Let

$$S = \mathbb{R}\backslash\{-1\} = \mathbb{R} - \{-1\} \qquad \text{and} \qquad T = \mathbb{R}\backslash\{2\} = \mathbb{R} - \{2\}.$$

Show that the function $f : S \to T$ defined by

$$f(x) = \frac{2x+1}{x+1}$$

is injective and surjective (hence bijective or a bijection).

Solution: Let $x, y \in S$ be such that $f(x) = f(y)$. Then,

$$\begin{aligned} \frac{2x+1}{x+1} &= \frac{2y+1}{y+1} \\ (2x+1)(y+1) &= (2y+1)(x+1) \\ 2xy + 2x + y + 1 &= 2xy + 2y + x + 1 \\ x &= y \end{aligned}$$

To show surjectivity, let $y \in T$. Let $x = \frac{y-1}{2-y}$. Notice that $x \in \mathbb{R}$ since $y \neq 2$. Further, notice that $x \neq -1$, for otherwise, $-1 = \frac{y-1}{2-y}$ and so $y - 2 = y - 1$ hence $-2 = -1$, a contradiction. Thus, $x \in S$ and,

$$\begin{aligned} f(x) &= \frac{2x+1}{x+1} \\ &= \frac{2\left(\frac{y-1}{2-y}\right)+1}{\frac{y-1}{2-y}+1} \\ &= \frac{\frac{2y-2+2-y}{2-y}}{\frac{y-1+2-y}{2-y}} \\ &= y \end{aligned}$$

Napkin Computation:

$$\begin{aligned} \frac{2x+1}{x+1} &= y \\ 2x + 1 &= xy + y \\ 2x - xy &= y - 1 \\ x &= \frac{y-1}{2-y} \end{aligned}$$