{k-1} \left(\frac{7}{4} + 1\right) \\
&= \left(\frac{7}{4}\right)^{k-1} \left(\frac{11}{4}\right) \\
&< \left(\frac{7}{4}\right)^{k-1} \frac{49}{16} \\
&= \left(\frac{7}{4}\right)^{k-1} \frac{7^2}{4^2} \\
&= \left(\frac{7}{4}\right)^{k+1}
\end{aligned}$$

Thus, $f_{k+1} < \left(\frac{7}{4}\right)^{k+1}$ and hence, the statement $f_n < \left(\frac{7}{4}\right)^n$ is true for all $n \in \mathbb{N}$ by Strong Induction.